Theory of a Superfluid Fermi Liquid. I. General Formalism and Static Properties*

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The microscopic theory of a superfluid Fermi liquid at finite temperature is developed for the case of a pure system with S-wave pairing, and applied to the calculation of the static properties. As a function of $\theta \equiv T/T_c$ these properties are determined entirely by the Landau parameters F_0, F_1, Z_0, etc., characterizing quasiparticle interactions in the normal phase. In particular the spin susceptibility χ and the density of the normal component ρ_n are given by

$$\chi(\theta)/\chi(1) = (1+\tfrac{1}{4}Z_0)\ f(\theta)/[1+\tfrac{1}{4}Z_0 f(\theta)],$$
$$\rho_n/\rho = (1+\tfrac{1}{3}F_1)\ f(\theta)/[1+\tfrac{1}{3}F_1 f(\theta)],$$

where the universal function $f(\theta) \equiv -[\nu(0)]^{-1}\Sigma_{\mathbf{p}}(dn/dE_{\mathbf{p}})$ is the "effective density of states near the Fermi surface" relative to its value $\nu(0)$ in the normal phase. Thus the often-quoted expression $\rho_n = \frac{1}{3}\Sigma_{\mathbf{p}}\mathbf{p}^2(dn/dE_{\mathbf{p}})$ is valid for an interacting system only in the limit $T \to 0$. In the latter part of the paper a simple phenomenological theory of "Fermi-liquid" effects on χ and ρ_n is developed for arbitrary conditions (including the presence of impurities and pairing with $l \neq 0$); it is found that under most circumstances explicit expressions for χ and ρ_n may be obtained which involve only the Landau parameters and a suitably generalized effective density of states. The theory should apply to the possible superfluid phase of He^3 and to most superconductors. It is suggested that the Knight shift in nontransition-metal superconductors should display some "Fermi-liquid" effects. The weak-field dc penetration depth $\lambda(T)$ is shown to be insensitive to such effects both in the Pippard limit and near T_c; however, in a London superconductor at lower temperatures the correction to $\lambda(T)$ should be observable and yield a direct estimate of F_1.

I. INTRODUCTION

IN the theory of superconductivity and of the possible superfluidity of liquid He^3, it is generally assumed not only that the pair interaction between particles of opposite momentum, which is specifically responsible for superfluidity, is weak, but also that all other interparticle interactions are weak. This second assumption is by no means justifiable. Indeed, if we denote by F a characteristic interparticle interaction multiplied by the density of states at the Fermi surface, then we know from experiment that $F \sim 1$ for He^3, while for metallic superconductors theoretical estimates predict values of the order of 0.1–0.5. It is therefore of interest to develop a theory in which these interactions are taken into account.

As is well known, in a normal system at sufficiently low temperatures the interactions between quasiparticles may conveniently be handled by the Fermi-liquid theory of Landau[1,2]; this theory can be applied directly to liquid He^3 and, with certain refinements, also to the electron "liquid" in a metal[3,4]—provided, at least, one is content to neglect the effects of anisotropy.[5] At the cost of introducing a few parameters whose values must be taken from experiment, the Landau theory is able to give an essentially exact account of the behavior of the system in the region of long wavelengths and low frequencies. The key to its success is that at sufficiently low temperatures, all thermal excitations are close to the Fermi surface. Consequently, the interaction between two quasiparticles may be regarded as a function only of the angle between their momenta (or quasimomenta) and their relative spin state. To put it more precisely, let us define a temperature T_0 such that the scattering amplitude $\Gamma(\mathbf{p},\mathbf{p}',\epsilon,\epsilon':\mathbf{k},\omega)$, considered as a function of $\epsilon,\epsilon',\Delta p,\Delta p'$ ($\Delta p \equiv |p| - p_F$) for[6] given small values of the momentum and energy transfer $(\mathbf{k},\omega)$ and given relative direction of $\mathbf{p}$ and $\mathbf{p}'$, is effectively constant in the region $v\Delta p, v\Delta p', \epsilon, \epsilon' \sim T_0$, where v is the Fermi velocity. Then the Landau theory will be a good description of the long-wavelength, low-frequency response of the system for temperatures $T \lesssim T_0$.

Correspondingly, one might expect that it is possible to develop a similar semiphenomenological description of the superfluid phase provided that the critical temperature $T_c \lesssim T_0$ and provided also that the pairing interaction itself is slowly varying over a range of ϵ and $v\Delta p$ comparable to T_0. A system for which these conditions are satisfied will be called a "superfluid Fermi liquid."[7] The description of such a system will require, in general, a specification of the usual Landau parameters and also of the pairing interaction as a function of spins and scattering angle. Actually, however, many interesting properties can be obtained from a knowledge only of the Landau parameters and of the gap Δ (or,

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[1] L. D. Landau, Zh. Eksperim. i Teor. Fiz. **30**, 1058 (1956) [English transl.: Soviet Phys.—JETP **3**, 920 (1957)].
[2] L. D. Landau, Zh. Eksperim. i Teor. Fiz. **35**, 95 (1958) [English transl.: Soviet Phys.—JETP **8**, 70 (1959)].
[3] V. P. Silin, Zh. Eksperim. i Teor. Fiz. **33**, 495 (1957) [English transl.: Soviet Phys.—JETP **6**, 387 (1958)].
[4] D. Pines and P. Nozières, *Theory of Quantum Liquids* (W. A. Benjamin, Inc., New York, 1965).
[5] Formally, a theory of the Landau type can be constructed which will take into account the effects of anisotropy; in practice, however, it usually involves too many parameters to be useful.
[6] We put $\hbar = k = 1$ throughout. Energies are measured relative to the chemical potential.
[7] Note that the so-called "strong-coupling" superconductors do not conform to this definition.

equivalently, T_c), i.e., for many purposes we can neglect all harmonics of the pairing interaction except the most attractive one. Thus the theory is quite economical as regards phenomonological parameters.

For the case of zero temperature and S-wave pairing, such a theory has already been developed by Larkin and Migdal,[8] and subsequently applied by Larkin[9] to the electrodynamics of superconductors. However, there are some interesting "Fermi-liquid" effects which show up only at finite temperatures; for instance, it was shown in Ref. 10 that the temperature dependence of the paramagnetic spin susceptibility of a superfluid Fermi liquid may be quite different from that predicted by a "gas" model.

The main purpose of this paper is to develop a general microscopic theory of the superfluid Fermi liquid at finite temperatures, which can be used to derive the response functions throughout the range $\omega,kv \lesssim T_0$. Thus, in principle at least, the theory can describe the static properties, the response to a space- and time-varying external field, and also the kinetic coefficients. Here it is applied to the calculation of the static properties as a function of temperature.[11] The model considered is that of a pure, isotropic, neutral system with S-wave pairing; translation invariance is not assumed. Of course, any real system which is not translationally invariant will *ipso facto* also display anisotropic behavior; however, we may hope that it is a good approximation in many cases to neglect the latter. Since the effect of the long-range Coulomb interactions can be taken into account at the end of the calculation by the method of Larkin,[9] it should be possible to apply the results, at least qualitatively, to some real superconductors. Also, although it is improbable that liquid He^3 will condense into an S state, these results should give a qualitative guide to the kind of behavior to be expected in the superfluid phase of He^3 if it exists. However, quantitative comparison with experiment of the most interesting predictions derived here, namely the temperature dependence of the spin susceptibility and London penetration depth, is prevented in many cases by the presence in the experimental situation of factors not allowed for by the simple model used (e.g., impurities). Accordingly, having derived these predictions for the simple model by rigorous microscopic methods, we go on to generalize them to a rather wider range of conditions by a more phenomenological argument. It is shown that the interactions between quasiparticles lead to "molecular fields" of various types; as a result, if the response to an external field of a given type is temperature-dependent in the "gas" model, the form of the temperature dependence is changed by the quasiparticle interactions, in a way which depends solely on Landau's dimensionless F parameters. Thus, under favorable conditions we can obtain the values of some of the F's directly from the experimental curves.

In Sec. II a general expression for the autocorrelation functions is derived in terms of the phenomenological parameters. In Sec. III the density, spin, current, and spin-current autocorrelation functions are evaluated explicitly in the static limit ($\omega=0, \mathbf{k}\to 0$). In this way the compressibility, spin susceptibility, and London penetration depth are obtained as a function of temperature, and the consistency of the theory is checked by verifying the longitudinal sum rules in this limit. In Sec. IV a less rigorous but more general treatment of the static properties of a superfluid Fermi liquid is given, and in Sec. V applications to real system are discussed. Section VI briefly summarizes the results of the paper. In an Appendix, the microscopic derivation of the Landau theory for normal systems at finite temperature is given.

Readers not interested in the details of the microscopic formulation may find it convenient to skip directly to Secs. IV and V, which are logically independent of the earlier sections and do not require an understanding of field-theoretic techniques.

II. AUTOCORRELATION FUNCTIONS

Most static, dynamic and kinetic properties of interest for a many-particle system may be obtained from a complete knowledge of the long-wavelength and low-frequency behavior of the autocorrelation functions

$$K_\xi(\mathbf{k},\omega)=\sum_{\mathbf{p}\mathbf{p}'\sigma\sigma'}\xi(p,\sigma)\ll a_{p+k/2,\sigma}^\dagger a_{p-k/2,\sigma}:\, a_{p'-k/2,\sigma}^\dagger a_{p'+k/2,\sigma}\gg_-(\omega)\xi(\mathbf{p}',\sigma'). \quad (1)$$

In Eq. (1) the "bare vertex" $\xi(\mathbf{p},\sigma)$ is some function of $\mathbf{p}$ and[12] σ (e.g., $1,p_\mu,\sigma,p_\mu\sigma$), and $\ll A:B\gg_\pm(\omega)$ denotes a retarded Green's function:

$$\ll A:B\gg_\pm(\omega)=-i\int_{-\infty}^{\infty}\theta(t)\langle A(t)B(0)\pm B(0)A(t)\rangle \times\exp i\omega t\, dt. \quad (2)$$

In this section a general expression will be derived for $K_\xi(\mathbf{k},\omega)$ for a superfluid Fermi liquid under the assumption that $\omega, vk\lesssim T_0$, where T_0 is the quantity introduced in the last section. It will always be assumed that T_0 is small compared to the chemical potential μ. The expression will contain the same phenomenological parameters as appear in the zero-temperature theory of Larkin and Migdal,[8] namely, the usual Landau parameters plus a function characterizing the pairing interaction.

[8] A. I. Larkin and A. B. Migdal, Zh. Eksperim. i Teor. Fiz. **44**, 1703 (1963) [English transl.: Soviet Phys.—JETP **17**, 1146 (1963)].

[9] A. I. Larkin, Zh. Eksperim. i Teor. Fiz. **46**, 2188 (1964) [English transl.: Soviet Phys.—JETP **19**, 1478 (1964)].

[10] A. J. Leggett, Phys. Rev. Letters **14**, 536 (1965).

[11] I hope to devote a subsequent paper to a consideration of the collective oscillations.

[12] In all cases of physical interest we can choose the representation so that $\xi(\mathbf{p},\sigma)$ is diagonal in the spin indices.

The first—and most tedious—step consists in the separation of "quasiparticle" and "nonquasiparticle" contributions to $K_\xi(\mathbf{k},\omega)$. In the mathematics this separation is made whether or not the quantity being considered is conserved; one then proves that for a conserved quantity the "nonquasiparticle" contribution is identically zero. The physics of the separation, however, is most easily understood by considering a quantity, such as the spin current, which is not conserved. We consider therefore the spin-current autocorrelation function[13] $K_{p\sigma}$ [$\xi(\mathbf{p},\sigma)=p_\mu\sigma/m$ in Eq. (1)] for small $\mathbf{k}$ and ω, and suppose for simplicity that the system is translationally invariant. We can make a spectral expansion of Eq. (1):

$$K_{p\sigma}(\mathbf{k},\omega)=2\sum_m e^{-E_m/T} \times\sum_n \frac{|\langle n|J_{\mathbf{k}\mu}{}^{sp}|m\rangle|^2(E_n-E_m)}{(\omega+i\delta)^2-(E_n-E_m)^2}, \quad (3)$$

$$(\delta\to+0)$$

where

$$J_{\mathbf{k}\mu}{}^{sp}\equiv\sum_{\mathbf{p}\sigma}(p_\mu\sigma/m)a_{\mathbf{p}+\mathbf{k}/2,\sigma}{}^\dagger a_{\mathbf{p}-\mathbf{k}/2,\sigma}.$$

By hypothesis all low-lying energy eigenstates of the system, and hence all states $|m\rangle$ occurring with appreciable thermal weight in (3), are describable in terms of Landau quasiparticles. Consider now those states $\langle n|$ occurring in (3) for which $E_n-E_m\gg T_0$. Such states will not in general be describable in terms of Landau quasiparticles; moreover, they will be highly degenerate. Let us use this last fact to choose the energy eigenstates so that they are *not* eigenstates of the total momentum. Then, we may expect the matrix elements $\langle n|J_{\mathbf{k}\mu}{}^{sp}|m\rangle$ to be smooth functions of $\mathbf{k}$; the same will be true of the matrix elements of the spin density fluctuation $\langle n|\rho_{\mathbf{k}}{}^{sp}|m\rangle$ where

$$\rho_{\mathbf{k}}{}^{sp}\equiv\sum_{\mathbf{p}\sigma}\sigma a_{\mathbf{p}+\mathbf{k}/2,\sigma}{}^\dagger a_{\mathbf{p}-\mathbf{k}/2,\sigma}.$$

From the spin-continuity equation we have, expanding $\rho_{\mathbf{k}}$ in a series in $\mathbf{k}$,

$$\langle n|\mathbf{k}\cdot\mathbf{J}_{\mathbf{k}}{}^{sp}|m\rangle=(E_n-E_m)\langle n|\rho_{\mathbf{k}}|m\rangle =(E_n-E_m)\{\langle n|S_z|m\rangle +\langle n|\mathbf{k}\cdot\boldsymbol{\alpha}|m\rangle+\cdots\}, \quad (4)$$

where S_z is just the operator of the total z component of spin of the system and $\boldsymbol{\alpha}$ is some operator independent of $\mathbf{k}$. Since S_z is conserved (so that $\langle n|S_z|m\rangle\equiv0$), it follows from (4) that $\langle n|\mathbf{J}_{\mathbf{k}}{}^{sp}|m\rangle$ is independent of $\mathbf{k}$ for small $\mathbf{k}$ and hence is equal to $\langle n|\mathbf{J}_0{}^{sp}|m\rangle$, where $\mathbf{J}_0{}^{sp}$ is the total spin-current operator of the system. Therefore, if the total spin current were conserved, it would follow that the "nonquasiparticle" contribution to $K_{p\sigma}(\mathbf{k},\omega)$ [Eq. (3)], that is, the contribution from states $\langle n|$ such that $E_n-E_m\gg T_0$, would be identically zero.[14] As it is, there will be a finite contribution which we label $\Phi_{p\sigma}$; this, however, is only weakly dependent on $\mathbf{k}$. It also follows from an inspection of Eq. (3) that $\Phi_{p\sigma}$ is only weakly dependent on ω. The $(\mathbf{k},\omega)$ dependence of the "quasiparticle" contributions is in general much sharper. Finally, since only low-lying states of the system are appreciably affected by the superfluid transition, $\Phi_{p\sigma}$ is independent of the transition, and hence *a fortiori* of T. Thus $\Phi_{p\sigma}(\mathbf{k},\omega,T)\equiv\Phi_{p\sigma}=\text{const}$.

A second, indeed complementary, consequence of the lack of conservation of the spin current is that the average spin current carried by a quasiparticle of momentum $\mathbf{p}$ and spin σ is *not* simply $(\mathbf{p}\sigma/m)$; we must in fact multiply this expression by some "renormalization factor" $R_{p\sigma}$. It is obvious from symmetry considerations that $R_{p\sigma}$ must be independent of $\mathbf{p}$ and σ for $|p|$ close to p_F. Thus, we should expect the spin-current autocorrelation function to have the form

$$K_{p\sigma}(\mathbf{k},\omega)=\Phi_{p\sigma}+R_{p\sigma}{}^2I_{p\sigma}(\mathbf{k},\omega),$$

where $\Phi_{p\sigma}$ and $R_{p\sigma}$ are constants, and $I_{p\sigma}(\mathbf{k},\omega)$ is some expression which involves only quasiparticle states. (When we write down a field-theoretic expression for K, this means that $I_{p\sigma}$ will involve only integrals taken close to the Fermi surface.)

Very similar remarks apply to the particle current autocorrelation function in the case where total momentum is not conserved. In this case the quasiparticle states are, of course, labeled by values of the pseudomomentum and are not eigenstates of the true momentum; the "nonquasiparticle" contribution Φ_p then includes the effect of interband transitions.

