Equivalence of Different Pair-Breaking Mechanisms in Superconductors*

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Paramagnetic impurities in a superconductor and a magnetic field parallel to a thin superconducting film with short electronic mean free path are known to have similar effects on many physical quantities. By investigating the properties of vertex functions and their renormalization factors we have derived necessary and sufficient conditions for such a physical quantity to be equally affected by the two pair-breaking mechanisms. We find that different expressions are obtained in the two theories only for those quantities which involve s-wave, spin-triplet vertices with momentum transfer $q \lesssim (l\xi_0)^{-1/2}$, where l is the electron mean free path and ξ_0 is the coherence distance. With the help of those results different response functions are analyzed and it is shown that the spin susceptibility is the only one of the quantities usually considered which satisfies the above requirements and hence gives different results for the field and paramagnetic-impurity cases.

I. INTRODUCTION

SIMILARITIES in the theories of paramagnetic impurities in a superconductor as developed by Abrikosov and Gorkov[1] and of a magnetic field[2] (or a current[3]) parallel to a superconducting film with short electronic mean free path, as developed by one of the authors (K.M.), have been of considerable interest. So it has been noted that not only the single-particle excitation spectrum and hence the tunneling density of states, but also certain response functions such as electrical and thermal conductivity,[4–6] are the same in both theories.[7] The latter can be expressed in terms of a quantity ξ, which characterizes the strength of the depairing mechanism. This seemed to suggest that all response functions might have identical expressions in both theories. Our recent calculation of the spin susceptibility,[8] which is measured for example in Knight-shift experiments, showed, however, that this is not the case. Different expressions for the spin susceptibility were obtained for the magnetic-field and paramagnetic-impurity cases. This raises the question of what criteria allow one to distinguish in a simple way response functions with different expressions in both theories. The problem is solved by investigating the properties of vertex functions in the presence of a field or paramagnetic impurities.

Response functions can generally be written in terms of averages over the Gibbs' ensemble of products of four field operators.[9] For superconductors this average can be decomposed into products of two Green's functions which were originally introduced by Gorkov.[10] In the presence of impurity scattering these products have to be averaged with respect to the positions of impurity atoms and the Green's functions have to be modified accordingly. The averaging process is most easily carried out by introducing vertex functions into the theory. We take account of the effect of scattering by renormalizing both the self-energy in the Green's function and the vertex functions.

The renormalization of the self-energy, which amounts to a replacement of ω and the order parameter Δ in the Green's function by $\tilde{\omega}(\omega)$ and $\tilde{\Delta}(\omega)$, has already been discussed by Abrikosov and Gorkov[1] for the paramagnetic-impurity case and by K. Maki[2] for the magnetic-field case. The aim of this communication is to present calculations of the renormalization factors for the vertices in both theories and to analyze the above problem in terms of these factors. This will be done in the next two sections where we will first treat the magnetic-field case and subsequently the case of paramagnetic impurities. Special attention will be paid to vertices belonging to s waves. The importance of the latter is connected with the fact that their renormalization coefficients are larger by a factor $(\tau\Delta)^{-1}$ as compared with higher l-wave vertices. τ is here the mean free time of an electron and $(\tau\Delta)^{-1} \gg 1$ for the cases of interest. Furthermore, we expect in the presence of paramagnetic impurities different renormalization coefficients for spin-singlet and spin-triplet vertex functions, while we have no physical reason to expect such a difference in the case of a magnetic field.

In the last section we will discuss the results of those calculations, establish criteria for the nonequivalence of

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the magnetic-field and paramagnetic-impurity cases, and analyze various response functions in the light of those criteria.

II. VERTEX FUNCTIONS IN THE PRESENCE OF A FIELD

In the following calculations of the renormalized vertex functions we will make use of Nambu's representation of temperature Green's functions in a superconductor.[11] Observable quantities can be obtained in this formalism by analytic continuation as extensively discussed in Ref. 9. In the presence of a field and in the limit of a short mean free path the Green's function can be written as

$$G(\mathbf{p},\omega_n)=(i\tilde{\omega}_n+\mathbf{q}\cdot\mathbf{v}-\xi_p\rho_3-\tilde{\Delta}_n\sigma_2\rho_1)^{-1}$$
$$=-\frac{i(\tilde{\omega}_n-i\mathbf{q}\cdot\mathbf{v})+\xi_p\rho_3+\tilde{\Delta}_n\sigma_2\rho_1}{(\tilde{\omega}_n-i\mathbf{q}\cdot\mathbf{v})^2+\xi_p{}^2+\tilde{\Delta}_n{}^2}. \quad (1)$$

