Theory of the Motion of Vortices in Superconductors*

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The theory of the motion of vortex lines in the mixed state of type II superconductors is derived on the basis of a local model that is a generalization of the London theory. It is believed the model simulates reasonably well the behavior of relatively pure superconductors ($l>\xi_0$), giving a vortex line with a normal core. It is found that if the force on a line is produced by a uniform transport current $\mathbf{J}_T$, electric fields generated by the motion drive the current through the core, so that the total current flow is $\mathbf{J}_T+\mathbf{J}_0(\mathbf{r}\text{-}\mathbf{v}_Lt)$, where $\mathbf{J}_0(\mathbf{r})$ is the circulation of a stationary vortex and $\mathbf{v}_L$ is the velocity of the line. In part $\mathbf{J}_T$ represents superfluid flow and in part normal flow. Expressions derived for the viscosity and flow resistivity are nearly identical with empirical laws of Kim and co-workers. The Hall angle expected in the mixed state is the same as in the normal state for a magnetic field equal to that in the core.

I. INTRODUCTION

IN the mixed state of type II superconductors, in accordance with the theory of Abrikosov, flux enters in the form of quantized flux lines or vortices. The mixed state exists between a lower critical field H_{c1} and an upper critical field H_{c2}, above which superconductivity in the bulk is destroyed. The circulating current of a vortex gives a magnetic field along the axis. The total flux is quantized; normally each vortex has a circulation corresponding to a single flux quantum, $hc/2e$. If the flux lines are pinned by imperfections so that they can not move, current flow in the mixed state will be subject to vanishing resistance. However, with a sufficient driving force produced by a transport current, vortices can become unpinned. Their motion is subject to a viscous drag, giving rise to dissipation. This shows up as a macroscopic resistivity as measured by conventional methods. The flux flow resistance has been studied by a number of observers, perhaps most completely by Kim, Hempstead, and Strnad.[1–3] Recently it has been observed that type II superconductors exhibit a Hall effect as well, with a Hall angle comparable to that of the normal metal.[4]

Note added in proof. Kim and co-workers[1] proposed an empirical expression for the flux flow resistivity and pointed out that the dissipation can be accounted for approximately if the transport current flows directly through the essentially normal cores of the vortex lines. Rosenbaum and Cardona[5] had earlier proposed such a picture in their interpretation of data on the microwave surface resistance of type II superconductors. Our theoretical model gives just this result.

We discuss here the free motion of vortices, when they are not subject to pinning forces. The main considerations which give the basis for the present theory have been given in earlier notes.[6] The essential ideas concerning the nature of supercurrent flow go back to London,[7] who pointed out that there can be no interior stresses associated with supercurrent flow, since this is an equilibrium situation. By definition, the normal component is the nonequilibrium part of the flow. Any stresses must be associated with the normal component or act at the boundaries. A flux line moves adiabatically so that the supercurrent distribution moves as a whole through the metal. One can regard the distribution as a single quantum state which moves as a rigid body. Applied forces act on the system as a whole; electric fields generated by the motion give rise to dissipation and to forces acting near the core of the line.

An improved and more detailed account of the theory is given here. We include effects associated with Hall fields which can give rise to helicon-type motions when the Hall angle of the normal metal is large. These were neglected in our earlier notes. The importance of the Hall fields in the mixed state has been recognized by Volger[8] and worked out independently by van Vijfeijken and Niessen.[9] While their conclusions are similar, our theory is more complete and differs from theirs in some important details.

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[1] C. F. Hempstead and Y. B. Kim, Phys. Rev. Letters **12**, 145 (1964); A. R. Strnad, C. F. Hempstead, and Y. B. Kim, *ibid.* **13**, 794 (1964); Y. B. Kim, C. F. Hempstead, and A. R. Strnad, Phys. Rev. **139**, A1163 (1965).

[2] W. F. Vinen, Rev. Mod. Phys. **36**, 48 (1964); P. H. Borcherds, C. E. Gough, W. F. Vinen, and A. C. Warren, Phil. Mag. **10**, 349 (1964).

[3] J. Volger, F. A. Staas, and A. J. van Vijfeijken, Phys. Letters **9**, 303 (1964).

[4] W. A. Reed, E. Fawcett, and Y. B. Kim, Phys. Rev. Letters **14**, 790 (1965); and A. K. Niessen and F. A. Staas, Phys. Letters **15**, 26 (1965).

[5] B. Rosenblum and M. Cardona, Phys. Rev. Letters **12**, 657 (1964).

[6] M. J. Stephen and J. Bardeen, Phys. Rev. Letters **14**, 112 (1965); J. Bardeen, *ibid.* **13**, 747 (1964).