Our task, then, is to represent the correlation functions in the form

$$K_\xi(\mathbf{k},\omega)=\Phi_\xi+R_\xi{}^2I_\xi(\mathbf{k},\omega), \quad (5)$$

where Φ_ξ is some constant (which may be zero), R_ξ is also a constant, and $I_\xi(\mathbf{k},\omega)$ is some expression which involves only integrals taken close to the Fermi surface, and therefore is completely specified by the phenomenological parameters of the theory. It is essential to the theory to be developed here that the constants Φ_ξ and R_ξ are the same for the superfluid and normal phases; this of course is just a formalization of the basic hypothesis that the superfluid state can be described in terms of Landau quasiparticles whose structure is insensitive to the transition.

For the zero-temperature case the field-theoretic techniques leading to expressions of the type (5) are by

[13] We label the various autocorrelation functions by the functions of $\mathbf{p}$ and σ which appear in $\xi(\mathbf{p},\sigma)$; thus, K_1, K_σ, K_p, $K_{p\sigma}$ represent, respectively, density, spin density, current and spin current.

[14] Cf. D. Pines, in *Proceedings of the Ninth International Conference on Low-Temperature Physics* (Plenum Press, Inc., New York, 1965).

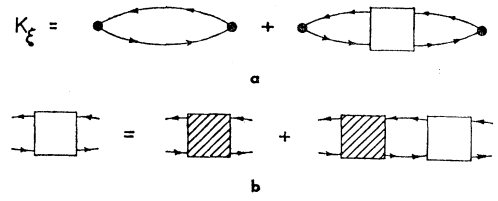

FIG. 1. Diagrammatic representation of (a) Eq. (6) and (b) Eq. (7). Open blocks represent Γ; shaded blocks, $\Gamma^{(0)}$; heavy dots, the "bare vertices" $\xi(\mathbf{p},\sigma)$. The two arrows on each internal line emphasize particle-number conservation in the normal phase.

now well known.[15,16] However, for finite temperatures there does not seem to exist in the literature a complete derivation even for normal systems (cf., however, Refs. 17 and 18), so we give the details of the calculation in an Appendix. Here we just sketch the main features and then go on to discuss the generalization to the superfluid case.

The basis of the derivation, as of all similar ones at $T=0$, is the equations expressing $K_\xi(\mathbf{k},\omega)$ in terms of the complete vertex part Γ, and Γ in terms of an "irreducible" vertex part $\Gamma^{(0)}$: in symbolic form

$$K_\xi=\xi GG\xi+\xi GG\Gamma GG\xi, \tag{6}$$

$$\Gamma=\Gamma^{(0)}+\Gamma^{(0)}GG\Gamma. \tag{7}$$

The graphical representation of Eqs. (6) and (7) is given in Fig. 1. In the finite-temperature case relations (6) and (7) hold between "temperature" functions defined in the complex energy plane and must be analytically continued to the real axis. Physically more fundamental than $\Gamma^{(0)}$ is Landau's "quasiparticle-irreducible" vertex part $\Gamma^\omega(\mathbf{pp}',\sigma\sigma')$ which by definition contains no pairs of lines corresponding to the propagation of a quasiparticle-quasihole pair; it is a slowly varying function of energy and magnitude of momentum near the Fermi surface. Introducing certain quantities $Q_{ij}{}^\omega$ related to Γ^ω (see below) we find the required expression in the form

$$K_\xi=\Phi_\xi+R_\xi{}^2\,\mathrm{Tr}\left\{\xi\frac{\lambda_i g_i}{1-Q_{ij}{}^\omega g_j}\xi\right\}. \tag{8}$$

In Eq. (8) the notation is as follows (see Appendix for further details). The quantities $g_i(\mathbf{p},\epsilon:\mathbf{k},\omega)$ $(i=1,2,3)$ are various combinations of the *singular* ("quasiparticle") parts of the one-particle retarded and advanced Green's functions:

$$\begin{aligned} g_1(\mathbf{p},\epsilon:\mathbf{k},\omega)&\equiv G_R(\mathbf{p}+\mathbf{k}/2,\ \epsilon+\omega)G_R(\mathbf{p}-\mathbf{k}/2,\ \epsilon)\,,\\ g_2(\mathbf{p},\epsilon:\mathbf{k},\omega)&\equiv G_R(\mathbf{p}+\mathbf{k}/2,\ \epsilon+\omega)G_A(\mathbf{p}-\mathbf{k}/2,\ \epsilon)\,,\\ g_3(\mathbf{p},\epsilon:\mathbf{k},\omega)&\equiv G_A(\mathbf{p}+\mathbf{k}/2,\ \epsilon+\omega)G_A(\mathbf{p}-\mathbf{k}/2,\ \epsilon)\,. \end{aligned} \tag{9}$$

In the normal case we have

$$G_R(\mathbf{p},\epsilon)=[G_A(p,\epsilon)]^*=a(\epsilon-\epsilon_\mathbf{p}+i\gamma_\mathbf{p})^{-1}\theta(\epsilon_0-|\epsilon_\mathbf{p}|)\,, \tag{10}$$

where $\epsilon_\mathbf{p}$ is the quasiparticle excitation energy and $\gamma_\mathbf{p}$ the half-width, and $a\leqslant 1$. The cutoff ϵ_0 in the definition of G is chosen so that $T\ll\epsilon_0\ll\mu$. In Eq. (8) the quantities $\lambda_i(\epsilon,\omega)$ $(i=1,2,3)$ are given by

$$\begin{aligned} \lambda_1&=\tanh(\epsilon/2T)\,,\\ \lambda_2&=\tanh(\epsilon+\omega/2T)-\tanh(\epsilon/2T)\,,\\ \lambda_3&=-\tanh(\epsilon+\omega/2T)\,. \end{aligned} \tag{11}$$

Summation over repeated indices i, j is implied. Moreover, the second term in (4) is to be understood as a matrix expression, with matrix multiplication defined by the relation

$$(AB)(\mathbf{pp}',\sigma\sigma',\epsilon\epsilon')\equiv\frac{1}{4\pi i}\sum_{\mathbf{p}''\sigma''}\int d\epsilon''\,A(\mathbf{pp}'',\sigma\sigma'',\epsilon\epsilon'')\times B(\mathbf{p}''\mathbf{p}',\sigma''\sigma',\epsilon''\epsilon')\,, \tag{12}$$

and

$$\mathrm{Tr}A\equiv\frac{1}{4\pi i}\sum_{\mathbf{p}\sigma}\int d\epsilon A(\mathbf{pp},\sigma\sigma,\epsilon\epsilon)\,. \tag{13}$$

Finally, it is shown in the Appendix that the values of the constants Φ_ξ, R_ξ are as follows: For any conserved quantity (density, spin and, in the case of translational invariance, current)

$$\Phi_\xi=0\,,\quad R_\xi=a^{-1}; \tag{14}$$

while (whether or not momentum is conserved), the constants in the current autocorrelation function are

$$\Phi_p=\frac{N}{m}\left\{\frac{m}{m^*}(1+\tfrac{1}{3}F_1)-1\right\}\,,\quad R_p=a^{-1}\frac{m}{m^*}(1+\tfrac{1}{3}F_1)\,, \tag{15}$$

in the usual notation of Fermi-liquid theory (cf. Appendix). The corresponding constants for the spin-current are obtained from (15) by replacing F_1 by $Z_1/4$.

Examination of Eqs. (8)–(13) reveals that the integrals over ϵ and $\mathbf{p}$ in (8) are effectively taken over a range $\epsilon,(|p|-p_F)v\sim\max(\omega,vk,T)$. Thus we have attained our object of expressing $K_\xi(\mathbf{k},\omega)$ in the general form (5).

It remains to examine the quantities $Q_{ij}{}^\omega$ in Eq. (8). It is shown in the Appendix that in the range of interest we can write all $Q_{ij}{}^\omega$ except $Q_{22}{}^\omega$ in the form

$$Q_{ij}{}^\omega(\mathbf{pp}',\sigma\sigma',\epsilon\epsilon':\mathbf{k}\omega)=\lambda_j(\epsilon',\omega)\Gamma^\omega(\mathbf{pp}',\sigma\sigma')\,, \tag{16}$$

where $\Gamma^\omega(\mathbf{pp}',\sigma\sigma')$ is the usual function introduced by Landau.[2] Equation (16) is also correct for $Q_{22}{}^\omega$ in the limit $\omega\gg\gamma$; in the opposite ("hydrodynamic") limit, however, we must take into account a pure imaginary term in $Q_{22}{}^\omega$, which we write as $\mathfrak{D}(\mathbf{pp}',\sigma\sigma',\epsilon\epsilon')$. The physical interpretation of $\mathfrak{D}$ is given in the Appendix. Since however, according to Eqs. (8) and (11), all terms involving $Q_{22}{}^\omega$ are proportional to ω in the limit $\omega\to 0$,

[15] A. Abrikosov, L. P. Gor'kov, and I. Dzyaloshinskii, *Methods of Quantum Field Theory in Statistical Physics* (English transl.: Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963).

[16] P. Nozières, *Theory of Interacting Fermi Systems* (W. A. Benjamin Inc., New York, 1964).

[17] G. M. Eliashberg, Zh. Eksperim. i Teor. Fiz. **41**, 1241 (1961) [English transl.: Soviet Phys.—JETP **14**, 886 (1962)].

[18] P. Nozières and J. M. Luttinger, Phys. Rev. **127**, 1431 (1962).

this term is only important in the absorption region ($\omega\sim\gamma$) or when we consider the kinetic coefficients [when in general we may be interested in $\lim_{\omega\to 0}\omega^{-1}\{K(\mathbf{k},\omega)-K(k,0)\}$]. For all other purposes we can write (8) as

$$K_\xi(\mathbf{k},\omega)=\Phi_\xi+R_\xi^2\,\mathrm{Tr}\left\{\xi\frac{\lambda g}{1-\Gamma^\omega\lambda g}\xi\right\}, \tag{17}$$

(where the sum over i has been incorporated into the matrix notation).

In the normal case the evaluation of (λg) according to Eqs. (9)–(11) is trivial; having dropped $\mathfrak{D}$, we must of course also put $\gamma_p\to 0$ in (10). We then find

$$K_\xi(\mathbf{k},\omega)=\Phi_\xi+a^2R_\xi^2\nu(0)\sum_\sigma\int\frac{d\Omega}{8\pi}\left\{\xi\frac{\mathbf{v}\cdot\mathbf{k}}{\omega-\mathbf{v}\cdot\mathbf{k}-\mathbf{v}\cdot\mathbf{k}F}\xi\right\}, \tag{18}$$

where

$$F=a^2\nu(0)\Gamma^\omega,\quad \nu(0)=3m^*N/p_F^2. \tag{19}$$

F is still a matrix with respect to direction on the Fermi surface and spin. Equation (18) is a correct description of the response of the system to an external field (and hence also of its free collective oscillations) both for $\omega=0$ and for $\omega\gg\gamma$; it is easy to verify that the results agree with the usual ones for $T=0$.[1,2,16] For other purposes we must return to the exact Eq. (8).

Let us now consider the generalization to the superfluid case. The treatment given below should be compared with that of Larkin and Migdal[8] for $T=0$; the results obtained reduce to theirs in that limit.

In the normal case the "Dyson" Eq. (7) has the graphical representation shown in Fig. 1(b); in that figure the intermediate propagators $G(\mathbf{p}+\mathbf{k}/2,\epsilon+\omega)$ and $G(\mathbf{p}-\mathbf{k}/2,\epsilon)$ are written with two arrows to emphasize that they conserve particle number or, in graphical terms, direction of the line. In the superfluid case we must consider also "anomalous" propagators corresponding to the creation or annihilation of pairs with opposite spin and momentum; if we choose a representation in which the gap Δ is real we have (cf. Ref. 8)

$$\begin{aligned}&\ll a_{p\uparrow}a_{-p\downarrow}\gg=\ll a_{p\downarrow}{}^\dagger a_{-p\uparrow}{}^\dagger\gg=F,\\&\ll a_{p\downarrow}a_{-p\uparrow}\gg=\ll a_{p\uparrow}{}^\dagger a_{-p\downarrow}{}^\dagger\gg=-F.\end{aligned} \tag{20}$$

(For the moment we need not specify precisely what kind of Green's functions we are interested in, since the formal structure of the equations is the same for all.) Owing to the nonconservation of particle number, we now also have to deal with a large number of different complete vertex parts Γ. Neglecting the question of spin dependence for the moment, let us label these vertex parts by the directions of the arrows involved; the leftward direction being counted as positive and the upper line being represented first. Thus, the right-hand side of the graphical equation above is labeled $^{+-}\Gamma^{+-}$. There are then 16 quantities corresponding to the possible directions of the arrows (they are not, of course, all numerically different); let us regard these as the elements of a 4×4 matrix whose rows and columns are labeled by the indices $--,++,-+,+-$ (in that order). Also we shall regard the pairs of intermediate propagators GG,GF,FF, etc. as forming the elements of a matrix $\hat{g}$ in the space (hereafter called the "arrow space"). Then the generalization of Eq. (3) is simply

$$\hat{\Gamma}=\hat{\Gamma}^{(0)}+\hat{\Gamma}^{(0)}\hat{g}\hat{\Gamma} \tag{21}$$

the circumflexes denoting matrices in the arrow space. The explicit expression for the elements of $\hat{g}$ in terms of the G's and F's will be given below. Note that in this formalism we must not count separately elements of $\Gamma^{(0)}$ which differ only in having an upper and a lower line on the same side interchanged. Thus it is consistent to put the $(+-,-+)$ element, for instance, equal to zero. Since the "particle-nonconserving" elements of $\Gamma^{(0)}$ are at most of order Δ/μ relative to the "conserving" elements, $\Gamma^{(0)}$ may be taken as a diagonal matrix to this order.

The generalization of Eq. (6) proceeds similarly. For the normal case (6) has the graphical representation shown in Fig. 1(a). The "bare vertex" $\xi(\mathbf{p},\sigma)$ conserves particle number and hence, in the superfluid as in the normal case, at each end of any graph contributing to K_ξ one arrow must enter the dot representing ξ and one leave it. It is now important to notice that for any physical vertex ξ in which we are likely to be interested, the contribution $\xi(\mathbf{p},\sigma)$ will be either even or odd under the substitution $\mathbf{p}\to-\mathbf{p}$, $\sigma\to-\sigma$. Since when a line changes its "direction" it makes just this substitution, we can verify that a correct generalization of (6) is

$$K_\pm=\mathrm{Tr}\{\xi\hat{P}_\pm(\hat{g}+\hat{g}\hat{\Gamma}\hat{g})\xi\}, \tag{22}$$

where the plus sign is to be taken in the case $\xi(\mathbf{p},\sigma)=\xi(-\mathbf{p},-\sigma)$ and the minus sign in the case $\xi(\mathbf{p},\sigma)=-\xi(-\mathbf{p},-\sigma)$. $\hat{P}_\pm$ is an idempotent operator in the arrow space of the form (see above for the order of labeling of the vectors of the space)

$$P_\pm=\begin{pmatrix}0&0\\0&\Pi_\pm\end{pmatrix},\quad \Pi_+=\tfrac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix},\quad \Pi_-=\tfrac{1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}. \tag{23}$$

We now consider the question of spin dependence and of the correct sign assignments in the elements of $\hat{g}$. Let us imagine that we have expanded K in an infinite series of diagrams containing the various elements of $\hat{g}$ and $\hat{\Gamma}^{(0)}$. The analysis is somewhat simplified by the fact that $\xi(\mathbf{p},\sigma)$ is diagonal in the spin index (cf. footnote 12). Since both the irreducible vertex parts $\Gamma^{(0)}$ and the propagators (both normal and anomalous) conserve spin, we always deal, at any subsequent point in the graph, with total z component of spin $S_z=0$ and with total spin $S=0$ or 1 according as $\xi(\mathbf{p},\sigma)=\pm\xi(\mathbf{p},-\sigma)$. Some straightforward though rather

tedious analysis then shows that we may consistently take $\hat{g}$ in the form

$$\hat{g}=\begin{pmatrix} -G^{-}G & -FF & G^{-}F & FG \\ -FF & -GG^{-} & -FG^{-} & -GF \\ G^{-}F & -FG^{-} & G^{-}G^{-} & -FF \\ FG & -GF & -FF & GG \end{pmatrix}, \tag{24}$$

[where $G^{-}(\mathbf{p},\epsilon_n)\equiv G(-\mathbf{p},-\epsilon_n)$], and

$$\hat{\Gamma}^{(0)}=\begin{pmatrix} \Gamma_P{}^{(0)}\mathbf{1} & 0 \\ 0 & \Gamma_L{}^{(0)}\mathbf{1} \end{pmatrix}+O(\Delta/\mu)\,, \tag{25}$$

provided that the irreducible pair scattering amplitude $\Gamma_P{}^{(0)}$ is taken to be the singlet amplitude if $S=0$ and the triplet amplitude if $S=1$, while the irreducible "particle-hole" amplitude $\Gamma_L{}^{(0)}$ is taken to be the spin-independent term for $S=0$ and the spin-dependent term for $S=1$. Thus if we write $\Gamma^{(0)}(\mathbf{pp}',\sigma\sigma',\epsilon\epsilon':\mathbf{k},\omega)$ in the form

$$\Gamma^{(0)}(\sigma,\sigma')=A+B\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' \tag{26}$$

(assuming, as usual, that the spin dependence is due to exchange effects) then we must take $\Gamma_L{}^{(0)}$ in (25) to be A for $S=0$ and $B/4$ for $S=1$. These prescriptions apply equally well, of course, to the "renormalized" quantities we shall deal with subsequently. Finally, ξ in Eq. (22) is now to be taken as a function only of $\mathbf{p}$[$\xi(\mathbf{p})\equiv\xi(\mathbf{p},\sigma)\equiv\xi(\mathbf{p},-\sigma)$ for $S=0$, $\xi(\mathbf{p})\equiv\xi(\mathbf{p},\sigma)\equiv-\xi(\mathbf{p},-\sigma)$ for $S=1$]; and in the matrix multiplication, etc., $\sum_{p\sigma}$ is everywhere to be replaced by $2\sum_{\mathbf{p}}$. Thus the spin indices do not enter the problem from now on.