Here ρ_0, $\rho_1\cdots\rho_3$ and σ_0, $\sigma_1\cdots\sigma_3$ are the unit matrix and the Pauli spin matrices, respectively, which operate in two different spaces as explained below, $\xi_p=p^2/2m-\mu$ where μ is equal to the chemical potential and $\mathbf{v}=\mathbf{p}/m$. Furthermore, $\mathbf{q}=e\mathbf{A}$, where here and in the following we set $\hbar=c=k_B=1$. In this notation we use a four-component space in which the electron-field operators are written in the form

$$\Psi(x)=\begin{pmatrix}\psi_\uparrow(\mathbf{x})\\ \psi_\downarrow(\mathbf{x})\\ \psi_\uparrow{}^\dagger(\mathbf{x})\\ \psi_\downarrow{}^\dagger(\mathbf{x})\end{pmatrix}; \quad (2)$$
$$\Psi^\dagger(\mathbf{x})=(\psi_\uparrow{}^\dagger(\mathbf{x}),\psi_\downarrow{}^\dagger(\mathbf{x}),\psi_\uparrow(\mathbf{x}),\psi_\downarrow(\mathbf{x})).$$

Although in the case of a magnetic field it would be sufficient to work in a two-dimensional space, we have additionally introduced σ_i matrices in order to facilitate the comparison of the pair-breaking effect of the magnetic field with that of paramagnetic impurities. The products $\rho_j\sigma_i$ can then be expressed as

$$\rho_0\sigma_i=\begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix},\qquad \rho_1\sigma_i=\begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix},$$
$$\rho_2\sigma_i=\begin{pmatrix}0 & -i\sigma_i\\ i\sigma_i & 0\end{pmatrix},\quad \rho_3\sigma_i=\begin{pmatrix}\sigma_i & 0\\ 0 & -\sigma_i\end{pmatrix}. \quad (3)$$

$\tilde{\omega}_n$ and $\tilde{\Delta}_n$ were calculated in Ref. 2 and are given by

$$\tilde{\omega}_n=\omega_n+\frac{u_n}{2\tau(1+u_n{}^2)^{1/2}}+\tfrac{3}{2}\zeta\Delta\frac{u_n}{(1+u_n{}^2)^{3/2}},$$
$$\tilde{\Delta}_n=\Delta+\frac{1}{2\tau(1+u_n{}^2)^{1/2}}+\frac{\zeta}{2}\Delta\frac{(1-2u_n{}^2)}{(1+u_n{}^2)^{3/2}}, \quad (4)$$

where $u_n=\tilde{\omega}_n/\tilde{\Delta}_n$, $\omega_n=\pi T(2n+1)$ (n is an integer), and $\zeta=2\tau_{\rm tr}/3\Delta v^2e^2\langle\mathbf{A}^2(\mathbf{x})\rangle$. The brackets indicate that the spatial average of the square of the vector potential $\mathbf{A}$ in London's gauge has to be taken. τ and $\tau_{\rm tr}$ are the mean free time and transport mean free time, respectively. For simplicity we will restrict ourselves first to the case of zero momentum transfer $\mathbf{q}$. In that case a vertex can be characterized by angular-momentum quantum numbers l, m and a matrix $\sigma_i\rho_j$. The integral equation for the renormalized vertex function $(\sigma_i\rho_j)_{l,\rm ren}Y_{lm}(\theta)$ in the presence of a field is given by

$$(\sigma_i\rho_j)_{l,\rm ren}Y_{lm}(\theta)$$
$$=\sigma_i\rho_jY_{lm}(\theta)+n\int\frac{d^3p'}{(2\pi)^3}|u(\mathbf{p}\cdot\mathbf{p}')|^2\rho_3G(p',\omega_+)$$
$$\times(\sigma_i\rho_j)_{l,\rm ren}Y_{lm}(\theta')G(p',\omega_-)\rho_3. \quad (5)$$

Here n is the impurity concentration and $\omega_\pm=\omega\pm\omega_0/2$, where ω_0 is the energy transfer. The quantities $\omega_\pm$ are again half-integers multiplied by $2\pi T$. By decomposing $|u(\mathbf{p}\cdot\mathbf{p}')|^2$ in terms of spherical harmonics

$$|u(p\cdot p')|^2=\frac{(2\pi)^2}{np_0m}\sum_{l=0}^{\infty}\frac{(2l+1)}{\tau_l}P_l(\cos\theta) \quad (6)$$

with $\cos\theta=(\mathbf{p}\cdot\mathbf{p}')/p_0{}^2$ we can simplify Eq. (5) to the form

$$(\sigma_i\rho_j)_{l,\rm ren}=\sigma_i\rho_j+\frac{4\pi^2}{p_0m\tau_l}\int\frac{d^3p'}{(2\pi)^3}\rho_3G(\mathbf{p}',\omega_+)(\sigma_i\rho_j)_{l,\rm ren}$$
$$\times G(\mathbf{p}',\omega_-)\rho_3. \quad (7)$$

Here we have neglected terms of the order $(\tau\Delta)$ containing other angular momenta which come into the equation because of mixing due to $(\mathbf{v}\cdot\mathbf{A})^2$ terms. In general we have 16 combinations $\sigma_i\rho_j$. Since the integral kernel in Eq. (5) mixes $\sigma_i\rho_j$ only with $\sigma_2\sigma_i\rho_1\rho_j$, we can reduce the above equations into a set of eight independent equations of rank 2. In those equations the following matrices are coupled to each other:

$$(1,\rho_1\sigma_2),$$
$$(\rho_3,\rho_2\sigma_2),$$
$$(\sigma_1\rho_3,\rho_2\sigma_3),\quad(\sigma_2\rho_3,i\rho_2),\quad(\sigma_3\rho_3,i\rho_2\sigma_1),$$
$$(\sigma_1,i\rho_1\sigma_3),\quad(\sigma_2,\rho_1),\quad(\sigma_3,i\rho_1\sigma_1). \quad (8)$$

The first two pairs correspond to spin-singlet vertex functions while the last 2×3 pairs correspond to spin-triplet vertex functions with spin pointing into x, y, z direction.[12] Indeed it can be seen easily that, for example, 1 and σ_3 belong to a spin-singlet and spin-triplet

[11] See, for example, J. R. Schrieffer, *Theory of Superconductivity* (W. A. Benjamin, Inc., New York, 1964).

