

Bloch Electrons in External Electric and Magnetic Fields

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The Schrödinger equation for a Bloch electron in external electric and magnetic fields is rewritten with the help of redundant variables, the wave function being obtained when a limiting process is carried out. The connection with the standard effective-mass approximation is discussed, and an application to the case of crossed electric and magnetic fields is carried out for the simple case of two parabolic bands in order to examine the singularity $\frac{1}{2}m^*(E/B)^2$ that appears in the energy in the effective-mass approximation when $B \rightarrow 0+$. We conclude that below a certain value of E/B the character of the solution changes and the effective-mass formula is then not applicable.

I. INTRODUCTION

THE behavior of a Bloch electron in the presence of externally applied fields has been examined extensively in the last few years. Two approaches have been used in most of the published work. Luttinger and Kohn¹ introduced a suitable set of basis functions and expanded the perturbed solution in terms of these functions, so the problem reduced to the computation of the expansion coefficients. However, in a series of papers, Wannier² replaced the original Schrödinger equation for the problem, a partial differential equation in three variables after the usual assumptions are made, by another partial differential equation of higher dimensionality whose solutions and eigenvalues bear a definite relation to the solutions and eigenvalues of the original problem. We employ Wannier's approach, but the equation we obtain differs in several respects from the equation proposed by him.

When an electron is acted on solely by the periodic potential $V(r)$ we can find three translation operators that commute with each other and with the Hamiltonian, and which provide a convenient way of cataloging the wave functions. Most important, these translation operators disentangle that part of the solution which has the lattice periodicity from the part that does not have this periodicity. In this simple case the wave functions are the well-known Bloch functions

$$\psi_B(\mathbf{r}) = \exp(i\mathbf{k} \cdot \mathbf{r})u_n(\mathbf{r}, \mathbf{k})$$

with $u_n(\mathbf{r}, \mathbf{k})$ satisfying the periodicity requirement

$$u_n(\mathbf{r} + \mathbf{R}, \mathbf{k}) = u_n(\mathbf{r}, \mathbf{k}),$$

where \mathbf{R} is any lattice vector, \mathbf{k} labels the eigenvalues of the three translation operators and n labels the different bands.

When we add constant and homogeneous electric and magnetic fields to the periodic potential, the above procedure is no longer valid. Questions then arise as to what is the meaning of a band in this case and how to catalogue the wave functions. It is not clear *a priori* whether part of the wave function shows

the lattice periodicity or not. This is most clearly seen in the Bloch functions $\psi(\mathbf{r}_1, \mathbf{r}_2)$ where the periodicity is manifested as $\psi(\mathbf{r}_1, \mathbf{r}_2 + \mathbf{R}) = \psi(\mathbf{r}_1, \mathbf{r}_2)$, the Bloch function being reproduced when we set $\mathbf{r}_1 = \mathbf{r}_2$, so that $\psi_B(\mathbf{r}) = \psi(\mathbf{r}_1 = \mathbf{r}, \mathbf{r}_2 = \mathbf{r})$. We will show, by explicitly constructing the proper functions, that a partial periodicity is still present in this case, although we can no longer expect the simple splitting that occurs in $\psi_B(\mathbf{r})$. Our procedure will lead us naturally to a function $b(\mathbf{r}, \mathbf{k})$ that contains the variable \mathbf{r}_2 explicitly and is the Fourier transform of $\psi(\mathbf{r}_1, \mathbf{r}_2)$ with respect to \mathbf{r}_1 , with the understanding that now \mathbf{k} is a variable in its own right and in general cannot be identified with the variable k that appears in $\psi_B(\mathbf{r})$. Once we have obtained the equation that defines $b(\mathbf{r}, \mathbf{k})$, whose eigenvalues correspond to the ones in the original problem, the most natural approach is to expand the solutions in terms of the periodic part of the unperturbed Bloch functions. Since we are starting from an exact equation, we are in a better position to appreciate the range of validity of the approximations involved. From this, the effective-mass approximation as well as less common, but sometimes better, approximations can be obtained readily.