Once we have Eqs. (21) and (22) we can go through all the procedure of the Appendix, with two minor caveats to be discussed below. As in the normal case, the crucial step is the separation of the one-particle propagators into "polar" and "nonpolar" (incoherent) parts; the form of the polar parts in the case of interest is given below. The "nonpolar" part of G is the same as in the normal phase in our approximation ($\Delta/\mu\ll1$) while the nonpolar part of F must be set equal to zero. Then it is not difficult to see that the gross renormalization factors Φ_ξ and R_ξ (which are determined by integrals over the whole range of energy and momentum) are indeed the same as in the normal phase [Eqs. (14) and (15)]. Also we must take the particle-nonconserving elements of $\hat{Q}_{ij}{}^{\omega}$ (i.e., the off-diagonal elements in the arrow space) equal to zero for all $\hat{Q}_{ij}{}^{\omega}$ except $\hat{Q}_{22}{}^{\omega}$; the latter, as in the normal case, contains an imaginary part which, though small ($\lesssim(\Delta/\mu)^2$) must nevertheless be kept for certain purposes, and this term may in general have off-diagonal elements of the same order as the diagonal ones.

We arrive then at the following generalization of Eq. (8):

$$K_\xi(\mathbf{k},\omega)=\Phi_\xi+R_\xi{}^2\,\mathrm{Tr}\left\{\xi\hat{P}_{\pm}\frac{\lambda\hat{g}}{1-\hat{Q}^{\omega}\hat{g}}\xi\right\}, \tag{27}$$

where all quantities are matrices with respect to the arrow space, the i-j space, $\mathbf{p}$ and ϵ (the dependence on σ was eliminated above). $\hat{g}$ now denotes only the product of polar parts of the single-particle functions.

In Eq. (27) we must be a little cautious about the definition of the quantities $\hat{g}_i$ in terms of "retarded" and "advanced" Green's functions [cf. Eq. (9)]. In fact, inspection of the method of analytic continuation from the complex plane (see Appendix, and Ref. 17) which gave rise to the combinations (9) in the normal case reveals that whenever a G^{-} occurs we must change "advanced" to "retarded" and vice versa. (Equivalently, we must insure that all elements of $\hat{g}_1$ have poles only in the lower half-plane and all elements of $\hat{g}_3$ only in the upper, etc.)

Equation (27), together with a knowledge of the form of the one-particle functions G and F, provides in principle a complete solution of the problem of obtaining the autocorrelation functions for a superfluid Fermi liquid in the region $\omega,kv\ll\mu$ and thus of deriving not only the static and dynamic response but also the kinetic coefficients. However, once we have to take "damping" terms into account the forms of $\hat{Q}^{\omega}$ and G become quite complicated. Let us therefore specialize at once to the limit of infinite lifetimes; this will be sufficient for consideration of the static properties and of the response of the system in the region $\omega\gg\gamma$, except in the very small temperature region close to T_c where $\gamma\sim\Delta$. In the infinite-lifetime limit the one-particle propagators have the simple form:

$$G_R(\mathbf{p},\epsilon)=\frac{a(\epsilon+\epsilon_{\mathbf{p}})}{(\epsilon+i\gamma)^2-E_{\mathbf{p}}{}^2}, \qquad G_A(-\mathbf{p},-\epsilon)=\frac{a(-\epsilon+\epsilon_{\mathbf{p}})}{(\epsilon+i\gamma)^2-E_{\mathbf{p}}{}^2},$$

$$F_R(\mathbf{p},\epsilon)=F_A(-\mathbf{p},-\epsilon)=\frac{-a\Delta}{(\epsilon+i\gamma)^2-E_{\mathbf{p}}{}^2}, \tag{28}$$

$$G_A=G_R{}^{*} \qquad F_A=F_R{}^{*}$$

where $E_{\mathbf{p}}{}^2=\epsilon_{\mathbf{p}}{}^2+\Delta^2$ ($\epsilon_{\mathbf{p}}$ is the normal-state excitation energy and Δ the gap) and $\gamma\to+0$. Moreover, in this limit we must for consistency neglect the imaginary part of $\hat{Q}^{\omega}$ and hence put

$$\hat{Q}_{ij}{}^{\omega}(\mathbf{pp}',\epsilon\epsilon':\mathbf{k},\omega)=\lambda_j(\epsilon')\hat{\Gamma}^{\omega}(\mathbf{p},\mathbf{p}')\,. \tag{29}$$

In the arrow space the matrix $\hat{\Gamma}^{\omega}$ has the form

$$\hat{\Gamma}=\begin{pmatrix} \Gamma^{\varphi}1 & 0 \\ 0 & \Gamma^{\omega}1 \end{pmatrix}. \tag{30}$$

$\Gamma^{\varphi}(\mathbf{p},\mathbf{p}')$ is the renormalized pairing amplitude, which must be taken to refer to the singlet or triplet state according as $\xi(\mathbf{p},\sigma)=\pm\xi(\mathbf{p},-\sigma)$. $\Gamma^{\omega}(\mathbf{p},\mathbf{p}')$ is just the usual Landau amplitude which occurs in the theory of the normal Fermi liquid; it is to be taken as the spin-

independent or spin-dependent part according as $\xi(\mathbf{p},\sigma)=\pm\xi(\mathbf{p},-\sigma)$ [cf. Eq. (26)]. There is a slight difficulty connected with the definition of Γ^φ; see below.

Using (29) and defining a matrix quantity

$$\tilde{g}(\mathbf{p}:\mathbf{k},\omega)\equiv\frac{\nu(0)}{4\pi i}\int_{-\epsilon_0}^{\epsilon_0}d\epsilon_{\mathbf{p}}\int_{-\infty}^{\infty}d\epsilon\sum_i\lambda_i(\epsilon)\hat{g}_i(\mathbf{p},\epsilon:\mathbf{k},\omega)\,, \quad (31)$$

we can write (27) in the form:

$$K_\xi(\mathbf{k},\omega)=\Phi_\xi+R_\xi^2\,\mathrm{Tr}\left\{\xi\hat{P}_\pm\left(\frac{\tilde{g}}{1-\hat{\Gamma}^\omega\tilde{g}}\right)\xi\right\}\,, \quad (32)$$

where all quantities are now matrices only with respect to the arrow space and direction on the Fermi surface; as regards the latter, matrix multiplication is defined by

$$(AB)(\mathbf{n},\mathbf{n}')=\int\frac{d\Omega''}{4\pi}A(\mathbf{n},\mathbf{n}'')B(\mathbf{n}'',\mathbf{n}')\,, \quad (33)$$

and we also have

$$\mathrm{Tr}A=\int\frac{d\Omega}{4\pi}A(\mathbf{n},\mathbf{n})\,. \quad (34)$$

There is one last point to clear up. As we have defined the quantities $\tilde{g}$ above, the elements associated with GG^- (and G^-G) depend logarithmically on the cutoff ϵ_0 (the dependence of all other elements is much weaker and can certainly be neglected). Also, the gap equation, which in our notation reads (from now on the quantities GG, GF, etc., are to be understood as the corresponding elements of $\tilde{g}$)

$$1=-\Gamma^\varphi(GG^-+FF)_{\mathbf{k}=0,\omega=0} \quad (35)$$

(where Γ^φ is the singlet amplitude) appears itself to depend on the cutoff. Of course there is in reality no such dependence, since it is easy to show[8] that the ϵ_0 dependence of Γ^φ is just such as to cancel that of (GG^-+FF). However, it is more convenient to get rid of ϵ_0 altogether by a further renormalization: we define

$$\Gamma_r^\varphi(T)=\Gamma^\varphi-\Gamma^\varphi(GG^-)_n\Gamma_r^\varphi\,, \quad (36)$$

where $(GG^-)_n$ is the relevant element of (31) evaluated in the *normal* state, with $\mathbf{k}=0$, $\omega=0$. In terms of Γ_r^φ the gap equation becomes

$$1=-\Gamma_r^\varphi(GG^--(GG^-)_n+FF)_{\mathbf{k}=0,\omega=0} \quad (37)$$

and we may obviously substitute Γ_r^φ for Γ^φ in (30) provided that at the same time we understand by the GG^- (or G^-G) element of (31) the "subtracted" quantity $(GG^-)-(GG^-)_n$. [$(GG^-)_n$ is to be evaluated with $\mathbf{k}=0$, $\omega=0$ in all cases.] From now on we shall do this and drop the subscript r on Γ_r^φ. The integral over $\epsilon_{\mathbf{p}}$ in (31) may now be extended to infinity since there is no longer any sharp dependence on the upper limit. We shall see that, in the cases of interest to us, Γ^φ actually drops out of the final formulas and we can express all results in terms of $\Delta(T)$. The relation of $\Delta(0)$ to T_c and also the ratio $\Delta(T)/\Delta(0)$ as a function of T/T_c is of course given equally well by (35) or (37) and is the same as predicted by the usual Bardeen-Cooper-Schrieffer (BCS) theory; we therefore actually have—apart from the Landau parameters describing the normal phase—only one undetermined parameter (T_c) in the theory. As a function of the reduced temperature, the effects we shall investigate are completely determined by the Landau parameters.

Let us collect the results of this section for easy reference. We have

$$K_\xi(\mathbf{k},\omega)=\Phi_\xi+R_\xi^2\,\mathrm{Tr}\left\{\xi\hat{P}_\pm\left(\frac{\tilde{g}}{1-\hat{\Gamma}^\omega\tilde{g}}\right)\xi\right\}\,,$$

$$\tilde{g}(\mathbf{p}:\mathbf{k},\omega)=\frac{\nu(0)}{4\pi i}\int_{-\infty}^{\infty}d\epsilon_p\int_{-\infty}^{\infty}d\epsilon\times\sum_i\lambda_i(\epsilon,\omega)g_i(\mathbf{p},\epsilon:\mathbf{k},\omega)\,,$$

$$\tilde{g}=\begin{pmatrix}-G^-G & -FF & G^-F & FG\\ -FF & -GG^- & -FG^- & -GF\\ G^-F & -FG^- & G^-G & -FF\\ FG & -GF & -FF & GG\end{pmatrix}, \quad (38)$$

$$\hat{\Gamma}^\omega=\begin{pmatrix}\Gamma^\varphi 1 & 0\\ 0 & \Gamma^\omega 1\end{pmatrix},$$

$$\hat{P}_\pm=\begin{pmatrix}0 & 0\\ 0 & \Pi_\pm\end{pmatrix},\quad \Pi_+=\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix},$$

$$\Pi_-=\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}.$$

Here $\nu(0)=3m^*N/p_F^2$ is the density of states at the Fermi surface in the normal phase, and, for instance, $G^-F=G(-(\mathbf{p}+\mathbf{k}/2),-(\epsilon+\omega))F(\mathbf{p}-\mathbf{k}/2,\epsilon)$. The form of the functions G and F is given by (28) and the constants R_ξ and Φ_ξ by (14) and (15).

To conclude this section, let us make explicit the correspondence between the formalism used here and that of Larkin and Migdal.[8] To do this we use the following symmetry relations between the elements of $\tilde{g}$, which hold for arbitrary temperature and arbitrary values of $\mathbf{k}$ and ω:

$$GG^-=G^-G,\quad FG=-GF,\quad FG^-=-G^-F. \quad (39)$$

Now let us examine Eqs. (53), (56), and (32) of Larkin and Migdal's paper,[8] using also their Eqs. (50) and (55)[19] and the symmetry properties of the operators $\mathbf{p}$ and σ. Then it can be seen that the diagonal elements

[19] The quantity $\partial G^{-1}/\partial\epsilon$ used in Ref. 8 is of course just a^{-1}.

(in Cartesian space) of (53) and (56) are particular cases of a relation which in our formalism is written

$$K_\xi(\mathbf{k},\omega)=a^{-1}\,\mathrm{Tr}\{\xi\tilde{g}\tilde{T}_\xi\}\,, \tag{40}$$

where $\tilde{T}_\xi$ is a diagonal matrix in the arrow space. Similarly, Eq. (32) of Ref. 8 [with (50) and (55)] corresponds to the relation

$$\tilde{T}_\xi=a^{-1}\{\xi+\hat{\Gamma}^\omega\tilde{g}\tilde{T}_\xi\}\,. \tag{41}$$

(To convince oneself of the truth of these statements it is simplest to effect a partial diagonalization of $\tilde{g}$ along the lines of the next section.[20]) Since in the translation-invariant case considered in Ref. 8 we have $\Phi_\xi=0$ $R_\xi=a^{-1}$ for both the spin and current autocorrelation functions, Eqs. (40) and (41) agree formally with the first of Eqs. (38). Furthermore, some tedious algebra shows that the elements of $\tilde{g}$ calculated from our definition [second of Eqs. (38)] correspond in the limit $T\to 0$ to those obtained from the zero-temperature definition of Ref. 8. Thus the theory developed here agrees with that of Larkin and Migdal in the limit of zero temperature.

III. THE STATIC RESPONSE FUNCTIONS

Let us now specialize to the case $\omega=0$. In this case the quantities $\tilde{g}(\mathbf{p}:\mathbf{k})$ take the form

$$\tilde{g}(\mathbf{p}:\mathbf{k})=\frac{\nu(0)}{2\pi i}\int d\epsilon_{\mathbf{p}}\int d\epsilon\,\tanh(\epsilon/2T)\hat{g}_1(\mathbf{p},\epsilon:\mathbf{k})\,. \tag{42}$$

Since there is now no danger of confusion with the "unintegrated" quantities, we shall hereafter write $\tilde{g}$ as $\hat{g}$ to emphasize its matrix nature with respect to the arrow space. Substituting the explicit forms of G and F from (28), we can verify not only the equalities (39) but also the following equality which is peculiar to the case $\omega=0$:

$$GF=G^-F\,,\quad G^-G^-=GG\,. \tag{43}$$

As a result the matrix $\hat{g}$ contains only four independent parameters. Let us transform the basis of the representation (38) with the unitary unimodular matrix $\hat{\tau}$: $\hat{A}\to\hat{\tau}\hat{A}\hat{\tau}^{-1}$, where

$$\tau=\begin{pmatrix}\hat{\theta} & 0\\ 0 & \hat{\theta}\end{pmatrix},\quad \hat{\theta}=2^{-1/2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}.$$

Then (after a trivial rearrangement) we have

$$\hat{g}=\begin{bmatrix} FF-GG^- & 0 & 0 & 0\\ 0 & -(GG^-+FF) & 2GF & 0\\ 0 & 2GF & GG+FF & 0\\ 0 & 0 & 0 & GG-FF\end{bmatrix}, \tag{44}$$

$$\hat{P}_\pm=\begin{pmatrix}0 & 0\\ 0 & \Pi_\pm\end{pmatrix},\quad \Pi_+=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},\quad \Pi_-=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \tag{45}$$

$$\hat{\Gamma}^\omega=\begin{pmatrix}\Gamma^\varphi 1 & 0\\ 0 & \Gamma^\omega 1\end{pmatrix}, \tag{46}$$

while the trace defininining K_ξ is of course invariant under the transformation:

$$K_\xi=\Phi_\xi+R_\xi^2\,\mathrm{Tr}\left\{\xi\hat{P}_\pm\left(\frac{\hat{g}}{1-\hat{\Gamma}^\omega\hat{g}}\right)\xi\right\}. \tag{47}$$

From Eqs. (44)–(47) it is obvious that the quantity $FF-GG^-$ is of no further interest to us. Moreover, we can immediately evaluate the autocorrelation function for any "even" operator ($\xi(\mathbf{p},\sigma)=\xi(-\mathbf{p},-\sigma)$) since it is clear that

$$\mathrm{Tr}\left\{\xi\hat{P}_+\left(\frac{\hat{g}}{1-\hat{\Gamma}^\omega\hat{g}}\right)\xi\right\}=\int\frac{d\Omega}{4\pi}\left\{\xi\frac{GG-FF}{1-\Gamma^\omega(GG-FF)}\xi\right\}. \tag{48}$$

In the limit $\mathbf{k}\to 0$ we have for arbitrary T, as is easily shown by direct evaluation

$$(GG-FF)_{\mathbf{k}=0}=-a^2\nu(0)\,.$$

Hence we have

$$\lim_{\mathbf{k}\to 0}K_\xi(\mathbf{k},0)=\Phi_\xi-a^2R_\xi^2\nu(0)\times\int\frac{d\Omega}{4\pi}\left\{\xi\left(\frac{1}{1+a^2\nu(0)\Gamma^\omega}\right)\xi\right\}, \tag{49}$$

if

$$\xi(\mathbf{p},\sigma)=\xi(-\mathbf{p},-\sigma)\,,$$

which is just the normal-state result [cf. Eqs. (18) and (19)]. Therefore we can state our first (trivial) result:

The value of any "even" autocorrelation function is unaffected by the superfluid transition in the limit $\mathbf{k}\to 0$.