[12] We would like to remark that we deal here with general two-electron states, which means electron-electron, electron-hole as well as hole-hole states.

vertex, respectively, by writing

$$\Psi^\dagger(\mathbf{x})1\Psi(\mathbf{x}')=\psi_\uparrow^+(\mathbf{x})\psi_\uparrow(\mathbf{x}')+\psi_\downarrow^+(\mathbf{x})\psi_\downarrow(\mathbf{x}')+\psi_\uparrow(\mathbf{x})\psi_\uparrow^+(\mathbf{x}')+\psi_\downarrow(\mathbf{x})\psi_\downarrow^+(\mathbf{x}'),$$

$$\Psi^\dagger(\mathbf{x})\sigma_3\Psi(\mathbf{x}')=\psi_\uparrow^+(\mathbf{x})\psi_\uparrow(\mathbf{x}')-\psi_\downarrow^+(\mathbf{x})\psi_\downarrow(\mathbf{x}')+\psi_\uparrow(\mathbf{x})\psi_\uparrow^+(\mathbf{x}')-\psi_\downarrow(\mathbf{x})\psi_\downarrow^+(\mathbf{x}').$$

In the limit $\mathbf{x}'\to\mathbf{x}$ the first expression reduces to the electron density, and the second expression to the z component of the electronic spin. In order to write the solutions of Eq. (5) in a simple form we have to introduce the following matrices

$$\Pi_i=(1,\sigma_2\rho_1,\sigma_2\rho_2,\rho_3),\quad (i=0,1,\cdots,3).$$

Furthermore we make the ansatz

$$\begin{aligned}(\Pi_i)_{\rm ren}&=(A_i^1+iB_i^1\Pi_1)\Pi_i,\\ (\Pi_i\sigma\rho_3)_{\rm ren}&=(A_i^3+iB_i^3\Pi_1)\Pi_i\sigma\rho_3\end{aligned}\tag{9}$$

for the renormalized vertex functions $(\Pi_i)_{\rm ren}$ and $(\Pi_i\sigma\rho_3)_{\rm ren}$. The coefficients A_i^1, A_i^3, B_i^1, B_i^3 can be determined from Eqs. (5) after straightforward algebraic calculations. In stating the results we distinguish between the cases $l\neq0$ and $l=0$.

In the former case we obtain

$$\begin{aligned}\begin{pmatrix}A_0^1\\A_1^1\end{pmatrix}&=\left[1-\frac{\tau_0}{2\tau_l}(1\pm C_-)\right]D_l^{-1},\\ \begin{pmatrix}A_2^1\\A_3^1\end{pmatrix}&=\left[1-\frac{\tau_0}{2\tau_l}(1\mp C_+)\right]D_l^{-1},\\ B_0^1&=-B_1^1=\frac{i\tau_0}{2\tau_l}E_+D_l^{-1},\\ B_2^1&=B_3^1=\frac{\tau_0}{2\tau_l}E_-D_l^{-1},\end{aligned}\tag{10}$$

and

$$A_i^3=A_i^1,\quad B_i^3=B_i^1.$$

Here the following functions have been introduced:

$$\begin{aligned}D_l&=1-\frac{\tau_0}{\tau_l},\quad C_\pm=\frac{uu_+\pm1}{(u^2+1)^{1/2}(u_+^2+1)^{1/2}},\\ E_\pm&=\frac{u\pm u_+}{(u^2+1)^{1/2}(u_+^2+1)^{1/2}}.\end{aligned}\tag{11}$$

It is apparent that for $l=0$ these solutions would diverge and that a more careful analysis in this case is required. It can be shown that the denominator in the expressions for the coefficients A_i^1, A_i^3, B_i^1, B_i^3 becomes now of order $(\tau\Delta)$, thus resulting in an enhancement of the renormalization factors by a factor of the order $(\tau\Delta)^{-1}$. This leads to the result for $l=0$:

$$\begin{aligned}\begin{pmatrix}A_0^1\\A_1^1\end{pmatrix}&=(1\mp C_-)(2D_-^1)^{-1},\\ \begin{pmatrix}A_2^1\\A_3^1\end{pmatrix}&=(1\pm C_+)(2D_+^1)^{-1},\\ B_0^1&=-B_1^1=iE_+(2D_-^1)^{-1},\\ B_2^1&=B_3^1=E_-(2D_-^1)^{-1},\end{aligned}\tag{12}$$

and

$$A_i^3=A_i^1,\quad B_i^3=B_i^1.$$

The quantities $D_\pm^1$ are defined as

$$D_\pm^1=\tau\Delta[(1+u^2)^{1/2}+(1+u_+^2)^{1/2}-\zeta(1-C_\pm)].\tag{13}$$

Thus our calculations show that in the case of a magnetic field the renormalization factors are the same for spin-singlet and spin-triplet vertex. This behavior is physically to be expected in the present case but it will turn out not to hold in the paramagnetic-impurity case.