In Sec. II we develop the general formalism. In Sec. III we consider the problem of crossed electric and magnetic fields and apply the previous results to an idealized two-band model in order to study the behavior of the apparent singularity $\frac{1}{2}m^*(E/B)^2$ that appears in the energy when the effective-mass formula is applied to nondegenerate bands.

II. MODIFIED WAVE EQUATION

The Hamiltonian for a Dirac particle of mass m^* and charge q (the charge of the electron is taken to be $q = -e$) in the low-energy limit is given by³ (correct to order ϵ/mc^2 , ϵ being the energy).

$$H = \frac{1}{2m} \left(\mathbf{p} - q\mathbf{A} - \frac{q\hbar}{4mc^2} \boldsymbol{\sigma} \times \mathbf{E} \right)^2 + q\Phi(\mathbf{r}) - \frac{q\hbar}{4m} g_0 \boldsymbol{\sigma} \cdot \mathbf{B} + \frac{q\hbar^2}{8m^2c^2} \nabla \cdot \mathbf{E},$$

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¹ J. M. Luttinger and W. Kohn, Phys. Rev. **97**, 869 (1955).

² G. H. Wannier, Rev. Mod. Phys. **34**, 645 (1962); other references are listed here.

³ B. Kursunoglu, *Modern Quantum Theory* (W. H. Freeman and Company, San Francisco, 1962), p. 305.

where σ are the Pauli matrices, $\mathbf{A}(\mathbf{r})$ is the vector potential, $\Phi(\mathbf{r})$ is the scalar potential, $\mathbf{E} = -\nabla\Phi$ is the time-constant electric field, $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic field, g_0 is the free-particle g factor and we use mksa units throughout.

In what follows the spin terms will be neglected since they do not add any essential features. Therefore the Schrödinger equation, for a particle in a periodic scalar potential $(1/q)V(\mathbf{r})$, vector potential $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$, and scalar potential $\varphi = -\mathbf{E} \cdot \mathbf{r}$ corresponding to the externally applied constant and homogeneous fields, reduces to

$$\left\{ \frac{\hbar^2}{2m} \left(-\nabla_r - \frac{q}{2\hbar} \mathbf{B} \times \mathbf{r} \right)^2 + V(\mathbf{r}) - q\mathbf{E} \cdot \mathbf{r} - \epsilon \right\} \psi(\mathbf{r}) = 0. \quad (2.1)$$

As is well known, this equation is not invariant when the variable \mathbf{r} is changed to $\mathbf{r} + \mathbf{R}_n$, and the resulting equation is

$$\left\{ \frac{\hbar^2}{2m} \left[-\nabla_r - \frac{q}{2\hbar} \mathbf{B} \times (\mathbf{r} + \mathbf{R}_n) \right]^2 + V(\mathbf{r}) - q\mathbf{E} \cdot (\mathbf{r} + \mathbf{R}_n) - \epsilon \right\} \times \psi(\mathbf{r} + \mathbf{R}_n) = 0. \quad (2.2)$$

Instead of attempting a solution of Eq. (2.1) by expanding $\psi(\mathbf{r})$ in some complete set of functions, we try to remove from Eq. (2.2) the terms that contain $(\mathbf{r} + \mathbf{R}_n)$. This is accomplished if we consider the functions

$$\exp[-i\mathbf{k} \cdot (\mathbf{r} + \mathbf{R}_n)] \psi(\mathbf{r} + \mathbf{R}_n), \quad (2.3)$$

where \mathbf{k} is a real variable. The equation that these functions satisfy is, making use of Eq. (2.2),

$$\left\{ \frac{\hbar^2}{2m} \left(-\nabla_r + \mathbf{k} + \frac{q}{2i\hbar} \mathbf{B} \times \nabla_k \right)^2 + V(\mathbf{r}) + \frac{1}{i} q\mathbf{E} \cdot \nabla_k - \epsilon \right\} \times \exp[-i\mathbf{k} \cdot (\mathbf{r} + \mathbf{R}_n)] \psi(\mathbf{r} + \mathbf{R}_n) = 0. \quad (2.4)$$