Thus, in particular, the compressibility is unaffected by the phase transition, and the spin current sum rule continues to be obeyed as in the normal phase.

Now we examine the less trivial case of "odd" operators. We now have to evaluate the expression

$$\int\frac{d\Omega}{4\pi}\,\mathrm{Tr}\left\{\xi\left(\frac{\hat{g}}{1-\hat{\Gamma}^\omega\hat{g}}\right)\xi\right\}, \tag{50}$$

where it is now convenient to take all quantities as matrices in a 2×2 "arrow" space:

$$\hat{g}=\begin{pmatrix}-(GG^-+FF) & 2GF\\ 2GF & GG+FF\end{pmatrix},$$

$$\hat{\Gamma}^\omega=\begin{pmatrix}\Gamma^\varphi & 0\\ 0 & \Gamma^\omega\end{pmatrix},\quad \hat{\xi}=\begin{pmatrix}0 & 0\\ 0 & \xi\end{pmatrix}. \tag{51}$$

[20] The quantity Γ^ξ of Ref. 8 differs from our Γ^φ by a minus sign as well as by a logarithmic factor due to the cutoff; cf. Eq. (19) of Ref. 8.

Since GF is proportional to $\mathbf{v}\cdot\mathbf{k}$ in the limit $\mathbf{k}\to 0$, it is tempting to take this limit immediately and replace (50) by the expression

$$\int \frac{d\Omega}{4\pi}\left\{\xi\frac{GG+FF}{1-\Gamma^{\omega}(GG+FF)}\xi\right\}. \tag{52}$$

However, this clearly cannot be correct in general, since $GG+FF\to 0$ as $T\to 0$ and hence if we used (52) for the longitudinal current autocorrelation function we should violate the longitudinal sum rule.

The difficulty lies, of course, in the fact that the matrix $1-\hat{\Gamma}^{\omega}\hat{g}$ may be singular in the limit $\mathbf{k}\to 0$, since according to the gap equation (37) its (1,1) element as well as its off-diagonal elements may tend to zero (or rather, some harmonic of it may).[16] However, it is clear that this difficulty does not apply to "spin-1" operators $[\xi(\mathbf{p},\sigma)=-\xi(\mathbf{p},-\sigma)]$ since, according to the prescriptions of Sec. II, Γ^{φ} in such cases is to be taken as the triplet scattering amplitude, whereas it is the singlet amplitude which enters the gap equation. Thus we can immediately see that for such operators (50) may indeed be replaced by (52). (This is, of course, the mathematical expression of the fact that any collective mode of the Bogolyubov-Anderson type will have spin zero.) For spin-zero odd operators (e.g., the longitudinal and transverse current) we must evaluate $K_{\xi}(\mathbf{k},0)$ for small but finite $\mathbf{k}$ and only then take the limit $\mathbf{k}\to 0$; whether or not the result is given by expression (52) then turns out to depend critically on the symmetry of the operator in question.

According to (50) and (51) we have to evaluate the expression

$$Q_{\xi}(\mathbf{k})\equiv\int\frac{d\Omega}{4\pi}\xi(\mathbf{n})\Phi_{22}(\mathbf{k},\mathbf{n}), \tag{53}$$

where $\mathbf{n}$ is a unit vector on the Fermi surface ($\mathbf{v}\equiv v\mathbf{n}$) and the 2×2 matrix $\hat{\Phi}$ satisfies

$$\hat{\Phi}(\mathbf{k}:\mathbf{n})=\hat{g}(\mathbf{k}:\mathbf{n})\hat{\Psi}(\mathbf{k}:\mathbf{n}), \tag{54a}$$

$$\hat{\Psi}(\mathbf{k}:\mathbf{n})-\int\frac{d\Omega'}{4\pi}\Gamma^{\omega}(\mathbf{n}\cdot\mathbf{n}')\hat{g}(\mathbf{k}:\mathbf{n}')\hat{\Psi}(\mathbf{k}:\mathbf{n}')=\hat{\xi}(\mathbf{n}), \tag{54b}$$

where $\hat{g}$, $\hat{\Gamma}^{\omega}$, and $\hat{\xi}$ are given by (51). Since we are dealing with spin-zero operators, Γ^{φ} must be taken as the singlet pairing amplitude and Γ^{ω} as the spin-independent part of $\Gamma^{\omega}(\mathbf{n}\cdot\mathbf{n}',\sigma\sigma')$. Let us write for small $\mathbf{k}$:

$$\begin{aligned} g_{11}(\mathbf{k}:\mathbf{n})&=A_0+\alpha(\mathbf{v}\cdot\mathbf{k})^2,\\ g_{12}(\mathbf{k}:\mathbf{n})&=g_{21}(\mathbf{k}:\mathbf{n})=\beta\mathbf{v}\cdot\mathbf{k},\\ g_{22}(\mathbf{k}:\mathbf{n})&=\gamma(=(GG+FF)_{\mathbf{k}=0}), \end{aligned} \tag{55}$$

where the coefficients $\alpha(T)$,$\beta(T)$, and $\gamma(T)$ will be evaluated below. It will turn out from what follows that there is no need to expand to any higher order than is done in (55).

Writing out (54b) explicitly, we have (omitting for brevity the dependence of Ψ on $\mathbf{k}$)

$$\Psi_{12}(\mathbf{n})-\int\frac{d\Omega'}{4\pi}\Gamma^{\varphi}(\mathbf{n}\cdot\mathbf{n}')\beta\mathbf{v}'\cdot\mathbf{k}\Psi_{22}(\mathbf{n}')-\int\frac{d\Omega}{4\pi}\Gamma^{\varphi}(\mathbf{n}\cdot\mathbf{n}')[A_0+\alpha(\mathbf{v}'\cdot\mathbf{k})^2]\Psi_{12}(\mathbf{n}')=0, \tag{56a}$$

$$\Psi_{22}(\mathbf{n})-\int\frac{d\Omega'}{4\pi}\Gamma^{\omega}(\mathbf{n}\cdot\mathbf{n}')\beta\mathbf{v}'\cdot\mathbf{k}\Psi_{12}(\mathbf{n}')-\int\frac{d\Omega'}{4\pi}\Gamma^{\omega}(\mathbf{n}\cdot\mathbf{n}')\gamma\Psi_{22}(\mathbf{n}')=0. \tag{56b}$$

Inspection of (56a) reveals that unless $\Psi_{12}(\mathbf{n})$ is such that the homogeneous equation,

$$\Psi_{12}(\mathbf{n})-\int\frac{d\Omega'}{4\pi}\Gamma^{\varphi}(\mathbf{n}\cdot\mathbf{n}')A_0\Psi_{12}(\mathbf{n}')=0, \tag{57}$$

is satisfied, Ψ_{12} is at most of order $(vk)\Psi_{22}$ and so can be neglected in (56b). Also in that case, from (54a), $\Phi_{22}(\mathbf{k}:\mathbf{n})\approx\gamma\Psi_{22}(\mathbf{k}:\mathbf{n})$ and so $Q_{\xi}(0)\equiv\lim_{\mathbf{k}\to 0}Q_{\xi}(\mathbf{k})$ is given by the expression (52). Now, in virtue of (37), (57) can be satisfied if and only if $\Psi_{12}(\mathbf{n})$ is a scalar with respect to $\mathbf{n}$ [we assume of course that there is no exact degeneracy among the harmonics of $\Gamma^{\varphi}(\mathbf{n}\cdot\mathbf{n}')$]. But then, substituting Ψ_{12} back into (56a), we see that as $\mathbf{k}\to 0$

$$\Psi_{12}(\mathbf{n})=-(\beta/\alpha)(\mathbf{v}\cdot\mathbf{k})^{-1}\Psi_{22}(\mathbf{n}), \tag{58}$$

and hence $\Psi_{22}(\mathbf{n})$ must be proportional to $\mathbf{n}\cdot\hat{k}$ (where $\hat{k}$ is a unit vector in the direction of $\mathbf{k}$). Hence, finally, from (56b), $\xi(\mathbf{n})$ must also be proportional to $\mathbf{n}\cdot\hat{k}$; i.e., it must be a "longitudinal" operator: $\xi(\mathbf{p},\sigma)=\xi(\mathbf{p},-\sigma)=\xi_0\mathbf{n}\cdot\hat{k}$. Thus, for any "odd" form of $\xi(\mathbf{p},\sigma)$ except a longitudinal one, Eq. (52) must be valid:

$$Q_{\xi}(0)=\int\frac{d\Omega}{4\pi}\left\{\xi\frac{\gamma}{1-\gamma\Gamma^{\omega}}\xi\right\},\qquad \text{if}\quad \xi(\mathbf{p},\sigma)=-\xi(-\mathbf{p},-\sigma)\neq\xi_0\mathbf{n}\cdot\hat{k}. \tag{59}$$

Finally let us consider the case of "longitudinal" operators $\xi(\mathbf{p},\sigma)=-\xi(-\mathbf{p},-\sigma)=\xi_0\mathbf{n}\cdot\hat{k}$. Putting $\Psi_{22}=C\mathbf{n}\cdot\hat{k}$ and substituting in (56a), we get $\Psi_{12}=-(\beta C/\alpha)(vk)^{-1}$; substitution of this in (56b) then gives

$$C\left\{\mathbf{n}\cdot\hat{k}+\frac{\beta^2}{\alpha}\int\frac{d\Omega'}{4\pi}\Gamma^{\omega}(\mathbf{n}\cdot\mathbf{n}')\mathbf{n}'\cdot\hat{k}-\gamma\int\frac{d\Omega'}{4\pi}\Gamma^{\omega}(\mathbf{n}\cdot\mathbf{n}')\mathbf{n}'\cdot\hat{k}\right\}=\xi_0\mathbf{n}\cdot\hat{k} \tag{60}$$

so that

$$C=\xi_0[1+\tfrac{1}{3}\Gamma_1^{\omega}(\beta^2\alpha^{-1}-\gamma)]^{-1}$$

[where Γ_1^{ω} is the first harmonic of $\Gamma^{\omega}(\mathbf{n}\cdot\mathbf{n}')$]. Hence, finally, from (53) and (54a) we get

$$Q_{\xi}(0)=\frac{-\tfrac{1}{3}\xi_0^2(\beta^2/\alpha-\gamma)}{1+\tfrac{1}{3}\Gamma_1^{\omega}(\beta^2/\alpha-\gamma)},\qquad \text{if}\quad \xi(\mathbf{p},\sigma)=\xi_0\mathbf{n}\cdot\hat{k}. \tag{61}$$

Let us now evaluate the quantities $\alpha(T)$, $\beta(T)$ and $\gamma(T)$ [Eqs. (55)]. We have (the subscript zero denotes $\mathbf{k}=0$, $\omega=0$):

$$\gamma(T)\equiv\frac{\nu(0)}{2\pi i}\int\int d\epsilon d\epsilon_p (GG+FF)_0(\mathbf{p},\epsilon)\tanh(\epsilon/2T)$$
$$=-a^2\nu(0)\int_0^\infty d\epsilon_p (2T)^{-1}\cosh^{-2}(E_p/2T)$$
$$\equiv -a^2\nu(0)f(T)\,, \tag{62}$$

where $f(T)$ is a universal function of $\theta\equiv T/T_c$:

$$f(T)\equiv -[\nu(0)]^{-1}\sum_p\left(\frac{dn}{dE_p}\right)=(2\theta)^{-1}$$
$$\times\int_0^\infty dx\cosh^{-2}\left\{\frac{1}{2\theta}\left(x^2+\frac{\Delta^2(\theta)}{T_c^2}\right)^{1/2}\right\}, \tag{63}$$

with $f(T_c)=1$, $f(0)=0$. According to the definition (63), $f(T)$ may be regarded as the "effective density of single-particle levels" near the Fermi surface relative to the normal-state value $\nu(0)$. In the BCS model and related models, $f(T)$ is also the relative density of the normal component as defined from the transverse current response in the local limit; however, it is an essential conclusion of this paper that this identification cannot be made in general for a superfluid Fermi liquid (see below).

Let us define for convenience

$$\lambda(T)\equiv\frac{\nu(0)}{2\pi i}\int\int d\epsilon d\epsilon_p (FF)_0(\mathbf{p},\epsilon)\tanh(\epsilon/2T)$$
$$=\tfrac{1}{2}(\nu(0)a^2+\gamma(T))=\tfrac{1}{2}a^2\nu(0)[1-f(T)]\,. \tag{64}$$

Then it may easily be shown from (28) and (55) that

$$\beta(T)=-\lambda/\Delta\,, \tag{65}$$

while after some calculation we get for $\alpha(T)$

$$\alpha(T)\equiv\lim_{\mathbf{k}\to 0}(\mathbf{v}\cdot\mathbf{k})^{-2}\frac{\nu(0)}{2\pi i}\int\int d\epsilon d\epsilon_p\Big\{(GG^-+FF)(\mathbf{p},\epsilon:0)$$
$$-(GG^-+FF)(\mathbf{p},\epsilon:\mathbf{k})\tanh(\epsilon/2T)\Big\}$$
$$=-a^2\nu(0)\int_0^\infty d\epsilon_p\frac{1}{4E_p}\frac{d}{dE_p}\left\{\frac{\tanh(E_p/2T)}{E_p}\right\}$$
$$=\lambda/2\Delta^2\,. \tag{66}$$

From Eqs. (64)–(66) it follows that

$$(\beta^2/\alpha)-\gamma=a^2\nu(0) \tag{67}$$

for arbitrary T.

Thus, finally, collecting Eqs. (47), (49), (59), (61), and (66), and writing $F\equiv a^2\nu(0)\Gamma^\omega$,[21] we get the following expressions for the static correlation functions in the limit $\mathbf{k}\to 0$:

$$\lim_{\mathbf{k}\to 0}K_\xi(\mathbf{k},0)=\Phi_\xi-a^2R_\xi^2\nu(0)\int\frac{d\Omega}{4\pi}\left\{\xi\frac{1}{1+F}\xi\right\},$$
if
$$\xi(\mathbf{p},\sigma)=\xi(-\mathbf{p},-\sigma)\,; \tag{68a}$$

$$\lim_{\mathbf{k}\to 0}K_\xi(\mathbf{k},0)=\Phi_\xi-a^2R_\xi^2\nu(0)\int\frac{d\Omega}{4\pi}\left\{\xi\frac{1}{1+F}\xi\right\},$$
if
$$\xi(\mathbf{p},\sigma)=\xi_0\mathbf{n}\cdot\mathbf{k}\,; \tag{68b}$$

$$\lim_{\mathbf{k}\to 0}K_\xi(\mathbf{k},0)=\Phi_\xi-a^2R_\xi^2\nu(0)\int\frac{d\Omega}{4\pi}\left\{\xi\frac{f(T)}{1+f(T)F}\xi\right\}, \tag{68c}$$

otherwise.

Equations (68) constitute the principal results of this paper, as far as the microscopic theory goes.

Since (68b) is the same as the normal-state result [Eq. (18)] we see that, at least in the limit $\mathbf{k}\to 0$, the usual longitudinal sum rule is satisfied in the superfluid as in the normal phase, which confirms the consistency of the calculations.