We want to discuss now the case of nonzero momentum transfer ($\mathbf{q}\neq0$). In this case a vertex can no longer be specified by a quantum number l since the $\mathbf{q}$-dependent integral kernel in Eq. (5) is no longer invariant under rotations in momentum space. Nevertheless, it still is permissible to consider the angular momentum as a good quantum number as long as $(lq)\ll1$ where $l=v_F\tau$. In that case terms of the form $(\tau\Delta)^2\zeta(lq)^2$ can also be dropped. A calculation similar to the one performed for the $\mathbf{q}=0$ limit reveals that the only change occurs in the expression for the determinant $D_l(\mathbf{q})$ which is now of the form

$$D_l(\mathbf{q})=1-\tau_0 n\int\frac{d\Omega}{(2\pi)^3}\frac{|u|^2|Y_{lm}|^2}{1-i(lq)z}\quad\text{for}\quad l=0,\tag{14}$$

where $z=\cos(\mathbf{q}\cdot\mathbf{p}/|q||p_0|)$ and

$$D_{l=0,\pm}(\mathbf{q})=1-[1/(lq)]\arctan(lq)+D_\pm^1(0)\quad\text{for}\quad l=0.\tag{15}$$

For the definition of $D_\pm^1(0)$ see Eq. (13). We see that for $l\neq0$ vertices under the assumption $(lq)\ll1$ the dependence of the renormalization factor on momentum transfer $\mathbf{q}$ can be disregarded. Inspection of Eq. (15) shows, however, that for $l=0$ vertices the $\mathbf{q}$-dependent term gives an important contribution and becomes dominant as soon as q becomes larger than $(l\xi_0)^{-1/2}$, where ξ_0 is the coherence length. In the limit $ql\gg1$ the determinant $D_{l=0,\pm}(\mathbf{q})$ approaches unity.

III. VERTEX FUNCTIONS IN THE PRESENCE OF PARAMAGNETIC IMPURITIES

In the presence of paramagnetic impurities one has to consider the possibility of electron spin flip due to impurity scattering which may result in different re-

normalization factors for spin-singlet and spin-triplet vertices.

The Green's function is again given by Eq. (1) with $\mathbf{q}=0$, where, according to Abrikosov and Gorkov, the following relations for $\tilde{\omega}_n$, $\tilde{\Delta}_n$ now hold:

$$\tilde{\omega}_n=\omega_n+\frac{1}{2}\left(\frac{1}{\tau}+\frac{1}{\tau_s}\right)\frac{u_n}{(1+u_n^2)^{1/2}},$$
$$\tilde{\Delta}_n=\Delta+\frac{1}{2}\left(\frac{1}{\tau}-\frac{1}{\tau_s}\right)\frac{1}{(1+u_n^2)^{1/2}}. \tag{16}$$

The quantity $\zeta=1/\tau_S\Delta$ is a measure of the strength of the depairing mechanism. τ_S can be expressed in terms of the spin-flip part of the impurity scattering potential u' as

$$\frac{1}{\tau_S}=\frac{nmp_0}{4\pi}S(S+1)\int\frac{d\Omega}{4\pi}|u'(\mathbf{p}\cdot\mathbf{p}')|^2, \tag{17}$$

where S is the impurity spin. The equation for the renormalization vertex function is now of the form

$$\begin{aligned}(\sigma_i\rho_j)_{l,\mathrm{ren}}Y_{lm}(\theta)&=\sigma_i\rho_jY_{lm}(\theta)+n\int\frac{d^3p'}{(2\pi)^3}|u(\mathbf{p}\cdot\mathbf{p}')|^2\rho_3G(p',\omega_+)\\&\times(\sigma_i\rho_j)_{l,\mathrm{ren}}Y_{lm}(\theta')G(\mathbf{p}',\omega_-)\rho_3+\tfrac{1}{4}S(S+1)n\\&\times\int\frac{d^3p'}{(2\pi)^3}|u'(\mathbf{p}\cdot\mathbf{p}')|^2\\&\times\boldsymbol{\alpha}G(\mathbf{p}',\omega_+)(\sigma_i\rho_j)_{l,\mathrm{ren}}Y_{lm}(\theta')G(\mathbf{p}',\omega_-)\boldsymbol{\alpha}.\end{aligned} \tag{18}$$