It is clear that the functions [Eq. (2.3)] we have just defined are only particular solutions of Eq. (2.4), which is now invariant with respect to the change of variable $\mathbf{r} \rightarrow \mathbf{r} + \mathbf{R}_n$. Therefore, among the set of functions that satisfy Eq. (2.4) there should exist some that are periodic in \mathbf{r} .

Since the set of functions [Eq. (2.3)] is closed with respect to all the possible lattice translations \mathbf{R}_n , it should be possible to obtain the periodic solutions of Eq. (2.4) as a linear combination of the functions [Eq. (2.3)].

The only possibility is

$$b(\mathbf{r}, \mathbf{k}) = \left(\frac{1}{2\pi} \right)^{3/2} \sum_n \exp(-i\mathbf{k} \cdot (\mathbf{r} + \mathbf{R}_n)) \psi(\mathbf{r} + \mathbf{R}_n). \quad (2.5)$$

Therefore the periodicity in \mathbf{r} singles out a unique solution of Eq. (2.4). {It should be noted that closure of a set with respect to a certain operation does not insure

completeness. If, however, we required solutions other than the ones given by Eq. (2.3), then Eq. (2.1) could not possibly be satisfied with the requirements of finiteness and continuity normally imposed on the wave function that determine the eigenvalues ϵ . Therefore we believe, but have not proved, that the functions [Eq. (2.3)] are complete to the extent required to produce the desired wave functions.}

Finally, the function $b(\mathbf{r}, \mathbf{k})$ must satisfy

$$\left\{ \frac{\hbar^2}{2m} \left(-\nabla_r + \mathbf{k} + \frac{q}{2i\hbar} \mathbf{B} \times \nabla_k \right)^2 + V(\mathbf{r}) + \frac{1}{i} q\mathbf{E} \cdot \nabla_k - \epsilon \right\} \times b(\mathbf{r}, \mathbf{k}) = 0 \quad (2.6)$$

with the requirements that $b(\mathbf{r}, \mathbf{k})$ must be finite, continuous and have the lattice periodicity in the variable \mathbf{r} .

Equation (2.5) can be written as

$$b(\mathbf{r}, \mathbf{k}) = \left(\frac{1}{2\pi} \right)^{3/2} \int d\rho \exp(-i\mathbf{k} \cdot \rho) \psi(\rho) F(\rho - \mathbf{r}), \quad (2.7)$$

where the function $F(\rho - \mathbf{r})$ is defined as follows:

$$F(\rho - \mathbf{r}) = \sum_n \delta(\rho - \mathbf{r} - \mathbf{R}_n). \quad (2.8)$$

Equation (2.7) shows that $b(\mathbf{r}, \mathbf{k})$ is the Fourier transform of a function $B(\mathbf{r}, \mathbf{r}_1)$

$$\begin{aligned} B(\mathbf{r}, \mathbf{r}_1) &= \psi(\mathbf{r}_1) F(\mathbf{r}_1 - \mathbf{r}) \\ &= \left(\frac{1}{2\pi} \right)^{3/2} \int d\mathbf{k} \exp(i\mathbf{k} \cdot \mathbf{r}_1) b(\mathbf{r}, \mathbf{k}) \end{aligned} \quad (2.9)$$

and the wave function $\psi(\mathbf{r})$ is reproduced to within a constant factor by setting $\mathbf{r}_1 = \mathbf{r}$. In order to avoid the singular factors that arise when we use the definition Eq. (2.8) for the function $F(\rho - \mathbf{r})$, we will consider the limiting case of a function sharply peaked at the lattice sites and everywhere different from zero. Taking the Fourier transform of Eq. (2.6) we obtain

$$\left\{ \frac{\hbar^2}{2m} \left(-\nabla_r + \frac{1}{i} \nabla_\rho - \frac{q}{2\hbar} \mathbf{B} \times \rho \right)^2 + V(\mathbf{r}) - q\mathbf{E} \cdot \rho - \epsilon \right\} \times B(\mathbf{r}, \rho) = 0. \quad (2.10)$$