Physically interesting examples of correlation functions satisfying Eq. (68c) are the spin-spin correlation function $K_\sigma[\xi(\mathbf{p},\sigma)=\sigma=\pm\frac{1}{2}]$, which is related to the bulk spin susceptibility by

$$\chi=-\beta^2\lim_{\mathbf{k}\to 0}K_\sigma(\mathbf{k},0)\,,$$

where β is the appropriate gyromagnetic ratio, and the transverse current-current correlation function $K_\perp[\xi(\mathbf{p},\sigma)=p_\mu/m,\ k_\mu=0]$, which is related to the London constant $\Lambda(T)$ by

$$[\Lambda(T)]^{-1}=e^2\left\{\frac{N}{m}+\lim_{\mathbf{k}\to 0}K_\perp(\mathbf{k},0)\right\}.$$

Using Eqs. (14) and (15) for the constants Φ_ξ and R_ξ, and adopting the normal definition[22] of the spin-independent harmonics F_l and spin-dependent harmonics Z_l of the quantity $F(\mathbf{n}\cdot\mathbf{n}',\sigma\sigma')$, we get

$$\chi(T)=\tfrac{1}{4}\beta^2\nu(0)\,f(T)/[1+\tfrac{1}{4}Z_0f(T)]\,, \tag{69}$$

$$[\Lambda(T)]^{-1}=\frac{Ne^2}{m^*}(1+\tfrac{1}{3}F_1)$$
$$\times\left\{1-(1+\tfrac{1}{3}F_1)\frac{f(T)}{1+\tfrac{1}{3}F_1f(T)}\right\}$$
$$=\frac{Ne^2}{m^*}(1+\tfrac{1}{3}F_1)\frac{1-f(T)}{1+\tfrac{1}{3}F_1f(T)}\,. \tag{70}$$

[21] Thus F is a matrix with respect to direction on the Fermi surface, and has the same spin dependance as Γ^ω in any given case.

[22] A. A. Abrikosov and I. M. Khalatnikov, Rept. Progr. Phys. **22**, 329 (1959).

In the limit $T\to 0$, we have $f(T)\to 0$ and Eq. (20) agrees with the result of Larkin.[9] For comparison with the results of the "weak-coupling" theory we write the second equation as an equation for $1-[\Delta(0)/\Delta(T)]$; by definition this quantity is equal to the relative density of the normal component $\rho_n(T)/\rho$.[23] Also we express $\chi(T)$ in terms of the susceptibility χ_n of the normal phase. Thus

$$\chi(T)/\chi_n=(1+\tfrac{1}{4}Z_0)\,f(T)/[1+\tfrac{1}{4}Z_0 f(T)]\,, \tag{71}$$

$$1-\Delta(0)/\Delta(T)\equiv\rho_n(T)/\rho=(1+\tfrac{1}{3}F_1)\times f(T)/[1+\tfrac{1}{3}F_1 f(T)]\,. \tag{72}$$

Thus, in a superfluid Fermi liquid neither $\chi(T)$ nor the "density of the normal component" $\rho_n(T)$, as defined from the transverse current response, is simply proportional to $f(T)$, i.e., to the "effective density of single-particle levels."

Consider Landau's original argument[24] concerning the density of the normal component; for simplicity let us assume the system to be translationally invariant.[25] Landau *defines* the "density of the normal component" $\rho_n(T)$ by considering an experiment in which the system is made to come to thermal equilibrium relative to boundary conditions moving with velocity $\mathbf{V}$; then

$$\mathbf{P}\equiv\rho_n\mathbf{V}. \tag{73}$$

An example of such an experiment is a rotating-bucket or oscillating-disk experiment, so that clearly this definition is equivalent to the definition (72) based on the response to a transverse vector potential in the local limit. Landau then observes that we may *calculate* $\rho_n(T)$, *at sufficiently low temperatures*, by noting that at such temperatures the few excitations present form an ideal gas and the standard statistical mechanics of an ideal gas may be applied to them. In this way he derives the expression

$$\rho_n(T)=\tfrac{1}{3}\sum_{\mathbf{p}}\mathbf{p}^2\left(\frac{dn}{dE_{\mathbf{p}}}\right). \tag{74}$$

Now, in the case of a superfluid Fermi liquid all momenta are close to the Fermi surface, so that we may replace $\mathbf{p}^2$ in (74) by $p_F{}^2=3(m^*/m)[\nu(0)]^{-1}\rho$, where $\rho\equiv Nm$ is the total mass density. Moreover, in the translation-invariant case considered the Landau effective-mass relation[1] holds:

$$m^*/m=1+\tfrac{1}{3}F_1. \tag{75}$$

Thus we get from Eq. (74):

$$\rho_n(T)/\rho=\frac{m^*}{m}[\nu(0)]^{-1}\sum_{\mathbf{p}}\left(\frac{dn}{dE_{\mathbf{p}}}\right)=(1+\tfrac{1}{3}F_1)f(T)$$

which agrees with Eq. (72) in the limit $T\to 0$. At higher temperatures the excitations may no longer be regarded as an ideal gas and Eq. (74) is no longer valid.

Actually, Eqs. (71) and (72) have a very simple interpretation in terms of the two-fluid model. For instance, let us write the usual Landau expression[1] for the spin susceptibility of a normal Fermi system in a form explicitly involving the density of states $\nu(0)$:

$$\chi=\tfrac{1}{4}\beta^2\,\nu(0)/[1+\tfrac{1}{4}(a^2\nu(0)\Gamma_{\sigma,0}{}^{\omega})] \tag{76}$$

(where $\Gamma_{\sigma,0}{}^{\omega}\equiv a^2[\nu(0)]^{-1}Z_0$). Then we get (71) if we replace $\nu(0)$ by $\nu(0)f(T)$ not only in the numerator but also in the denominator of this expression. Thus, in considering the response of the system to probes which affect only the normal component, we must describe the latter as a "Fermi liquid" in its own right, with effective density of states $\nu(0)f(T)$, and apply to it the relevant results of Landau theory.

To conclude our discussion of the microscopic theory, let us derive a result which, though not directly connected with the results of this section, will be useful when we come to discuss possible applications to real systems: We consider the static transverse current-current correlation function in the limit $\mathbf{k}\to\infty\,(vk\gg\Delta)$. In this limit symmetry arguments again allow us to write

$$\lim_{\mathbf{k}\to 0}K_\perp(\mathbf{k},0)=\Phi_\xi+R_\xi{}^2\,\mathrm{Tr}\left\{\xi\frac{(GG+FF)(\mathbf{k})}{1-\Gamma^{\omega}(GG+FF)(\mathbf{k})}\xi\right\}. \tag{77}$$

Now in the limit $\mathbf{k}\to\infty$, $(GG+FF)(\mathbf{k})\to-a^2\nu(0)$; so by using Eq. (15) and after a certain amount of matrix algebra, we can rewrite (77) to lowest nonvanishing order in $(vk)^{-1}$ as

$$\begin{aligned}\lim_{\mathbf{k}\to\infty}K_\perp(\mathbf{k},0)&=-\frac{N}{m}+\left(\frac{m}{m^*}\right)^2\\&\quad\times\mathrm{Tr}\{\xi[(GG+FF)(\mathbf{k})+a^2\nu(0)]\xi\}\\&=-\frac{N}{m}+\left(\frac{1}{m^*}\right)^2\nu(0)\int p_\mu{}^2 f(T,\Delta/\mathbf{k}\cdot\mathbf{v})d\Omega\\&\approx-\frac{N}{m}+\left(\frac{1}{m^*}\right)^2\frac{\nu(0)\Delta}{kv}p_F{}^2\int_{-\infty}^{\infty}f(T,y)dy\\&=-\frac{N}{m}+\frac{p_F{}^2\Delta}{k}\psi(T)\,,\end{aligned} \tag{78}$$

where $\psi(T)$ is a universal function of T/T_c. Thus, in this limit $K_\perp(\mathbf{k},0)$ depends only on the Fermi momentum and the gap and is given by the same expression as in BCS theory; it is thus completely insensitive to

[23] The definition of $\rho_n(T)$ is of course somewhat arbitrary when the system is not translation invariant; however, for an impurity-free system it is most natural to choose it, as we have done, so that $\rho_n(T)\to\rho$ when $T\to T_c$ and $\to 0$ when $T\to 0$. [Thus $\rho_n(T)/\rho$ would represent the normal component of the quasiparticle excitations rather than of the liquid as a whole.]

[24] L. D. Landau, J. Phys. USSR **5**, 71 (1941).

[25] Although the original derivation was for a Bose liquid, the generalization to the Fermi case is easily made [J. Bardeen, Phys. Rev. Letters **1**, 399 (1959)].

Fermi-liquid effects. (In the case $T=0$ this was previously shown by Larkin.[9] Compare the normal-state anomalous skin effect, which is also completely unaffected by Fermi-liquid corrections.[26])

This concludes our discussion of the microscopic theory. The rest of this paper will be devoted to a semiphenomenological generalization of Eqs. (69) and (70) and to possible applications to real systems.

IV. PHENOMENOLOGICAL APPROACH

In this section we give a less rigorous but more general treatment of the effect of "Fermi-liquid" interactions on the static properties of a superfluid system. We shall consider explicitly only those properties which are affected by the superfluid transition, namely, the spin susceptibility $\chi(T)$ and the London constant $\Lambda(T)$. For simplicity we consider only the case where the normal phase is isotropic. It will be assumed, as always, that the system constitutes a "superfluid Fermi liquid" as defined in Sec. I.

Consider first the simpler case of the spin susceptibility $\chi(T)$.[27] According to our hypothesis (see Sec. I) the ground state and low-lying excited states of the system (those states excited for $T\lesssim T_0$) may be described in terms of linear combinations of Landau quasiparticle states. Also, since total spin isco nserved,[28] in calculating the susceptibility we need deal only with these low-lying states. Thus for our purposes the Hamiltonian can be written entirely in terms of Landau quasiparticle operators. Let us split it into two parts:

$$H=H_w+H_L\,,$$

$$H_w=\sum_{\mathbf{p}\sigma}\epsilon(\mathbf{p})\delta n(\mathbf{p},\sigma)+\sum_{\mathbf{p}\mathbf{p}'\sigma\sigma'}V(\mathbf{p}\mathbf{p}',\sigma\sigma')\times\alpha_{\mathbf{p}\sigma}{}^{\dagger}\alpha_{-\mathbf{p}\sigma'}{}^{\dagger}\alpha_{-\mathbf{p}'\sigma'}\alpha_{\mathbf{p}'\sigma}+U\,, \quad (79)$$

$$H_L=\tfrac{1}{2}\sum_{\mathbf{p}\mathbf{p}'\sigma\sigma}f(\mathbf{p}\mathbf{p}',\sigma\sigma')\delta n(\mathbf{p},\sigma)\delta n(\mathbf{p}',\sigma')\,. \quad (80)$$

Here $\alpha_{\mathbf{p}\sigma}{}^{\dagger}$ is the creation operator for a Landau quasiparticle, and $\delta n(\mathbf{p},\sigma)\equiv\alpha_{\mathbf{p}\sigma}{}^{\dagger}\alpha_{\mathbf{p}\sigma}-\theta(p_F-|p|)$. The quantities $\epsilon(\mathbf{p})$ and $V(\mathbf{p}\mathbf{p}',\sigma\sigma')$ are, respectively, the true (renormalized) single-particle energy and pairing interaction. In the case of a metal, $\mathbf{p}$ must be understood as the pseudomomentum. The operator U describes any other interactions which may be important in the problem, e.g., interactions with impurities. (It is assumed that all quasiparticle-quasiparticle scattering terms except the pairing interaction may be neglected.) Finally in (80) the function $f(\mathbf{p}\mathbf{p}',\sigma\sigma')$ is the usual Landau function used to describe quasiparticle interactions in the normal phase; it may be expanded in the usual way in Legendre polynomials of $\cos\theta$, where θ is the angle between $\mathbf{p}$ and $\mathbf{p}'$:

$$F(\mathbf{p}\mathbf{p}',\sigma\sigma')\equiv\nu(0)f(\mathbf{p}\mathbf{p}',\sigma\sigma')=\sum_l(F_l+Z_l\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}')P_l(\cos\theta)\,, \quad (81)$$

where $\nu(0)$ is the density of states at the Fermi surface in the normal phase.

We shall refer to the problem defined by H_w alone as the "weak-coupling" problem. The susceptibility calculated from H_w, which we denote by $\chi_w(T)$, depends of course on the precise nature of V and U and cannot be written down in general form. What we shall show is that under certain conditions, which will usually be met in practice, the true susceptibility $\chi(T)$ is entirely defined by $\chi_w(T)$ and the Landau parameter Z_0.

Notice first of all that H_L has no effect on the superfluid condensation itself nor on any "one-particle" property such as the effective density of single-particle levels. This follows from the fact that H_L is a function only of the quantity

$$\zeta(\mathbf{n},\sigma)\equiv\sum_{|p|}\langle\delta n(\mathbf{p},\sigma)\rangle\,,$$

which describes the distortion of the "average" Fermi surfaces, while the formation of Cooper pairs does not affect $\zeta(\mathbf{n},\sigma)$. Consequently, the thermodynamics of the system in zero external field may be calculated correctly from H_w; in particular, the BCS relation between the critical field $H_c(T)$ and the gap $\Delta(T)$ [and hence the ratio $H_c(T)/H_c(0)$ as a function of T/T_c] is unaffected by the presence of Fermi-liquid interactions.[29] However, when we consider the response to an external field, it is essential to take H_L into account, as we shall now see.

According to Eq. (81) we may expand H_L in terms of the harmonics of the quantities $\zeta(\mathbf{n},\sigma)$:

$$H_L=[2\nu(0)]^{-1}\{F_0(\delta N)^2+Z_0\mathbf{S}^2+F_1p_F{}^{-2}\mathbf{Q}^2+\cdots\}\,, \quad (82)$$

where $\mathbf{Q}\equiv\sum_{\mathbf{p}\sigma}\mathbf{p}\delta n(\mathbf{p},\sigma)$ is the total momentum if the system is translation-invariant. Thus, we can see that H_L gives rise to "molecular fields" of various kinds. To proceed further we make the assumption that *either* (a) the superfluid phase is isotropic *or* (b) all harmonics Z_l with $l>1$ are negligible. If either of these conditions is fulfilled, it is obvious from symmetry considerations that, for purposes of calculating the spin susceptibility, we need keep in H_L only the term in $\mathbf{S}^2$. Then our total Hamiltonian in the presence of an external magnetic field $\mathfrak{H}$ has the form

$$H=H_w+\tfrac{1}{2}[\nu(0)]^{-1}Z_0\mathbf{S}^2-\beta\mathbf{S}\cdot\mathfrak{H}\,, \quad (83)$$

[26] V. P. Silin, Zh. Eksperim. i Teor. Fiz. **33**, 1282 (1957) [English transl.: Soviet Phys.—JETP **6**, 985 (1958)].

[27] The treatment given here follows closely the lines of Ref. 10. A slight oversimplification in that reference will be pointed out.

[28] The results of this argument will actually be applied to some cases where total spin is *not* conserved, e.g., because of spin-orbit scattering. There is no inconsistency in this. All we need for present purposes is that the average spin carried by a Landau quasiparticle be effectively $\pm\frac{1}{2}$, and this will be true provided that $V\ll\mu$, where V is a typical spin-orbit interaction energy. On the other hand spin-orbit scattering will affect the superfluid-state spin susceptibility appreciably if $V\sim\Delta$.

[29] Although the normal-state susceptibility is of course affected, the absolute value of χ is so small anyway that the effect on $H_c(T)$ is completely negligible.

where β is the appropriate gyromagnetic ratio. Then, defining $\chi_w(T)$ as above, we easily find by minimizing the free energy first in the absence and then in the presence of the term containing Z_0 that

$$\chi=\chi_w(T)[1+\beta^{-2}[\nu(0)]^{-1}Z_0\chi_w(T)]^{-1} =\chi_w(T)[1+\tfrac{1}{4}Z_0\{\chi_w(T)/\chi_w(T_c)\}]^{-1}, \quad (84)$$

where we have used the fact that the normal-state susceptibility calculated from H_w, $\chi_w(T_c)$, is just the free-gas value $\frac{1}{4}\beta^2\nu(0)$. In the case where H_w is the BCS Hamiltonian, (84) reduces to (69).