Here α is defined as $\boldsymbol{\alpha}=(1+\rho_3)/2\boldsymbol{\sigma}+(1-\rho_3)/2\sigma_2\boldsymbol{\sigma}\sigma_2$. It can be checked that the last term on the right side of Eq. (18) does not change the property of the integral equation to mix $\sigma_i\rho_j$ only with $\sigma_2\sigma_i\rho_1\rho_j$. We therefore obtain again a set of eight independent equations of rank 2. It is therefore advantageous to introduce the quantities Π_i as defined in the last section and to make the ansatz (9) for the renormalized vertices. We again distinguish between the cases $l\neq0$ and $l=0$, and again state only the results of the straightforward calculations. For $l\neq0$ the coefficients A_i^1, A_i^3, B_i^1, B_i^3 are given by identically the same expressions as in the magnetic-field case [see Eq. (10)], implying that for $l\neq0$ the renormalization factors are the same for spin-singlet and spin-triplet vector as in the case of paramagnetic impurities. For $l=0$ the coefficients A_i^1, B_i^1 are of the same form as in the magnetic-field case [see Eq. (13)]. The coefficients A_i^3, B_i^3 on the other hand, turn out to be different. They can be obtained from A_i^1, B_i^1 simply by replacing $D_\pm^1$ by $D_\pm^3$, where

$$D_\pm^3=\tau\Delta[(1+u^2)^{1/2}+(1+u_+^2)^{1/2}-\zeta(1+\tfrac{1}{3}C_\pm)]. \tag{19}$$

Thus in the case of paramagnetic impurities the renormalization factor differs for $l=0$ for spin-singlet and spin-triplet vertex.

For nonzero momentum transfer ($\mathbf{q}\neq0$) a similar consideration as done in the last section holds, and again only s-wave vertex functions are affected and then only if q is of the order of or larger than $(l\xi_0)^{-1/2}$. In that case $D_\pm^1$ and $D_\pm^3$ have to be replaced by

$$\begin{pmatrix}D_\pm^1(q)\\D_\pm^3(q)\end{pmatrix}=1-(ql)^{-1}\arctan ql+\begin{pmatrix}D_\pm^1(q=0)\\D_\pm^3(q=0)\end{pmatrix}, \tag{20}$$

where $D_\pm^1(q=0)$ and $D_\pm^3(q=0)$ were defined in Eqs. (13) and (19), respectively.

IV. DISCUSSION

In the last two sections we have calculated the renormalized vertices belonging to different angular momenta and spin states, for both the magnetic-field and paramagnetic-impurity case. It was found that for zero momentum transfer $\mathbf{q}$ the renormalization factors for $l=0$ vertices were enhanced by a factor $(\tau\Delta)^{-1}$ as compared with those for $l\neq0$ vertices. Furthermore we found that in the presence of paramagnetic impurities the renormalization factors were different for spin-singlet and spin-triplet s-wave vertices in contrast to the magnetic-field case for which no such difference did occur. The results were shown to hold also in the $\mathbf{q}\neq0$ case as long as $q\lesssim(l\xi_0)^{-1/2}$. For larger q values the renormalization factor of a spin-singlet approaches that of a spin-triplet vertex. The last result can be derived under the assumption that the kernel in the integral equation for the renormalized vertex function does not mix different angular-momentum components, which holds for the q region of interest. This enables us to formulate as follows the necessary and sufficient conditions for distinguishing physical observables which behave differently in magnetic-field and paramagnetic-impurity cases.

Theorem: A physical quantity, if expressed in vertex functions, must have at least one component belonging to an s-wave spin-triplet vertex with momentum transfer smaller than $q\lesssim(l\xi_0)^{-1/2}$ in order to give different expressions for magnetic-field and paramagnetic-impurity cases.

With the help of this theorem we want to discuss various transport coefficients.

A. Electrical and Thermal Conductivity (p-Wave Vertices)

Both electrical and thermal conductivity are examples for p-wave vertices which give identical results for both pair-breaking mechanisms. The p-wave character of the vertices can be seen by writing down expressions for $\sigma_{ij}(\mathbf{p},\omega_0)$ and $\kappa_{ij}(\mathbf{p},\omega_0)$ in terms of Green's functions from which the physically observable electrical and thermal conductivity can be obtained by

analytical continuation to imaginary ω_0 values:

$$\sigma_{ij}(\mathbf{q},\omega_0)=\left(\frac{e}{m}\right)^2 T\sum_n \int \frac{d^3p}{(2\pi)^3} \times \mathrm{Tr}\{p_i G(\mathbf{p}_+,\omega_+)(\rho_0)_{1,\mathrm{ren}} p_j G(\mathbf{p}_-,\omega_-)\}\,,$$
$$\kappa_{ij}(\mathbf{q},\omega_0)=\frac{1}{m^2\omega_0}\sum_n \int \frac{d^3p}{(2\pi)^3}\omega_+\omega_- \times \mathrm{Tr}\{p_i\rho_3 G(p_+,\omega_+)(\rho_3)_{1,\mathrm{ren}} p_j G(\mathbf{p}_-,\omega_-)\}\,, \quad (21)$$

where $\mathbf{p}_\pm=\mathbf{p}\pm\mathbf{q}/2$, $\omega_+=\omega_n=\pi T(2n+1)$, $\omega_-=\omega_n-\omega_0$, and $\omega_0=2\pi T n_0$. Since the vector $\mathbf{p}$ transforms like a p wave under space rotations, only p-wave vertices appear in the above expressions. With the help of the expression derived for $(\rho_3)_{1,\mathrm{ren}}$ it is not difficult to verify the results obtained in Refs. 4, 5, and 6.