Since $B(\mathbf{r}, \rho)$ is the product of two functions, the approximation to the wave function is related to how well we are able to approximate Eq. (2.8). Although this is the proper criterion to use for the goodness of the approximation, it is not the most practical approach, and in most cases we will expand Eq. (2.10) in a convenient set of periodic functions in \mathbf{r} . This procedure will yield directly to an effective-mass equation for the modulation functions, that play the role of expansion coefficients.

One of the most immediate features of Eq. (2.10) is that we can look upon it as being the wave equation for a coupled system, and this is in last analysis what allows us to talk about an effective Hamiltonian after we

average over the "lattice" coordinates \mathbf{r} . We can recognize in Eq. (2.10) the Hamiltonian of two particles, one moving under the influence of the lattice and another subject to the external electric and magnetic fields. The coupling term is of an unusual type, being a bilinear expression in the kinetic momenta. It is characteristic in this problem that the coupling term is not a small perturbation, and when we perform a canonical transformation that eliminates the leading nondiagonal terms in $(1/i)\nabla_r$, we find that the main result is to replace the free mass m of the particle in the external fields by the effective mass m^* . Therefore, when we are dealing with light holes and electrons the ratio of the free-particle energy to the energy brought about by the coupling term is of the order m^*/m and to a good first approximation the free-particle energy can be neglected entirely. This is the approximation used by Wolf⁴ in his study of the matrix elements in bismuth and by Keldysh⁵ in the problem of deep levels in semiconductors. In the case of heavy holes, for example, the ratio m^*/m is of order one, which corresponds to the intermediate coupling case and the approximation is obviously not good.

Two more points we would like to make before closing this section are: First, from inspection of Eq. (2.6) it is clear that $b(\mathbf{r}, \mathbf{k}) \neq F(\mathbf{k})u(\mathbf{r})$, where $u(\mathbf{r})$ is the periodic part of a Bloch function, because the coupling term contains the momentum operator for the r particle and $[H_{\text{Bloch}}; \mathbf{p}] \neq 0$ so they cannot be diagonalized simultaneously. Therefore the best we can do is to use a combination of the u 's for which \mathbf{p} is diagonal to first order (effective-mass approximation), or otherwise consider a general expansion of the type $\sum_n F_n(\mathbf{k})u_n(\mathbf{r})$. Second, whenever $\mathbf{E} \cdot \mathbf{R}_n \neq 0$, where \mathbf{R}_n is any of the lattice vectors, the Stark ladder of eigenvalues $\epsilon + q\mathbf{E} \cdot \mathbf{R}_n$ are generated by the mutually orthogonal wave functions

$$\begin{aligned} \exp\left(\frac{i}{\hbar}\mathbf{R}_n \cdot \left(\mathbf{p} + \frac{q}{2}\mathbf{B} \times \mathbf{r}\right)\right)\psi(\mathbf{r}) \\ = \exp\left(+\frac{iq}{2\hbar}\mathbf{B} \times \mathbf{r} \cdot \mathbf{R}_n\right)\psi(\mathbf{r} + \mathbf{R}_n). \end{aligned} \quad (2.11)$$

This can be proved by noticing that the operators $\sigma = \exp((i/\hbar)\mathbf{R}_n \cdot (\mathbf{p} + \frac{1}{2}q\mathbf{B} \times \mathbf{r}))$ satisfy the commutation relation

$$[H; \sigma] = q\mathbf{E} \cdot \mathbf{R}_n \sigma, \quad (2.12)$$

and then it is a simple matter to prove Eq. (2.11).