Equation (84) is quite general, provided only that one of the conditions (a) or (b) above is satisfied. However, in one case of interest—the possible superfluid phase of liquid He^3—it is quite likely that neither condition is met. Let us therefore digress for a moment to examine this case, confining the consideration to a pure translation-invariant neutral system with singlet-spin pairing as discussed in Secs. II and III but allowing now the possibility of $l\neq 0$ pairing. The generalization of the theory developed in these sections is then trivial and we simply quote the generalization of Eq. (69):

$$\chi(T)=\tfrac{1}{4}\beta^2\nu(0)\int\frac{d\Omega}{4\pi}f(T:\mathbf{n}) \times\left\{1+\frac{1}{4}\int\frac{d\Omega'}{4\pi}Z(\mathbf{n},\mathbf{n}')f(T:\mathbf{n}')\right\}^{-1}, \quad (85)$$

where

$$f(T,\mathbf{n})\equiv\int_0^\infty d\epsilon_p(2T)^{-1}\cosh^{-2}(E_{\mathbf{p}}(\mathbf{n})/2T)\,, \qquad E_{\mathbf{p}}^2=\epsilon_{\mathbf{p}}^2+[\Delta(\mathbf{n})]^2, \quad (86)$$

and $Z(\mathbf{n}\cdot\mathbf{n}')$ is the spin-dependent part of $F(\mathbf{pp}',\sigma\sigma')$. It is also very plausible, in view of the work of Balian and Werthamer,[30] that the correct generalization to the case of triplet-spin pairing simply consists of the replacement in Eq. (85)

$$f(T:\mathbf{n})\rightarrow\tfrac{2}{3}+\tfrac{1}{3}f(T:\mathbf{n})$$

the quantity $f(T:\mathbf{n})$ remaining an even function of $\mathbf{n}$. However, we shall not attempt to prove this here. The evaluation of (85) is difficult in the general case; however, we can draw some important conclusions immediately. First, of course, (85) goes over into (84) in the limit $T\rightarrow 0$ [this would not be a trivial statement for triplet pairing, since in that case $\chi_w(0)=\frac{2}{3}\chi_w(T_c)$].[30] Moreover, the $l\neq 0$ harmonics of $f(T:\mathbf{n})$ can never exceed $1-f(T)$ in order of magnitude, and in particular they tend to zero as $1-f(T)$ for $T\rightarrow T_c$. Expanding (86) in powers of $1-f(T)$, we then see that the deviation of (85) from (84) is of second order in $1-f(T)$, while the deviation of $[\chi(T)/\chi(T_c)]$ from $[\chi_w(T)/\chi_w(T_c)]$ is of first order. Thus, (84) should predict the slope of $\chi(T)$ correctly in the limit $T\rightarrow T_c$. In the intermediate temperature region the higher harmonics Z_2, $Z_4\cdots$ will have some effect, but since in practice it is likely that Z_2 is considerably smaller than Z_0 for He^3, the effect is likely to be small. Thus, Eq. (84), though not exact for arbitrary pairing as was stated in Ref. 10, is nevertheless likely to be a good description even of a highly anisotropic superfluid.

Let us now turn to the London constant $\Lambda(T)$, or to be more precise, to the quantity $K(T)$[31] which we define (apart from constants) as the response in the local limit of the paramagnetic part of the current to a transverse vector potential:

$$K(T)=\lim_{\mathbf{q}\rightarrow 0}\langle\mathbf{J}_p(\mathbf{q})\rangle/\mathbf{A}(\mathbf{q})\,, \quad \mathbf{q}\cdot\mathbf{A}=0. \quad (87)$$

$K(T)$ is related to $\Lambda(T)$ by the equation

$$[\Lambda(T)]^{-1}=e^2\{(N/m)-K(T)\}\,. \quad (88)$$

To calculate the effect of Fermi-liquid interactions on $K(T)$, we shall treat the vector potential $\mathbf{A}(\mathbf{r})$ as constant in space and put in the "transversality" condition only through $K_w(T)$, the weak-coupling value of K. This is clearly justified since the "characteristic length" associated with Fermi-liquid interactions is of order k_F^{-1} and hence much smaller than any other characteristic length entering the problem. It will be assumed that the impurity concentration is too small to invalidate the Fermi-liquid description of the normal phase; in metals, this means roughly that the electronic mean free path must be large compared to the lattice spacing.

The argument runs closely parallel to the one used above for the spin susceptibility; however, there are complications owing to the possible lack of translational invariance.[32] In general the ground state and low excited states of the system are not eigenstates of the total current operator $\mathbf{J}$. However, we can split $\mathbf{J}$ into two parts:

$$\mathbf{J}=\mathbf{J}_f+\mathbf{J}_{QP},$$

where the operator $\mathbf{J}_f$ has no matrix elements between the low-lying states of the system (though in general it has matrix elements between low and high states), while $\mathbf{J}_{QP}$ is diagonal in the Landau quasiparticle representation. Clearly in the weak-field limit the contributions to $K(T)$ of $\mathbf{J}_f$ and $\mathbf{J}_{QP}$ are additive, and the contribution of $\mathbf{J}_f$ [which corresponds to the term Φ_ξ in Eq. (5)] is insensitive to the superfluid transition. Therefore we consider only the contribution of $\mathbf{J}_{QP}$, which we label $\tilde{K}(T)$: that is,

$$\tilde{K}(T)=\langle\mathbf{J}_{QP}\rangle/\mathbf{A}. \quad (89)$$

In the Landau quasiparticle representation it follows from the continuity equation and the kinetic equation

[30] R. Balian and N. R. Werthamer, Phys. Rev. **131**, 1553 (1963).

[31] $K(T)$ is the negative of the $K_1(T)$ of Sec. III.

[32] The discussion which follows may be compared with that at the beginning of Sec. II.

(cf. Ref. 1) that

$$\mathbf{J}_{\mathrm{QP}}=\sum_{\mathbf{p}\sigma}(1/m^*)(1+\tfrac{1}{3}F_1)\mathbf{p}\delta n(\mathbf{p},\sigma)$$
$$\equiv(1/m^*)(1+\tfrac{1}{3}F_1)\mathbf{Q}\,, \quad (90)$$

where $\mathbf{Q}$ is defined as in Eq. (82). Since we may now restrict our attention to low-lying states of the system, we can once again write the Hamiltonian in the form $H=H_w+H_L$, where H_w and H_L are given, respectively, by (79) and (80). Then, provided either (a) the superfluid phase is isotropic, or (b) all harmonics F_l [see Eq. (81)] are negligible for $l>2$, we may strike out [for purposes of calculating $K(T)$] all terms in H_L except the one containing $\mathbf{Q}^2$. Using (90), we obtain

$$H_L\rightarrow\frac{1}{2\nu(0)}F_1\left(\frac{m^*}{1+\frac{1}{3}F_1}\right)^2 p_F^{-2}\mathbf{J}_{\mathrm{QP}}{}^2. \quad (91)$$

Thus, again H_L provides a sort of "molecular field." In a weak external field, ignoring the term in $\mathbf{J}_f\cdot\mathbf{A}$ which contributes to the constant part of $K(T)$, we can write

$$H=H_w+H_L-\mathbf{J}_{\mathrm{QP}}\cdot\mathbf{A}. \quad (92)$$

We define $\tilde{K}_w(T)$ as the value of $\tilde{K}(T)$ calculated by setting $H_L=0$.[33] Then it follows from (91) and (92) that

$$\tilde{K}(T)=\tilde{K}_w(T)[1+\lambda\tilde{K}_w(T)]^{-1},$$
$$\lambda\equiv\frac{1}{\nu(0)p_F{}^2}\left(\frac{m^*}{1+\frac{1}{3}F_1}\right)^2 F_1. \quad (93)$$

It is easy to show that the value of $\tilde{K}_w$ in the normal phase [i.e., of $\tilde{K}_w(T_c)$] is just equal to $\frac{1}{3}(p_F/m^*)^2\times(1+\frac{1}{3}F_1)^2\nu(0)$; hence we can put (93) into a form analogous to (84):

$$\tilde{K}(T)=\tilde{K}_w(T)[1+\tfrac{1}{3}F_1\{\tilde{K}_w(T)/K_w(T_c)\}]^{-1}. \quad (94)$$

Moreover, it can also be shown (cf. Sec. II) that the "nonquasiparticle" contribution to $K(T)$ is the quantity $-\Phi_p$, where Φ_p is given by Eq. (15); thus

$$K(T)=\frac{N}{m}\left\{1-\frac{m}{m^*}(1+\tfrac{1}{3}F_1)\right\}$$
$$+\frac{\tilde{K}_w(T)}{1+\frac{1}{3}F_1\{K_w(T)/K_w(T_c)\}}. \quad (95)$$

In the case of a pure superfluid with S-wave pairing we have

$$\tilde{K}_w(T)=(N/m^*)(1+\tfrac{1}{3}F_1)^2 f(T)\,. \quad (96)$$

The constants arise from the fact that the current carried by the quasiparticles is renormalized according to Eq. (90) (cf. footnote 33). Therefore, using (88), we get agreement with Eq. (70). [The quantity $f(T)$ is defined in Eq. (63).]

In the case where the superfluid phase is highly anisotropic and the Landau parameters F_3, $F_5\cdots$ are not negligible, the same sort of remarks apply as we made concerning the susceptibility.

It should be noticed that nowhere in this section have we said anything about the presence or absence of long-range (Coulomb) forces. Actually, these can be handled by the method first proposed by Silin[3] (cf. also Ref. 16) in which the short-range effects are incorporated in the Landau parameters (and in the pairing interaction) while the long-range effects are put in at the end of the calculation by means of Maxwell's equations. These long-range effects are important only when we consider processes in which the total local charge density is changed; they do not affect the spin susceptibility or the quantity $K(T)$, which by definition measures the local response to a transverse vector potential (though they may, of course, mean that this quantity is not experimentally accessible).

The results of this section allow us to predict unambiguously the temperature dependence of the quantities $\chi(T)$ [Eq. (84)] and $K(T)$ [Eq. (95)] in the presence of Fermi-liquid interactions, provided we know the values $\chi_w(T)$ and $K_w(T)$ predicted by the appropriate "weak-coupling" theory. It should be emphasized that no assumptions have been made about the nature of the "weak-coupling" theory involved save the minimal ones that the superfluid transition does not change the "average" Fermi surfaces, and that the superfluid phase is isotropic (we saw that violation of the second condition is unlikely to affect the results very seriously). Thus the derivation used in this section is much more general than that of Secs. II and III, and should be applicable to almost all cases of physical interest.

V. APPLICATIONS TO REAL SYSTEMS

The only extended, isotropic (in the normal phase) and neutral Fermi system likely to display superfluid behavior is liquid He^3. This liquid is well described by Landau's theory of a normal Fermi liquid below about 0.05°K,[34] while T_c is certainly not higher than 0.0035°K.[35] Hence, from the point of view of this paper, He^3 in the superfluid phase, if it exists, should constitute a superfluid Fermi liquid. Of course, it is unlikely that the condensed phase is spherically symmetric; however, as we saw above, Eqs. (84) and (94) are likely to give a qualitatively good description even of a D- or F-wave paired system. If we assume a D state, the appropriate values of $\chi_w(T)$ and $K_w(T)$ have been calculated by

[33] This quantity differs from the value of $K(T)$ usually calculated in the literature by a constant factor due to the renormalization of the current carried by the quasiparticles.

[34] N. Bernardes and D. F. Brewer, Rev. Mod. Phys. **34**, 190 (1962).

[35] W. R. Abel, A. C. Anderson, W. C. Black, and J. C. Wheatley, Phys. Rev. Letters **14**, 129 (1965).

Anderson and Morel[36] and may be substituted in Eqs. (84) and (95) respectively. Then, (84) and (95) read, simply,

$$\chi(T)\approx\tfrac{1}{4}\beta^2\nu(0)\, f(T)/[1+\tfrac{1}{4}Z_0 f(T)]\,, \tag{97}$$

$$K(T)\approx\frac{N}{m}(1+\tfrac{1}{3}F_1)\, f(T)/[1+\tfrac{1}{3}F_1 f(T)]\,, \tag{98}$$

the approximate equality being due to neglect of the higher harmonics Z_l and F_l. If the pairing takes place in an F state (spin triplet), one may hope that the theory of Balian and Werthamer[30] may be generalized by requiring the gap $\Delta(\mathbf{n})$ to have a D-wave dependence. In that case we should expect (98) to be valid as it stands, while $f(T)$ in (97) must be replaced by $\frac{2}{3}+\frac{1}{3}f(T)$ (see Ref. 30). The function $f(T)\equiv[\nu(0)]^{-1}\times\sum_{\mathbf{p}}(dn/dE_{\mathbf{p}})$, though not identical with the BCS expression for a D-wave dependence of the gap, may be calculated numerically.[36,37] The spin susceptibility of liquid He^3, though very small in absolute magnitude, may be measured with reasonable accuracy by spin-echo experiments,[35] while the quantity $K(T)$ can be directly measured from experiments of the oscillating-disk (or rotating-bucket) type (cf. Ref. 34). Since for He^3 the Landau parameters are large ($Z_0=-2.8$, $F_1=5.46$ at low pressures),[38] it should be possible to verify Eqs. (98) and (97) (or its generalization to the case of triplet pairing) quite unambiguously; the curve $K(T)$ should be much less concave[39] and the curve $\chi(T)$ (in the case of singlet pairing) much more so, than the curve $f(T)$. In the case of triplet pairing $\chi(0)$ should be 0.37 rather than the value $\frac{2}{3}$ predicted by the weak-coupling theory.[30]

Let us now consider metallic superconductors. As we saw in the last section, the long-range nature of the Coulomb interaction is unimportant for our purposes. As to the electron-phonon interaction, for all superconductors except lead and mercury it seems that the gap Δ is small compared to a characteristic phonon frequency ω_0. This means that we can introduce an effective electron-electron interaction (due in general both to Coulomb and to phonon effects) which will be nearly constant over a region $\epsilon,(|p|-p_F)v\sim\Delta$. Thus, apart from the relatively unimportant question of anisotropy of the normal phase, the electrons in a superconductor constitute a superfluid Fermi liquid from the point of view of the present paper.

Observation of a Fermi-liquid correction to the spin susceptibility [Eq. (84)] is complicated by the fact that the Knight shift in the transition metals, where it has been most extensively investigated, almost certainly receives considerable contributions from sources unconnected with spin polarization of the conduction electrons.[40] Other superconductors which have been investigated are Hg, Al, and Sn. Hg, as a "strong-coupling" superconductor, lies outside the scope of this paper, while the present data for Al[41] unfortunately show too much scatter to allow any quantitative conclusions to be drawn. We are left with the data for Sn, which should fall under the present theory. If we assume that in Sn the Knight shift does indeed measure the spin susceptibility and nothing else, then to get a finite value of $\chi(0)$ we must assume that spin-flip scattering is important, and $\chi_w(T)$ should be given by the expression of Abrikosov and Gor'kov.[42] It is then possibly significant that the experimental curve for Sn[43] is definitely more convex than the curve of $\chi_w(T)$ predicted by these authors, whatever the value of the spin-flip scattering parameter (see Fig. 6 of Ref. 42). This is what we should expect from Eq. (84) if Z_0 has a small negative value, as is expected on theoretical grounds. However, probably we should not place too much reliance on this evidence, since a constant contribution to the Knight shift from sources other than spin polarization could upset this interpretation.

Prospects are somewhat brighter as regards the experimental detection of Fermi-liquid effects on the weak-field dc penetration depth $\lambda(T)$, which can be determined with great accuracy; in particular this has been done by Waldram and Pippard for tin.[44] Tin, like most elemental superconductors, is a Pippard superconductor for low impurity concentrations; that is, for all temperatures except very close to T_c we have $\xi_0\gg\lambda_L(T)$ where the London penetration depth $\lambda_L(T)$ is related to $\Lambda(T)$ [Eq. (88)] by $\lambda_L{}^2=c^2\Lambda/4\pi$. Let us first therefore consider whether we should expect Fermi-liquid effects to be noticeable in a Pippard superconductor. In the Pippard limit $\xi_0\gg\lambda_L$ the penetration depth $\lambda(T)$ is determined entirely by the behavior of $K(\mathbf{k},0)$ for $vk\gg\Delta$; as we saw at the end of Sec. III [Eq. (78)], this behavior depends only on the Fermi momentum and the gap and is entirely independent of the Landau parameters. Hence, throughout the temperature region where the Pippard limit applies, the predictions of the present theory coincide with those of BCS, and no Fermi-liquid effects are to be expected. However, in the limit $T\to T_c$ the opposite (London) limit applies. In that case we have for a pure specimen, from Eqs. (88), (95), and (96),

$$\lambda_L{}^{-2}(T)=\frac{4\pi Ne^2}{m^*c^2}\left\{(1+\tfrac{1}{3}F_1)\frac{(1-f(T))}{1+\tfrac{1}{3}F_1 f(T)}\right\}. \tag{99}$$

[36] P. W. Anderson and P. Morel, Phys. Rev. **123**, 1911 (1961).
[37] L. H. Nosanow and R. Vasudevan, Phys. Rev. Letters **6**, 1 (1961).
[38] D. Hone, Phys. Rev. **125**, 1494 (1962).
[39] In fact it may even have an inflection point.
[40] A. Clogston, A. Gossard, V. Jaccarino, and Y. Yafet, Rev. Mod. Phys. **36**, 170 (1964).
[41] R. H. Hammond and G. M. Kelley, Rev. Mod. Phys. **36**, 185 (1964).
[42] A. A. Abrikosov and L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **42**, 1088 (1962) [English transl.: Soviet Phys.—JETP **15**, 752 (1962)].
[43] G. M. Androes and W. D. Knight, Phys. Rev. **121**, 779 (1961).
[44] J. R. Waldram, Advan. Phys. **13**, 1 (1964).