B. Ultrasonic Attenuation (s- and d-Wave Vertices)

The ultrasonic-attenuation coefficients have been studied in detail by Kadanoff and Falko.[13] Using their analysis one can show that the transverse attenuation coefficient involves s and d waves. Since we deal in both cases with spin-singlet vertices only, there will be no distinction between the magnetic-field and the paramagnetic-impurity cases. In order to understand how s- and d-wave vertices enter into the problem we note that the attenuation coefficients can be written in terms of correlations between the electronic stress tensor (τ_{ij} in the notation of Ref. 13) and the electron density. Furthermore, we note that the trace of the stress tensor τ_{ii} and the density transform like an s wave, while the trace-free part of the stress tensor $\tau_{ij}-\frac{1}{3}\tau_{ll}\delta_{ij}$ transforms like a d wave. While the transverse attenuation coefficient can be expressed in terms of the tracefree part of the stress tensor only, the longitudinal attenuation coefficient contains, in addition, the trace of the stress tensor as well as the electron density. Thus the above statement follows. The reason for the appearance of spin-singlet vertices only, lies in the fact that there are no electron-spin interactions in ultrasonic-attenuation processes. Using our Eqs. (10)–(13) we obtain a slightly different expression for the attenuation as compared with Ref. 13. Our results have the same form as in Ref. 13 except for a replacement of the quantity X as defined in Eq. (56) of Ref. 13 by

$$X=lq[1+\tau\Delta((1+u^2)^{1/2}+(1+u_+^2)^{1/2}-\zeta(1-C_+))]. \quad (22)$$

The correction is of importance for $q<(l\xi_0)^{-1/2}$.

C. Spin Susceptibility (s-Wave, Spin-Triplet Vertex)

Detailed investigations of the spin susceptibility in the presence of a field or paramagnetic impurities were given in Refs. 8, 14, and 15, respectively. Since we deal in this case with an s-wave spin-triplet vertex the expressions for χ_S/χ_n differ in the two theories. The spin-triplet character of the vertex can be seen by expressing the susceptibility in terms of Green's functions as

$$\chi_S(\mathbf{q},\omega_0)=\frac{\mu_B^2}{4}T\sum_n \int \frac{d^3p}{(2\pi)^3} \times \mathrm{Tr}\{\sigma_3\rho_3 G(\mathbf{p}_+,\omega_+)(\sigma_3\rho_3)_{\mathrm{ren}} G(\mathbf{p}_-,\omega_-)\}\,. \quad (23)$$

Again the physically observable susceptibility has to be obtained by analytic continuation to imaginary ω_0 values. Although a general expression for $\chi_S(\mathbf{q},\omega_0)$ can be written down explicitly we want to restrict ourselves for simplicity to the special case $\chi_S(q,0)$. In that case we obtain

$$\frac{\chi_S(\mathbf{q},0)}{\chi_n(0)}=1-(ql)^{-1}\arctan ql\frac{\pi T}{\Delta}\sum_n (u_n^2+1)^{-1} \times\left[(u_n^2+1)\left(1-\frac{\zeta(1+2u_n^2)}{3(1+u_n^2)^{3/2}}\right)+\frac{1}{2\tau\Delta}(1-(ql)^{-1}\arctan ql)\right]^{-1} \quad (24)$$

for the susceptibility in the presence of paramagnetic impurities. $\chi_n(0)$ denotes in both cases the susceptibility in the normal state. Thus we have an explicit example of a physical quantity which leads to different expressions for the magnetic-field and paramagnetic-impurity cases.

D. Nuclear-Spin Relaxation (s-Wave, Spin-Triplet, Large-Momentum-Transfer Vertex)

In order to be able to express the ratio R_S/R_n between the nuclear-spin-relaxation rate in the superconducting and normal state we introduce first the frequency-dependent function $R_S(\omega_0)/R_n$ defined by

$$\frac{R_S(\omega_0)}{R_n}=\frac{T^2}{2N(0)^2\omega_0}\sum_n \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3} \times \mathrm{Tr}\{\boldsymbol{\sigma}\rho_3 G(\mathbf{p},\omega_+)(\boldsymbol{\sigma}\rho_3)_{\mathrm{ren}} G(\mathbf{p}',\omega_-)\}\,, \quad (25)$$

where $N(0)=mp_0/2\pi^2$. If we then continue analytically $R_S(\omega_0)$ to imaginary $\omega_0=i\omega$ values, R_S/R_n can be written as

$$R_S/R_n=\lim_{\omega\to 0}\mathrm{Im}R_S(\omega)/R_n\,, \quad (26)$$

where Im means that the imaginary part has to be taken. In the integral for $R_S(\omega_0)/R_n$ the main contribu-

[13] L. P. Kadanoff and I. I. Falko, Phys. Rev. **136**, A1170 (1964).

[14] L. P. Gorkov and A. I. Rusinov, Zh. Eksperim. i Teor. Fiz. **46**, 1363 (1964) [English transl.: Soviet Phys.—JETP **19**, 922 (1964)].