III. CROSSED ELECTRIC AND MAGNETIC FIELDS

Aronov⁶ has given the energy eigenvalues for the problem of a Bloch electron in crossed electric and mag-

⁴ P. A. Wolff, J. Phys. Chem. Solids **25**, 1057 (1964).

⁵ L. V. Keldysh, Zh. Eksperim. i Teor. Fiz. **45**, 364 (1963) [English transl.: Soviet Phys.—JETP **18**, 253 (1964)].

⁶ A. G. Aronov, Fiz. Tverd. Tela **5**, 552 (1963) [English transl.: Soviet Phys.—Solid State **5**, 402 (1963)].

netic fields, in the effective-mass approximation for the case of nondegenerate bands. His solution indicates that in a transition between two such bands the Landau levels are brought together by an amount $\frac{1}{2}(m_1 + m_2) \times (E/B)^2$, where m_1 and m_2 are the effective masses at the top and bottom of the valence and conduction bands, respectively. Experiments performed by Vrethen⁷ in the optical absorption in germanium below the direct-gap band edge indicate that this is actually the case for the values of E and B used in the experiment. They satisfy the condition

$$[\frac{1}{2}(m_1 + m_2)(E/B)^2]/E_g \ll 1,$$

where E_g is the energy gap. We would expect such a necessary condition on physical grounds since otherwise higher order interactions between the bands could not be neglected and nondegeneracy would not apply. This is the physical reason behind the singularity present in Aronov's solution when B approaches zero. Intuitively we may expect that below a certain value of the magnetic field the character of the solution will change from a "magnetic-type" solution to an "electric-type" solution.

In this section we show that indeed the character of the solutions change below a certain critical value of the magnetic field. In addition the effective-mass results are reproduced when the above inequality is satisfied since the idealization of the model used will make other conclusions questionable.

We will assume a two-band model for light holes and electrons where the energies, measured from the center of the gap, are, respectively,

$$\epsilon_1 = -\Delta - (\hbar^2 k^2 / 2m_1), \quad (3.1)$$

$$\epsilon_2 = \Delta + (\hbar^2 k^2 / 2m_2). \quad (3.2)$$

The magnetic field \mathbf{B} is taken along the z direction and the electric field \mathbf{E} along the x direction. Writing $\hat{e}_1, \hat{e}_2, \hat{e}_3$ for the three unit vectors along the coordinates axis, Eq. (2.10) in this case is

$$\begin{aligned} \left\{ \frac{\hbar^2}{2m} \left(-\nabla_r + \frac{1}{i} \hat{e}_3 \frac{\partial}{\partial \rho_3} \right)^2 + V(r) + \frac{\hbar^2}{2m} \left[\left(\frac{1}{i} \frac{\partial}{\partial \rho_1} + \frac{qB}{2\hbar} \rho_2 \right)^2 \right. \right. \\ \left. \left. + \left(\frac{1}{i} \frac{\partial}{\partial \rho_2} - \frac{qB}{2\hbar} \rho_1 \right)^2 \right] - qE\rho_1 + \frac{\hbar^2}{im} \left[\left(\frac{1}{i} \frac{\partial}{\partial \rho_1} + \frac{qB}{2\hbar} \rho_2 \right) \frac{\partial}{\partial x} \right. \right. \\ \left. \left. + \left(\frac{1}{i} \frac{\partial}{\partial \rho_2} - \frac{qB}{2\hbar} \rho_1 \right) \frac{\partial}{\partial y} \right] - \epsilon \right\} B(\mathbf{r}, \boldsymbol{\rho}) = 0. \end{aligned} \quad (3.3)$$

If, now, we choose $u_n(\mathbf{r})$ to be the periodic solutions of

$$\left[\frac{\hbar^2}{2m} \left(-\nabla_r + k_2 \hat{e}_2 + k_3 \hat{e}_3 \right)^2 + V(\mathbf{r}) - \epsilon_n \right] u_n(\mathbf{r}) = 0, \quad (3.4)$$