The BCS expression is

$$\lambda_L^{-2}(T)=\frac{4\pi Ne^2}{m^*c^2}[1-f(T)], \quad (100)$$

so that the two expressions coincide in the limit $T\to T_c$. Thus, for instance, the quantity $\lim_{y\to\infty} d\lambda(y)/dy$, $y(T)\equiv(1-T^4/T_c^4)^{-1/2}$, should be given by the BCS expression, as is in fact found with high accuracy by Waldram.[44] (It should perhaps be noted that the experimental determination of N and m^* from the normal-state specific heat and anomalous skin effect is unaffected by Fermi-liquid corrections.)[26] Thus in the regions $T_c-T\sim T_c$ and $T_c-T\ll T_c$, the penetration depth $\lambda(T)$ should not show any appreciable Fermi-liquid effects. In the intermediate region we should expect some small effect, and it is interesting that Waldram[44] does indeed find a small discrepancy from the BCS curve. He suggests that this may be due to a deviation of the temperature dependence of the gap $\Delta(T)$ from the BCS predictions, but recent measurements of the critical field in tin by Finnemore and Mapother[45] seem to indicate no such deviation. It is therefore tempting to speculate that the discrepancy may be due to Fermi-liquid effects—it is of the right sign for a positive value of F_1—but in view of the complexity of the calculations we shall not pursue this possibility here. In general the above discussion shows that Pippard superconductors would not be expected to show the effects of Fermi-liquid interactions in any dramatic way.

With a good London superconductor the situation should be quite different. In this case the electrodynamics is always local and the theory of Sec. IV may be used directly in all temperature regions. In order to give a general expression covering both pure and dirty London superconductors, let us define a quantity $z(T,l)$ in terms of $\lambda_L(T,l)$, the penetration depth predicted by the weak-coupling theory of Mattis and Bardeen[46] for the given temperature and normal-state mean free path l, and $\lambda_L(0,\infty)$, the London penetration depth of BCS theory in the limit $T=0$, $l=\infty$:

$$z(T,l)\equiv[\lambda_L(0,\infty)/\lambda_L(T,l)]^{-2} =1-[\tilde{K}_w(T,l)/\tilde{K}_w(T_c)], \quad (101)$$

with $\tilde{K}_w(T_c)=(N/m^*)(1+\frac{1}{3}F_1)^2$ [see the discussion of Eq. (93)]. The quantity $\rho z(T,l)$ is the effective density of the superfluid component in the weak-coupling model. In the limit $l\ll\xi_0$ it is given by $J(0,T)(l/\xi_0)\rho f(T)$ where $f(T)$ is defined by (63) and $J(R,T)$ is the usual function used in BCS theory (see e.g., Ref. 47, p. 221). Then from (88) and (95) we find

$$\lambda_L^{-2}(T)=\frac{4\pi Ne^2}{m^*c^2}(1+\tfrac{1}{3}F_1)\left\{\frac{z(T,l)}{1+\frac{1}{3}F_1(1-z(T,l))}\right\}. \quad (102)$$

From Eq. (102) it is obvious that the most spectacular effect on the penetration depth will occur for a pure London superconductor ($l\to\infty$). In this case (102) reduces to (99), which can be written

$$[\lambda(T)/\lambda(0)]^2=[\lambda(T)/\lambda(0)]^2_{\rm BCS}(1+\tfrac{1}{3}F_1)-F_1/3, \quad (103)$$

so that the temperature dependence of $\lambda(T)$ will enable us to determine F_1 directly as the parameter of an otherwise parameter-free curve. Pure elemental niobium and vanadium are probably the nearest things to a pure London superconductor found in nature, and it would therefore be of great interest to measure the penetration depth accurately for these metals. However, even in the usual case of a dirty London superconductor, where $z(0,l)\sim 0.1$ (say), the accuracy of penetration depth measurements should probably be sufficient to allow us to detect the effect of Fermi-liquid interactions [Eq. (102)] and even to get at least an order of magnitude value of F_1.

To obtain such a direct estimate of F_1, even for a very dirty superconductor, would be of more than academic interest. As has been shown by Silin,[26] there exists another way of obtaining F_1 for metals, namely a comparison of experimental data on the electronic specific heat, the anomalous skin effect and the dielectric constant in the infrared region. However, the values calculated by Silin from these data for various metals are very large (comparable to those for He^3) and cannot be reconciled with any theoretical estimate based on present ideas about the Coulomb and electron-phonon interactions (see, for example, Ref. 4). Thus it would be of great interest to obtain a value of F_1 directly from the penetration-depth curve, as suggested above, even for a dirty metal. The value should be at least of the same order of magnitude as in the pure metal, so that such an experiment should constitute a direct test of accepted ideas about the weakness of electron interactions. One caution is necessary: If the mean free path l becomes comparable to k_F^{-1} (which is of the order of the interatomic spacing a), then the Landau theory in its usual form presumably becomes meaningless for the normal state and F_1 is not well defined. Thus, it would be desirable to work with a specimen such that $a\ll l\ll\xi_0$.

VI. CONCLUSION

In this paper we have developed a formalism suitable for obtaining the properties of a neutral, isotropic superfluid Fermi liquid, and have applied it to the calculation of the static correlation functions in the long-wavelength limit. It has been shown that the "Fermi liquid" interactions have the effect of changing the temperature dependence of the spin susceptibility and London constant. Further, we have shown that this effect is not peculiar to the specific model considered, and in fact the true temperature dependence of the susceptibility and London constant may almost always be

[45] D. K. Finnemore and D. E. Mapother (unpublished).
[46] D. C. Mattis and J. Bardeen, Phys. Rev. 111, 412 (1958).
[47] J. R. Schrieffer, *Superconductivity* (W. A. Benjamin, Inc., New York, 1964).

obtained from a knowledge of their "weak-coupling" values and of the appropriate Landau parameters.

The theory should provide a good description of the properties of liquid He^3 in the superfluid phase (if it exists). As to metallic superconductors, it is suggested that the temperature dependence of the Knight shift in tin may be due to Fermi-liquid effects and that other nontransition superconductors may be expected to show similar effects. Also, the temperature variation of the weak-field dc penetration depth in very hard (London) superconductors should show the effect of the Fermi-liquid interactions unambiguously. Such experiments should in principle provide much more direct values of the Landau parameters Z_0 and F_1 than any other methods presently available, and this would be of considerable interest from a theoretical standpoint.

ACKNOWLEDGMENTS

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APPENDIX

The purpose of this Appendix is to derive an expression of the form (5) for the autocorrelation function $K_\xi(\mathbf{k},\omega)$; that is, to split it into two terms corresponding to "nonquasiparticle" and "quasiparticle" contributions (cf., the discussion at the beginning of Sec. II). The "nonquasiparticle" contribution Φ_ξ will be insensitive to the values of $\mathbf{k}$ and ω (for ω, $vk \ll \mu$) and also to the onset of superfluidity; the "quasiparticle" term will consist of some integral $I_\xi(\mathbf{k}\omega,)$ which is taken over regions near the Fermi surface, multiplied by a constant R_ξ^2 which is expressed in terms of integrals over all regions of energy and momentum. Thus all the dependence on $\mathbf{k},\omega$ and T is contained in the function $I_\xi(\mathbf{k},\omega)$ which may be expressed in terms of a few phenomenological constants of the Landau type; the constants Φ_ξ and R_ξ may be evaluated once and for all in the normal state at any convenient temperature (e.g., $T=0$). For the sake of clarity the derivation will be carried out explicitly for a normal system; the generalization to the superfluid case is then straightforward and is indicated in Sec. II of the text. The formalism used here is based on the work of Eliashberg,[17] to which we refer for details of the analytic continuation procedures.

We are interested in the quantity

$$K_\xi(\mathbf{k},\omega) \equiv \sum_{\mathbf{p}\mathbf{p}'\sigma\sigma'} \xi(\mathbf{p},\sigma) K^{\mathbf{p}\mathbf{p}',\sigma\sigma'}(\mathbf{k},\omega)\xi(\mathbf{p}',\sigma'), \quad \text{(A1)}$$

where

$$K^{\mathbf{p}\mathbf{p}',\sigma\sigma'}(\mathbf{k},\omega) \equiv \ll a_{\mathbf{p}+\mathbf{k}/2,\sigma}^\dagger a_{\mathbf{p}-\mathbf{k}/2,\sigma} : a_{\mathbf{p}'-\mathbf{k}/2,\sigma'}^\dagger a_{\mathbf{p}'+\mathbf{k}/2,\sigma'} \gg_-(\omega),$$

$$\ll A : B \gg_\pm(\omega) \equiv -i\int_{-\infty}^{\infty} \theta(t)\langle A(t)B(0) \pm B(0)A(t)\rangle \times \exp i\omega t\, dt.$$

(In the one-particle Green's functions occurring subsequently, which will be written without a subscript $\pm$, the $+$ sign must be taken, as usual.) From now on the spin indices will be omitted; they enter on an equal footing with $\mathbf{p}$.

The following statements are proved in the work of Eliashberg[17]: The quantity $K^{\mathbf{p}\mathbf{p}'}(\mathbf{k},\omega)$ is the analytic continuation of the "temperature" Green's function $K^{\mathbf{p}\mathbf{p}'}(\mathbf{k},\omega_m)$ from the upper half-plane of ω. Now $K^{\mathbf{p}\mathbf{p}'}(\mathbf{k},\omega_m)$ is related to the complete vertex part $\Gamma^{\mathbf{p}\mathbf{p}'}(\epsilon_n,\epsilon_{n'}:\mathbf{k},\omega_m)$ by an equation of the usual form, which we write symbolically as

$$K = GG + GG\Gamma GG.$$

When written out explicitly and continued analytically to the real axis this equation gives (we omit the indices $\mathbf{k},\omega$ for brevity where they are not specially emphasized)

$$K^{\mathbf{p}\mathbf{p}'}(\mathbf{k},\omega) = \frac{1}{4\pi i}\int \sum_{i=1}^{3} \lambda_i(\epsilon) K_i^{\mathbf{p}\mathbf{p}'}(\epsilon)\, d\epsilon, \quad \text{(A2)}$$

where

$$\begin{aligned} \lambda_1(\epsilon) &= \tanh(\epsilon/2T), \\ \lambda_2(\epsilon) &= \tanh(\epsilon+\omega/2T) - \tanh(\epsilon/2T), \\ \lambda_3(\epsilon) &= -\tanh(\epsilon+\omega/2T), \end{aligned} \quad \text{(A3)}$$

and the quantities $K_i^{\mathbf{p}\mathbf{p}'}(\epsilon)$ are given by

$$K_i^{\mathbf{p}\mathbf{p}'}(\epsilon) = g_i(\mathbf{p},\epsilon)\delta_{\mathbf{p}\mathbf{p}'} + \frac{1}{4\pi i}\sum_{j=1}^{3}\int d\epsilon' Q_{ij}^{\mathbf{p}\mathbf{p}'}(\epsilon,\epsilon') g_j(\mathbf{p},\epsilon'), \quad \text{(A4)}$$

where

$$\begin{aligned} g_1(\mathbf{p},\epsilon:\mathbf{k},\omega) &\equiv G_R(\mathbf{p}+\mathbf{k}/2,\ \epsilon+\omega) G_R(\mathbf{p}-\mathbf{k}/2,\ \epsilon), \\ g_2(\mathbf{p},\epsilon:\mathbf{k},\omega) &\equiv G_R(\mathbf{p}+\mathbf{k}/2,\ \epsilon+\omega) G_A(\mathbf{p}-\mathbf{k}/2,\ \epsilon), \\ g_3(\mathbf{p},\epsilon:\mathbf{k},\omega) &\equiv G_A(\mathbf{p}+\mathbf{k}/2,\ \epsilon+\omega) G_A(\mathbf{p}-\mathbf{k}/2,\ \epsilon), \end{aligned} \quad \text{(A5)}$$

where G_R and G_A are, respectively, the retarded and advanced single-particle Green's functions. (Strictly, the analytic continuation gives us the functions which are analytic in the upper and lower half-planes of ϵ, respectively; this must be remembered when we come to the generalization of the superfluid case.) The func-

tions $Q_{ij}(\mathbf{p},\mathbf{p}',\epsilon,\epsilon':\mathbf{k},\omega)$ are connected with the various analytic continuations of the vertex part in a way explained in detail in Ref. 17. For our purposes it is sufficient to know that they all have the form [see Eq. (12) of Ref. 17]

$$Q_{ij}(\epsilon,\epsilon':\omega)=\lambda_j(\epsilon',\omega)\Gamma_{ij}(\epsilon,\epsilon',\omega) +D_{ij}(\epsilon,\epsilon',\omega)\coth[L(\epsilon,\epsilon',\omega)/2T], \quad \text{(A6)}$$

where $L(\epsilon,\epsilon',\omega)$ is some linear combination of the quantities $\epsilon,\epsilon',\omega, L(\epsilon,\epsilon',\omega)=\pm\epsilon\pm\epsilon'(\pm\omega)$, $\Gamma_{ij}(\mathbf{pp}',\epsilon\epsilon':\mathbf{k},\omega)$ is some analytic continuation of the "temperature" vertex part $\Gamma(\epsilon_n,\epsilon_{n'}:\omega_m)$ and D_{ij} is the *difference* of two analytic continuations, which is always pure imaginary. In particular we have $D_{i2}\equiv D_{2i}\equiv 0$. When the argument of the hyperbolic cotangent in (A6) is zero the principal part is to be taken. For a proof of these statements we refer to Ref. 17.