[15] A. I. Larkin, Zh. Eksperim. i Teor. Fiz. **48**, 232 (1965) [English transl.: Soviet Phys.—JETP **21**, 153 (1965)].

tions come from $|\mathbf{p}|$, $|\mathbf{p}'|$ values close to p_0, thus implying large momentum transfer. It was shown before that the determinants $D_-^1(\mathbf{q})$ and $D_-^3(\mathbf{q})$ reduce to unity in that case. Hence we can replace $(\sigma\rho_3)_{\text{ren}}$ in Eq. (25) by $(\sigma\rho_3)$ and the depairing effect of the magnetic field or the impurities enters only in the coherence factors. We obtain, therefore, results equivalent to those derived by Ambegaokar and Griffin[16] with a "golden rule" calculation.

The above analysis shows that the spin susceptibility is the only physical quantity for which the pair-breaking effect of a magnetic field and paramagnetic impurities lead to different results. Other physical observables, which were not considered here, such as tunneling currents, give identical results in both cases since no renormalized vertices enter into their expressions. Finally we want to remark that in the presence of both a magnetic field and paramagnetic impurities, both pair-breaking effects can be added up in the quantity ζ, except for the spin susceptibility, in which case the essential determinant D_-^3 is replaced by $D_-^3=\tau\Delta[(u^2+1)+(u_+^2+1)-(\zeta_1+\zeta_2-(\zeta_1-\zeta_2/3)C_-)]$ where ζ_1 and ζ_2 are the parameters for the pair-breaking effects of the field and the impurities, respectively.

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APPENDIX

Let us consider vertex functions with zero momentum transfer in the presence of a magnetic field. With the ansatz Eq. (9) we obtain from Eq. (7) the following set of equations for $(\Pi_0)_{\text{ren}}$, for instance:

$$\left\{1-\frac{1}{2\tau_i\Omega(\omega)}\left[1-\frac{uu_+-1}{(1+u^2)^{1/2}(1+u_+^2)^{1/2}}\right]+\frac{\tau\Delta}{2\tau_i\Omega(\omega)}\zeta K_1(u,u_+)\right\}A_0^1$$
$$+\frac{1}{2\tau_i\Omega(\omega)}\left[\frac{u+u_+}{(1+u^2)^{1/2}(1+u_+^2)^{1/2}}-\tau\Delta\zeta K_3(u,u_+)\right]B_0^1=1;$$
$$-\frac{i}{2\tau_i\Omega(\omega)}\left[\frac{u+u_+}{(1+u^2)^{1/2}(1+u_+^2)^{1/2}}-\tau\Delta\zeta K_3(u,u_+)\right]A_0^1 \tag{A1}$$
$$+\left\{1-\frac{1}{2\tau_i\Omega(\omega)}\left[1+\frac{uu_+-1}{(1+u^2)^{1/2}(1+u_+^2)^{1/2}}\right]+\frac{\tau\Delta}{2\tau_i\Omega(\omega)}\zeta K_2(u,u_+)\right\}B_0^1=0.$$

Here we used the following abbreviations

$$\Omega(\omega)=(\tilde{\omega}^2+\tilde{\Delta}^2)^{1/2}+(\tilde{\omega}_+^2+\tilde{\Delta}_+^2)^{1/2},$$
$$K_1=\frac{1}{8}\left[\frac{10(uu_+-1)}{(1+u^2)^{1/2}(1+u_+^2)^{1/2}}\left(\frac{1}{1+u^2}+\frac{1}{1+u_+^2}+\frac{1}{(1+u^2)^{1/2}(1+u_+^2)^{1/2}}\right)\right.$$
$$\left.+2\left(\frac{1}{1+u^2}+\frac{1}{1+u_+^2}\right)+\frac{14}{(1+u^2)^{1/2}(1+u_+^2)^{1/2}}\right],$$
$$K_2=\frac{1}{8}\left[\frac{10(uu_+-1)}{(1+u^2)^{1/2}(1+u_+^2)^{1/2}}\left(\frac{1}{1+u^2}+\frac{1}{1+u_+^2}+\frac{1}{(1+u^2)^{1/2}(1+u_+^2)^{1/2}}\right)\right. \tag{A2}$$
$$\left.+6\left(\frac{1}{1+u^2}+\frac{1}{1+u_+^2}\right)+\frac{14}{(1+u^2)^{1/2}(1+u_+^2)^{1/2}}-4-\frac{4uu_+}{(1+u^2)^{1/2}(1+u_+^2)^{1/2}}\right],$$
$$K_3=\frac{1}{8}\left[\frac{u+u_+}{(1+u^2)^{1/2}(1+u_+^2)^{1/2}}\left(\frac{10uu_+}{(1+u^2)^{1/2}(1+u_+^2)^{1/2}}-10\left(\frac{1}{1+u^2}+\frac{1}{1+u_+^2}\right)+14\right)\right.$$
$$\left.-6\left(\frac{1}{(1+u^2)^{1/2}}+\frac{1}{(1+u_+^2)^{1/2}}\right)\left(\frac{u}{(1+u^2)^{1/2}}+\frac{u_+}{(1+u_+^2)^{1/2}}\right)\right].$$

[16] A. Griffin and V. Ambegaokar, *Proceedings of the Ninth International Conference on Low Temperature Physics, Columbus, Ohio, 1964* (Plenum Press Inc., New York, 1965).