⁷ Q. H. F. Vrethen, Phys. Rev. Letters **14**, 558 (1965).

and write

$$B(\mathbf{r}, \boldsymbol{\rho}) = \exp(ik_2\rho_2 + ik_3\rho_3 + (qB/2i\hbar)\rho_2\rho_1) \times [g_1(\rho_1)u_1(\mathbf{r}) + g_2(\rho_1)u_2(\mathbf{r})] \quad (3.5)$$

we find that the equations satisfied by the modulation functions are

$$\left[\epsilon_1 - \epsilon - \left(qE + \frac{qB}{\hbar} \frac{\partial \epsilon_1}{\partial k_2} \right) x + \frac{\partial \epsilon_1}{\partial k_1} \frac{1}{i} \frac{\partial}{\partial x} \right] g_1 + \left[\alpha_{12} \frac{1}{i} \frac{\partial}{\partial x} - \beta_{12} \frac{qB}{\hbar} x \right] g_2 = 0, \quad (3.6a)$$

$$\left[\alpha_{21} \frac{1}{i} \frac{\partial}{\partial x} - \beta_{21} \frac{qB}{\hbar} x \right] g_1 + \left[\epsilon_2 - \epsilon - \left(qE + \frac{qB}{\hbar} \frac{\partial \epsilon_2}{\partial k_2} \right) x + \frac{\partial \epsilon_2}{\partial k_1} \frac{1}{i} \frac{\partial}{\partial x} \right] g_2 = 0 \quad (3.6b)$$

when $\boldsymbol{\rho} = \mathbf{r}$, and the terms

$$(\hbar^2/2m)[-\partial^2/\partial\rho_1^2 + ((qB/\hbar)\rho_1)^2]g_i(\rho_1)$$

are neglected. Here we have used the notation:

$$\alpha_{ij} = (\hbar/m)\langle i | p_x | j \rangle, \quad \beta_{ij} = (\hbar/m)\langle i | p_y | j \rangle,$$

and the formula⁸

$$(\hbar/m)\langle i | \mathbf{p} | i \rangle = -(\hbar^2/m)\mathbf{k} + \nabla_{k\epsilon_i}(k).$$

Making use of the assumed parabolicity of the bands [Eqs. (3.1), (3.2)], Eqs. (3.6) reduced to

$$\left[\epsilon_1 - \epsilon - \left(qE - \frac{qB}{m_1} \hbar k_2 \right) x \right] g_1 + \left[\alpha_{12} \frac{1}{i} \frac{\partial}{\partial x} - \beta_{12} \frac{qB}{\hbar} x \right] g_2 = 0, \quad (3.7a)$$

$$\left[\alpha_{21} \frac{1}{i} \frac{\partial}{\partial x} - \beta_{21} \frac{qB}{\hbar} x \right] g_1 + \left[\epsilon_2 - \epsilon - \left(qE + \frac{qB}{m_2} \hbar k_2 \right) x \right] g_2 = 0. \quad (3.7b)$$

Furthermore, since the bands will be parabolic only in the neighborhood of the maximum or minimum of the energy, we will neglect the contribution of the terms $\hbar k_2(qB/m_i)$, and set $k_2 = 0$.

After some straightforward but tedious manipulations we can obtain from Eqs. (3.7) an equation for a function $f_1(x)$ that differs from the equation for $g_1(x)$ by the

absence of the first derivative:

$$\left\{ |\alpha_{12}|^2 \frac{d^2}{dx^2} + \frac{qB}{2i\hbar} (\alpha_{12}\beta_{21} - \alpha_{21}\beta_{12}) + (\epsilon_1 - \epsilon - qEx)(\epsilon_2 - \epsilon - qEx) - \frac{(qE)^2 |\alpha_{12}|^2}{2(\epsilon_2 - \epsilon - qEx)^2} - \frac{1}{4} \left[\frac{qB (\alpha_{12}\beta_{21} - \alpha_{21}\beta_{12}) x}{i\hbar |\alpha_{12}|} - \frac{qE |\alpha_{12}|}{\epsilon_2 - \epsilon - qEx} \right]^2 \right\} \times f_1(x) = 0. \quad (3.8)$$