To simplify the notation let us regard the quantities $Q_{ij}(\mathbf{pp}',\epsilon\epsilon':\mathbf{k},\omega)$ as matrices with respect to $\mathbf{p}$ (and σ), ϵ, and $i(i=1,2,3)$. Matrix multiplication will be defined by

$$(AB)_{ij}(\mathbf{pp}',\sigma\sigma',\epsilon\epsilon') =\frac{1}{4\pi i}\sum_{\mathbf{p}''\sigma''}\sum_{l}\int d\epsilon'' A_{il}(\mathbf{pp}'',\sigma\sigma'',\epsilon\epsilon'') \times B_{lj}(\mathbf{p}''\mathbf{p}',\sigma''\sigma',\epsilon''\epsilon'),$$

and the trace by

$$\mathrm{Tr}A=\frac{1}{4\pi i}\sum_{\mathbf{p}\sigma}\sum_{i}\int d\epsilon A_{ii}(\mathbf{pp},\sigma\sigma,\epsilon\epsilon).$$

Note that $\mathbf{k}$ and ω are parameters here. We may regard $g_i(\mathbf{p},\epsilon)$, $\lambda_i(\epsilon)$, and $\xi(\mathbf{p},\sigma)$ as diagonal matrices. Combining (A1), (A2) and (A4) we can now write concisely

$$K_\xi=\mathrm{Tr}\{\xi\lambda g(1+Qg)\xi\}. \quad \text{(A7)}$$

Now the matrix Q obeys the "Dyson" equation (cf. Ref. 17—note that our $Q^{(0)}$ is Eliashberg's $\mathcal{T}^{(1)}$ while his $\mathcal{T}^{(0)}$ is not used here):

$$Q=Q^{(0)}+Q^{(0)}gQ, \quad \text{(A8)}$$

where $Q^{(0)}$ is related to the usual irreducible vertex part $\Gamma^{(0)}$ in the same way as Q is to Γ. Now, following the original idea of Landau,[2] we introduce instead of $Q^{(0)}$ a "quasiparticle-irreducible" vertex part Q^ω, as follows. At least for $\mathbf{p}$ sufficiently close to the Fermi surface, the one-particle Green's function will have a pole near the real axis plus some incoherent background:

$$G_R(\mathbf{p},\epsilon)=[G_A(\mathbf{p},\epsilon)]^*=a(\epsilon-\epsilon_\mathbf{p}+i\gamma_\mathbf{p})^{-1}+G_{\mathrm{inc}}, \quad \text{(A9)}$$

where $\epsilon_\mathbf{p}$ is the quasiparticle excitation energy and $\gamma_\mathbf{p}$ the half-width, and the constant a is less than unity. Now we divide the products $g_i(\mathbf{p},\epsilon:\mathbf{k},\omega)$ into the part which is the product of two "polar" terms, which we label g_n, and all the rest, $g_f\equiv g-g_n$. It is convenient also to introduce a cutoff $\epsilon_0(\omega,vk, T\ll\epsilon_0\ll\mu)$ such that by definition $g_n(\mathbf{p})\equiv 0$ if $|\epsilon_\mathbf{p}|>\epsilon_0$. The rationale of the separation is that, within the limits of our approximation, all the dependence on $\mathbf{k}$ and ω (and also on the superfluid transition) is contained in g_n. It will turn out that the results are effectively independent of ϵ_0 in the normal case, while the dependence in the superfluid case is easily eliminated (see Sec. II of text). We now introduce Q^ω by the equation

$$Q^\omega=Q^{(0)}+Q^{(0)}g_fQ^\omega=Q^{(0)}+Q^\omega g_fQ^{(0)}. \quad \text{(A10)}$$

(The second equality follows from the first by a simple matrix identity.) It then follows from (A8) that

$$Q=Q^\omega+Q^\omega g_nQ=Q^\omega+Qg_nQ^\omega. \quad \text{(A11)}$$

$Q_{ij}{}^\omega(\epsilon,\epsilon':\omega)$ is obviously related to the analytic continuations of a quantity $\Gamma^\omega(\epsilon_n,\epsilon_{n'}:\omega_m)$ by the analog of Eq. (A6):

$$Q_{ij}{}^\omega(\epsilon,\epsilon':\omega)=\lambda_j(\epsilon',\omega)\Gamma_{ij}{}^\omega(\epsilon,\epsilon',\omega) +D_{ij}{}^\omega(\epsilon,\epsilon',\omega)\coth[L(\epsilon,\epsilon',\omega)/2T], \quad \text{(A12)}$$

where in particular $D_{i2}{}^\omega\equiv D_{2i}{}^\omega\equiv 0$. The real parts of all $\Gamma_{ij}{}^\omega$ are equal and will be assumed to be slowly varying functions of $\epsilon,\epsilon',|p|,|p'|,\mathbf{k}$ and ω for $\epsilon,\epsilon',\mathbf{p},\mathbf{p}'$ near the Fermi surface and $vk,\omega\ll\mu$ (this is of course an essential hypothesis of Fermi-liquid theory):

$$\mathrm{Re}\Gamma_{ij}{}^\omega\approx\Gamma^\omega(\mathbf{n}\cdot\mathbf{n}',\sigma,\sigma'),$$

where $\mathbf{n}$ and $\mathbf{n}'$ are unit vectors respectively parallel to $\mathbf{p}$ and $\mathbf{p}'$. We shall subsequently identify Γ^ω with the function introduced by Landau[2] at $T=0$.

Now we use the following matrix identity: if for any two matrices S,f we have $f=f_1+f_2$ and $S=S_0+S_0f_2S$, then

$$f(1+Sf)=f_1(1+S_0f_1) +(1+f_1S_0)(f_2{}^{-1}-S_0)^{-1}(1+S_0f_1). \quad \text{(A13)}$$

Applying this identity to (A7) with the help of (A11), we get

$$K_\xi=\mathrm{Tr}\{\xi\lambda g_f(1+Q^\omega g_f)\xi\}+\mathrm{Tr}\{\xi\lambda(1+g_fQ^\omega) \times[g_n(1-Q^\omega g_n)^{-1}](1+Q^\omega g_f)\xi\}. \quad \text{(A14)}$$

The first term in (A14) is a constant Φ_ξ; it is insensitive to $\mathbf{k},\omega$ and the superfluid phase transition, since it contains no terms referring to quasiparticles near the Fermi surface (except for negligibly small ones). The second term still needs some rearrangement in order to ensure that the central integral is taken only over a region of width $\sim\omega$ or T around the Fermi surface (rather than $\sim\epsilon_0$). Using the fact that g_1 and g_3 have two poles close together on the same side of the real

axis and therefore the rest of the integrand must be sharply varying in the region $\epsilon\sim\epsilon_0$ to give an appreciable contribution, we see that to order ϵ_0/μ the contribution to the integral $Q_{ij}{}^{\omega}g_j(j\neq 2)$ from the terms containing $D_{ij}{}^{\omega}$ of order $D_{ij}{}^{\omega}(\epsilon_0,\epsilon_0)$ at most. But $D_{ij}{}^{\omega}$, which is pure imaginary and related to the probability of a real collision process, is small for $\epsilon\sim\epsilon_0$ because of Pauli-principle restrictions. Hence we can neglect this contribution. Since the contribution of the terms in $\lambda g_f Q^{\omega}$ containing $D_{22}{}^{\omega}$ is also of order ϵ_0/μ at most relative to those containing $\Gamma_{i2}{}^{\omega}$, while $D_{i2}{}^{\omega}\equiv 0$, we can put $Q^{\omega}=\Gamma^{\omega}\lambda$ everywhere outside the square brackets in (A14). It should be noticed that this argument is also valid for the superfluid phase. Thus (A14) becomes

$$K_{\xi}=\Phi_{\xi}+\mathrm{Tr}\{\xi(1+\lambda g_f\Gamma^{\omega}) \times[\lambda g_n(1-Q^{\omega}g_n)^{-1}](1+\Gamma^{\omega}g_f\lambda)\xi\}\,. \quad \text{(A15)}$$

The integrals inside the square brackets are now effectively taken only over a region $\epsilon,v\Delta p\sim T$ (or ω, if $\omega>T$) (for the term in the denominator containing $D_{22}{}^{\omega}$, see the comments at the end of this appendix). Therefore, the factors outside the square brackets, which are slowly varying functions of $\mathbf{p}$ and ϵ, may be evaluated at $p=p_F$, $\epsilon=0$ and simply act to renormalize the "bare vertices" ξ:

$$\xi(1+\lambda g_f\Gamma^{\omega})=(1+\Gamma^{\omega}g_f\lambda)\xi=R_{\xi}\xi\,, \quad \text{(A16)}$$

where R_{ξ} is a diagonal matrix in i-j space. Inspection of the symmetry properties of the quantities $\Gamma_{ij}{}^{\omega}$ (see Ref. 17) then shows that $R_{11}=R_{33}=$ real, and explicit evaluation of (A15) in the limit $\omega\gg T$, $T\to 0$ and comparison with the well-known results of zero-temperature theory[15,16] then leads to the conclusion that R_{ξ} is just the unit matrix multiplied by the analogous renormalization factor of zero-temperature formalism:

$$R_{ii}\xi=R_{\xi}\xi=(1+(GG)^{\omega}\Gamma^{\omega})\xi\,, \quad \text{(A17)}$$

in the usual notation.[15] The comparison also leads to the conclusion:

$$\Phi_{\xi}=\mathrm{Tr}\{\xi(GG)^{\omega}(1+\Gamma^{\omega}(GG)^{\omega})\xi\}\,. \quad \text{(A18)}$$

[In (A17) and (A18) the matrix notation of course now refers only to $\mathbf{p},\sigma$ and ϵ.]

Thus our final result for the autocorrelation functions in the general case has the form

$$K_{\xi}(\mathbf{k},\omega)=\Phi_{\xi}+R_{\xi}^{2}\,\mathrm{Tr}\left\{\xi\frac{\lambda g_n}{1-Q^{\omega}g_n}\xi\right\}\,. \quad \text{(A19)}$$

The physical interpretation of the constants Φ_{ξ} and R_{ξ} is discussed in Sec. II of the text. They will be evaluated below.

In (A19) the functions $Q_{ij}{}^{\omega}$ have the form [see (A12)]

$$Q_{ij}{}^{\omega}(\epsilon,\epsilon':\omega)=\lambda_j(\epsilon',\omega)\Gamma_{ij}{}^{\omega}(\epsilon,\epsilon',\omega)+\mathfrak{D}_{ij}(\epsilon,\epsilon',\omega)\,, \quad \text{(A20)}$$

where $\mathfrak{D}_{ij}(\epsilon,\epsilon',\omega)$ is a pure imaginary function and $\mathfrak{D}_{i2}\equiv\mathfrak{D}_{2i}\equiv 0$. Since the integral containing Q^{ω} is taken only close to the Fermi surface, $\mathfrak{D}_{ij}$ is everywhere small and all the analytic continuations may be taken as the same real function $\Gamma^{\omega}(\mathbf{n}\cdot\mathbf{n}',\sigma,\sigma')$ defined above:

$$\Gamma_{ij}{}^{\omega}(\mathbf{pp}',\sigma\sigma',\epsilon\epsilon':\mathbf{k},\omega)\approx\Gamma^{\omega}(\mathbf{n}\cdot\mathbf{n}',\sigma,\sigma')\,.$$

From (A20) and (A3) it is clear that we may at once drop all the imaginary terms $\mathfrak{D}_{ij}$ in $Q_{ij}{}^{\omega}$ except for $\mathfrak{D}_{22}$, which may be important in the limit $\omega\to 0$ (since then $\lambda_2\to 0$ and $g_2^{-1}\sim\gamma$, which is of the same order of magnitude as $\mathfrak{D}_{22}$). Actually, it is easy to see that $\mathfrak{D}_{22}$ will be important only when we consider either the "absorption region" $\omega\sim\gamma$ or the kinetic coefficients. For all other purposes we may write

$$K_{\xi}=\Phi_{\xi}+R_{\xi}^{2}\sum_{\sigma}\int\frac{d\Omega}{8\pi}\left\{\xi\frac{\tilde{g}}{1-\Gamma^{\omega}\tilde{g}}\xi\right\}\,, \quad \text{(A21)}$$

where

$$\tilde{g}\equiv\frac{\nu(0)}{4\pi i}\int d\epsilon_{\mathrm{p}}\int d\epsilon\sum_{i=1}^{3}\lambda_i(\epsilon,\omega)g_i(\mathbf{p},\epsilon:\mathbf{k},\omega)\,,$$

with $\lambda_i(\epsilon,\omega)$ given by (A3) and $\nu(0)$ the density of states at the Fermi surface. For consistency we must of course take the limit $\gamma\to 0$ when calculating $\tilde{g}$. Then we easily get

$$\tilde{g}=a^2\nu(0)\,\mathbf{v}\cdot\mathbf{k}/[\omega-\mathbf{v}\cdot\mathbf{k}] \quad \text{(A22)}$$

and (A21) agrees with the usual zero-temperature results provided that we identify $\Gamma^{\omega}[\equiv\Gamma^{\omega}(\mathbf{n}\cdot\mathbf{n}',\sigma,\sigma')]$ with the quantity defined by Landau.[2]

It remains to sketch the derivation of the constants Φ_{ξ} and R_{ξ} for the cases of physical interest [$\xi(\mathbf{p},\sigma)=1,\sigma,\mathbf{p},(\mathbf{p}\sigma)$]. First, using (A18) it is easy to show[15] that for any "conserved" quantity $\xi(\mathbf{p},\sigma)$ (i.e., any quantity such that $\sum_{\mathbf{p}\sigma}\xi(\mathbf{p},\sigma)a_{\mathbf{p}\sigma}{}^{\dagger}a_{\mathbf{p}\sigma}$ commutes with the Hamiltonian) we have simply

$$\Phi_{\xi}=0\,,\quad R_{\xi}=a^{-1}\,. \quad \text{(A23)}$$

We shall assume that the total particle number and spin are conserved, but not necessarily the total momentum. Let us consider the autocorrelation function $K_p(\mathbf{k},\omega)$ corresponding to a given Cartesian component of current [$\xi(\mathbf{p},\sigma)\equiv p_{\mu}/m$]. To derive the corresponding constants Φ_p, R_p we notice that it follows from the continuity equation that for $\mathbf{p}$ parallel to $\mathbf{k}$ $K_p(\mathbf{k},\omega)$ is related to the density autocorrelation function $K_1(\mathbf{k},\omega)$ in the limit $\omega\gg kv$ by the relation

$$K_p(\mathbf{k},\omega)-K_p(\mathbf{k},0)=-\omega^2k^{-2}K_1(\mathbf{k},\omega)\,.$$

Therefore, evaluating K_p and K_1 according to (A21) and (A22), and using (A23) for K_1, we find

$$R_p=a^{-1}(m/m^*)[1+(F_1/3)]\,, \quad \text{(A24)}$$

in the standard notation [F_1 is the first harmonic of the

spin-independent part of the function $F(\mathbf{n}\cdot\mathbf{n}',\sigma\sigma')$ $\equiv a^2\nu(0)\Gamma^{\omega}(\mathbf{n}\cdot\mathbf{n}',\sigma\sigma')$ and $m^*\equiv p_F|\nabla_{\mathbf{p}}\epsilon_{\mathbf{p}}|^{-1}_{p=p_F}$]. Finally, using the longitudinal sum rule

$$K_p(\mathbf{k},0)=-N/m,$$

and Eqs. (A21), (A22), and (A24), we get

$$\Phi_p=(N/m)\{(m/m^*)[1+(F_1/3)]-1\}. \quad \text{(A25)}$$

[In the case of translational invariance, combination of (A25) and (A23) of course gives the well-known Landau effective-mass relation.[1]]

The spin-current renormalization constants [$\xi(\mathbf{p},\sigma)=p_\mu\sigma/m$] can be obtained in an entirely analogous way, since a corresponding continuity equation and sum rule exist in this case:

$$R_{p\sigma}=a^{-1}(m/m^*)[1+(Z_1/12)],$$

$$\Phi_{p\sigma}=(N/m)\{(m/m^*)[1+(Z_1/12)]-1\}, \quad \text{(A26)}$$

where Z_1 is the first harmonic of the coefficient of $\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'$ in $F(\mathbf{n}\cdot\mathbf{n}',\sigma\sigma')$.

To conclude, let us just illustrate the physical significance of the quantity $\mathfrak{D}_{22}$. (For more details, though in a rather different notation, see the work of Eliashberg.[17]) For definiteness consider the spin diffusion coefficient $D(T)$, which is proportional to

$$\lim_{\mathbf{k}\to 0}\lim_{\omega\to 0}(i\omega)^{-1}[K_{p\sigma}(\mathbf{k},\omega)-K_{p\sigma}(\mathbf{k},0)], \quad \text{(A27)}$$

where $K_{p\sigma}$ is the spin-current autocorrelation function. It is easy to show by means of a further "renormalization" of the quantities Q_{ij}^{ω} (cf. Ref. 17) that (A27) is proportional to

$$\lim_{\mathbf{k}\to 0}\lim_{\omega\to 0}(i\omega)^{-1}\,\mathrm{Tr}\left\{\mathbf{p}\sigma\frac{\lambda_2 g_2}{1-i\bar{\mathfrak{D}}g_2}\mathbf{p}\sigma\right\}$$

$$=\sum_{\mathbf{p}\sigma}\left\{\mathbf{p}\sigma(2\gamma-\bar{\mathfrak{D}})^{-1}\left(\frac{dn}{d\epsilon_{\mathbf{p}}}\right)\mathbf{p}\sigma\right\}, \quad \text{(A28)}$$

where $\bar{\mathfrak{D}}=-i(1+Z_1/12)\mathfrak{D}$, and we have used the fact that the trace of a product of several operators of which only one is nondiagonal is independent of their order. On the right-hand side of (A28) γ is the diagonal matrix $\gamma_{\mathbf{p}\sigma}$, and $\bar{\mathfrak{D}}$ is the matrix $\bar{\mathfrak{D}}_{\mathbf{p}\sigma,\mathbf{p}'\sigma'}\equiv\bar{\mathfrak{D}}(\mathbf{p}\mathbf{p}',\sigma\sigma',\epsilon_{\mathbf{p}}\epsilon_{\mathbf{p}'})$.

It is obvious from (A28) that the effect of $\bar{\mathfrak{D}}$ is to subtract from the "true" inverse lifetime (collision probability) $2\gamma_{\mathbf{p}}$ the part due to these collisions which conserve the spin current, and thereby to define a "diffusion lifetime" τ_D by the (matrix) equation

$$(2\gamma-\bar{\mathfrak{D}})^{-1}\left(\frac{dn}{d\epsilon_{\mathbf{p}}}\right)\mathbf{p}\sigma=\tau_D\frac{dn}{d\epsilon_{\mathbf{p}}}\mathbf{p}\sigma.$$

Indeed, by proceeding along these lines we recover the usual expression[48] for the spin diffusion coefficient of a normal Fermi liquid.

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[48] D. Hone, Phys. Rev. **121**, 669 (1961).