The above equations are solved and we find:

$$A_0{}^1=[1-(\tau_0/2\tau_l)(1+C_-)]D_l^{-1},$$
$$B_0{}^1=i(\tau_0/2\tau_l)E_+D_l^{-1}, \quad \text{for} \quad l\neq 0, \tag{A3}$$

and

$$A_0{}^1=(1-C_-)(2D_-{}^1)^{-1},$$
$$B_0{}^1=iE_+(2D_-{}^1)^{-1}, \quad \text{for} \quad l=0, \tag{A4}$$

where

$$D_l=1-(\tau_0/\tau_l)+0(\tau\Delta), \tag{A5}$$

and

$$D_-{}^1=\tau\Delta[(1+u^2)^{1/2}+(1+u_+{}^2)^{1/2}-\zeta(1-C_-)]. \tag{A6}$$

From the above expressions we see that the term proportional to $\tau\Delta\zeta$ in Eq. (A1) gives negligible correction (of the order of $\tau\Delta$) for the case of $l\neq 0$ while it gives an important contribution in the case of $l=0$. We obtain almost similar equations for the other amplitudes $\{A_i{}^1, B_i{}^1\}$ $(i\neq 0)$.

Now let us turn to the case of nonvanishing q. In this case we obtain equations similar to those in (A1), which can be obtained by simply replacing $\Omega(\omega)^{-1}$ by $\Omega(\omega)^{-1}(lq)^{-1}\arctan(ql)$ for the s-wave case. In a higher wave vertex such a momentum dependence is negligible as long as $ql\ll 1$.

In the case of paramagnetic impurities the equations involved are much more simple and we do not feel it is necessary to present them here.

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Linewidth of the Electron Paramagnetic Resonance of $(Al_2O_3)_{1-x}(Cr_2O_3)_x$†

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The linewidth variation of electron-paramagnetic-resonance (EPR) absorption in single crystals of $(Al_2O_3)_{1-x}(Cr_2O_3)_x$ with x equal to 0.01, 0.001, or 0.0001% has been investigated as a function of the angle between the C axis of the samples and the magnetic field. The observed width variation is practically independent of the values of x and of the choice of the axis of rotation perpendicular to both the C axis and the magnetic field. In order to account for the observed width variation, three alternative models have been considered: a mosaic model, a strain model, and a hybrid of the two models. The calculated width variation based on each of the three models is in qualitative agreement with the observed one, the degree of agreement being the best for the strain model. Quantitatively, however, significant discrepancies between the observed and calculated variations have been found for some regions of the angle of rotation. Some discussions of the discrepancies are presented.

I. INTRODUCTION

RECENT electron-paramagnetic-resonance (EPR) absorption studies[1,2] of the Cr^{3+} ion in single crystals of $(Al_2O_3)_{1-x}(Cr_2O_3)_x$ with x close to 0.03% have indicated that the width of the three EPR lines corresponding to the $(\frac{1}{2}\rightarrow -\frac{1}{2})$ and $(\frac{3}{2}\rightarrow \frac{1}{2})$ transitions is approximately 12 G, when the external magnetic field is parallel to the crystallographic (111) or C axis of the samples. For such small values of x, the spin-spin interactions among the Cr ions contribute insignificantly to the linewidth. For example, when x is taken to be 0.001%, the linewidth of the $(\frac{3}{2}\rightarrow\frac{1}{2})$ transition, if broadened solely by the spin-spin interaction, would be not larger than 0.5 G.[1] Under the circumstances, the hyperfine interactions between the electronic spin of each Cr ion and its neighboring Al^{27} nuclei of spin $\frac{5}{2}$ and of 100% abundance have to be considered as being responsible for the linewidth. According to Laurance *et al.*,[2] the hyperfine broadening mechanism would lead to a linewidth of 9.7 G. This is approximately 81% of the observed width of 12 G. Thus the hyperfine broadening mechanism appears to be the major, but not the sole, source of the linewidth for the case of the magnetic field being parallel to the C axis.

The linewidth is expected to vary depending upon the angle θ between the C axis and the magnetic field if the dipolar part of the hyperfine interactions is appreciable. The expected anisotropy of the linewidth may be calculated in terms of the hyperfine coupling constants measured by Laurance *et al.*[2] The hyperfine broadening mechanism then would predict a variation of the linewidth between 9.4 and 9.9 G [see Eq. (B 14) of Ref. 1], when the samples are rotated over the 90° range of θ about an axis which is perpendicular to the C axis. In the following, the axis of rotation is referred to as the a axis.

However, the experimental results obtained from the sample with x less than 0.01% investigated in the present work indicate that the linewidth varies by approximately a factor of 5 over the 90° range of θ,

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[2] N. Laurance, E. C. McIrvine, and J. Lambe, J. Phys. Chem. Solids **23**, 515 (1962).