This equation can be looked upon as a Schrödinger equation for a one-dimensional system moving in a potential

$$-|\alpha_{12}|^{-2} [(qE)^2 x^2 - (\epsilon_1 + \epsilon_2 - 2\epsilon)qEx] + \frac{(qE)^2}{2(\epsilon_2 - \epsilon - qEx)^2} + \frac{1}{4} \left[\frac{qB (\alpha_{12}\beta_{21} - \alpha_{21}\beta_{12}) x}{i\hbar |\alpha_{12}|^2} - \frac{qE}{\epsilon_2 - \epsilon - qEx} \right]^2$$

and with eigenvalues equal to

$$|\alpha_{12}|^{-2} [(qB/2i\hbar)(\alpha_{12}\beta_{21} - \alpha_{21}\beta_{12}) + (\epsilon_1 - \epsilon)(\epsilon_2 - \epsilon)].$$

It is clear that the sign of the quadratic term of this potential will determine whether the eigenvalues are discrete or continuous. If we wish to reproduce the Landau levels in some approximation, it is therefore necessary that

$$[(qB/2i\hbar)(\alpha_{12}\beta_{21} - \alpha_{21}\beta_{12})]^2 > (qE)^2 |\alpha_{12}|^2. \quad (3.9)$$

Equation (3.8) cannot be solved exactly, but we can approximate it, if the electric field is small enough, by replacing the terms that contain $(\epsilon_2 - \epsilon - qEx)^{-2}$ by some average value ϵ' . The approximate equation is then

$$\left\{ |\alpha_{12}|^2 \frac{d^2}{dx^2} + (\epsilon_1 - \epsilon)(\epsilon_2 - \epsilon) - \epsilon' + [(qE)^2 x^2 - (\epsilon_1 + \epsilon_2 - 2\epsilon)qEx] - [(qB/2i\hbar)(\alpha_{12}\beta_{21} - \alpha_{21}\beta_{12})x/|\alpha_{12}|]^2 \right\} f_1 = 0. \quad (3.10)$$

The eigenfunctions of this equation are the same as the one-dimensional harmonic oscillator, and the eigenvalues are given by

$$(\epsilon - \epsilon_1)(\epsilon - \epsilon_2) - \epsilon' + (a - b)R^2 = (2n + 1) |\alpha_{12}| (a - b)^{1/2}, \quad (3.11)$$

where

$$a = |\alpha_{12}|^{-2} [(qB/2i\hbar)(\alpha_{12}\beta_{21} - \alpha_{21}\beta_{12})]^2, \\ b = (qE)^2, \\ c = qE(2\epsilon - \epsilon_1 - \epsilon_2),$$

and

$$R = c/2(a - b).$$

In order to make connection with the results of the effective-mass formula we assume that $m_1 \approx m_2 \approx m^*$ so $\epsilon_1 + \epsilon_2 \approx 0$, consider only the low-lying eigenvalues, so

⁸ A. H. Wilson, *The Theory of Metals* (Cambridge University Press, New York, 1958), 2nd ed., p. 46.

$\epsilon - \epsilon_1 \approx 2\Delta = E_0$, neglect ϵ' , and make the identifications

$$(1/\Delta)(1/2i)(\alpha_{12}\beta_{21} - \alpha_{21}\beta_{12}) \approx \hbar^2/m^*$$

and

$$|\alpha_{12}|^2 \approx (1/2i)(\alpha_{12}\beta_{21} - \alpha_{21}\beta_{12}).$$

Then

$$\epsilon \approx \epsilon_2 + (\hbar q B/m^*)(n + \frac{1}{2}) - \frac{1}{2}m^*(E/B)^2, \quad (3.12)$$

and Eq. (3.9) becomes

$$[2m^*(E/B)^2]/E_0 \ll 1. \quad (3.13)$$

We have thus proved that the requirements for the validity of the effective-mass formula, guessed on physical grounds, are indeed appropriate.

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Quantum Statistical Derivation of Ferromagnetic-Resonance Equations

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From the quantum equation for the density matrix, equations are derived which characterize the behavior of magnetization and mean exchange-interaction energy of ferromagnets in an external magnetic field for a wide temperature region. In the limiting case $T \gg T_C$ (T_C is the Curie temperature), these equations lead to the Bloch-type ferromagnetic-resonance equations (for sufficiently weak magnetic field). In the low-temperature region $T \ll T_C$ these equations lead to the well-known Landau-Lifshitz equations. Expressions for the relaxation times are obtained which connect these parameters with the microscopic properties of ferromagnetic substances.

THE behavior of the magnetization of ferromagnetic substances placed in an alternating magnetic field is fully determined theoretically by solving ferromagnetic-resonance equations. It is usually supposed that ferromagnetic-resonance equations are differential equations connecting time derivatives of three magnetization components, on the one hand, and the values of these components and the value of the external magnetic field, on the other hand. Such, for example, are the well-known Landau-Lifshitz phenomenological equations used for the description of ferromagnetic resonance in the low-temperature region ($T < T_C$, where T_C is the Curie temperature). This equation has the form,¹

$$\frac{d\mathbf{M}}{dt} = \gamma [\mathbf{M} \times \mathbf{H}] - \frac{\chi_0 [\mathbf{M} \times (\mathbf{M} \times \mathbf{H})]}{\tau_L M^2}. \quad (1)$$

In the opposite high-temperature limiting case ($T \gg T_C$) the behavior of the magnetization of ferromagnetic substances is satisfactorily described either by the empirical equation obtained by Codrington, Olds, and Torrey,²

$$\frac{d\mathbf{M}}{dt} = \gamma [\mathbf{M} \times \mathbf{H}] - \frac{1}{\tau_2} \left[\mathbf{M} - \frac{(\mathbf{M} \cdot \mathbf{H})\mathbf{H}}{H^2} \right] + \frac{1}{\tau_1} \left[\chi_0 \mathbf{H} - \frac{(\mathbf{M} \cdot \mathbf{H})\mathbf{H}}{H^2} \right], \quad (2)$$

or by phenomenological equations derived by Skrotzki and Shmatov,³

$$\begin{aligned} dM_{x,y}/dt &= \gamma [\mathbf{M} \times \mathbf{H}]_{x,y} - (M_{x,y} - \chi_0 H_{x,y})/\tau_L, \\ dM_z/dt &= \gamma [\mathbf{M} \times \mathbf{H}]_z - (M_z - \chi_0 H_z)/\tau_{11}. \end{aligned} \quad (3)$$

In the above equations, \mathbf{M} is the magnetization, \mathbf{H} is the external magnetic field, χ_0 is the static magnetic susceptibility, and τ_L , τ_1 , τ_2 , τ_{\perp} , and τ_{11} are relaxation times.

Although Eqs. (1)–(3) are in accord with many experimental facts, they have nevertheless some deficiencies. Firstly, these equations are valid in the limiting cases of high or low temperature only. Secondly, the methods used for obtaining Eqs. (1)–(3) do not give possibilities for determining the relaxation constants involved in the above-mentioned equations. And finally, equations for magnetization components only cannot give, in the general case, the full description of ferromagnetic-resonance absorption. This is because of the necessity of taking into account the variation of the mean exchange energy during saturation, as we shall see further on.

These deficiencies are explained by the fact that magnetic-resonance equations were determined formerly either by means of empirical selection or by a phenomenological method. It is interesting therefore to consider ferromagnetic-resonance absorption from the micro-

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