[7] F. London, *Superfluids* (John Wiley & Sons, Inc., New York, 1950).

[8] J. Volger (private communication).

[9] A. J. van Vijfeijken and A. K. Niessen, Phys. Letters **16**, 23 (1965).

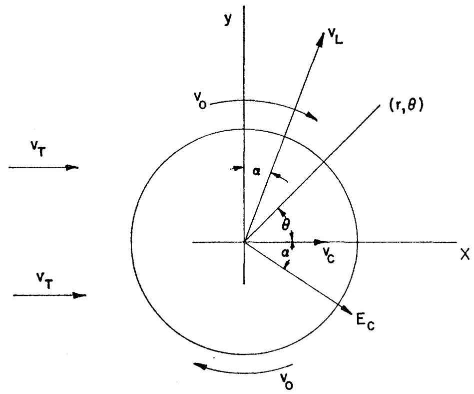

FIG. 1. Coordinates used to discuss vortex motion. The circulation of a stationary vortex outside of a normal core of radius a is indicated by v_0. This pattern is driven to move in the direction $\mathbf{v}_L$ by a uniform transport flow indicated by $\mathbf{v}_T$. A uniform electric field in the core, $\mathbf{E}_c$, produced by the motion, gives a drift velocity $\mathbf{v}_c$ at the Hall angle α relative to $\mathbf{E}_c$ and parallel to $\mathbf{v}_T$. The force on the vortex is in the y direction, perpendicular to $\mathbf{v}_T$, and $\mathbf{v}_L$ is at the Hall angle relative to the y direction.

A reasonable model for a vortex, suggested by a calculation of Caroli *et al.*,[10] is an essentially normal core, of radius approximately equal to the coherence distance ξ about which the supercurrents circulate. They showed that the density of quasiparticle states in the vicinity of the axis is given nearly correctly by this model. A schematic diagram showing a cross section of the vortex in a plane normal to the axis is given in Fig. 1. In order that the motion be in the positive y direction for a transport current in the positive x direction, we take the direction of the magnetic field in the negative z direction, $H_z=-H$. We assume that the penetration depth λ is large compared with ξ, so that the field can be taken to be uniform in the vicinity of the core.

It has been suggested by de Gennes and Matricon[11] that the motion of vortices of superconductors should be analogous to the motion of quantized vortices in superfluid helium, in which the concept of a Magnus force plays an important role. However, the differences in behavior which result from the presence of the lattice of positive ions in the metal are so marked that the analogy is not a very close one, at least as far as the motion of the vortex is concerned. The positive lattice has two important effects. First, it neutralizes the charge of the electrons, so that the vortex is subject to a driving force only when the transport current represents a real current flow relative to the lattice. The driving force on the vortex is just that determined by Ampere's rule. If the transport current density, assumed uniform, is $\mathbf{J}_T$ and $\boldsymbol{\varphi}_0$ is a vector representing the flux, the force per unit length is $(\mathbf{J}_T\times\boldsymbol{\varphi}_0)/c$. Second, the electrons are scattered primarily by the lattice; electron-lattice scattering dominates over electron-electron scattering in bringing about steady-state flow. The quasiparticle excitations relax to the lattice; they do not tend to come to a local equilibrium among themselves so as to give a normal current flow relative to the superfluid flow, as in the two-fluid model of He II. It is for this reason that second sound would be very difficult to observe in a superconductor.

In helium, by Galilean invariance, a vortex takes up the motion of the fluid and rides along with it. If the ground state is moving with velocity $\mathbf{v}$, and the velocity field of the vortex relative to the fluid is $V(r)$, the total velocity is $\mathbf{v}+\mathbf{V}(\mathbf{r}-\mathbf{v}t)$. Because of the positive lattice, this is not true for a superconductor. One can define a supercurrent flow $\mathbf{J}_s(\mathbf{r}-\mathbf{v}_Lt)$ corresponding to a vortex moving with velocity $\mathbf{v}_L$ relative to the lattice, where $\mathbf{J}_s(\mathbf{r})$ represents a possible supercurrent flow in the lattice frame. Because of relaxation processes which scatter electrons to the lattice, there is an additional component of flow $\mathbf{J}_n(\mathbf{r}-\mathbf{v}_Lt)$ resulting from the motion, and the total flow is $\mathbf{J}=\mathbf{J}_n+\mathbf{J}_s$. The normal component which gives rise to dissipation and viscous drag is, near $T=0$, large only near the core.

We shall show that for a local but otherwise rather general model of a superconductor the steady-state flow is a superposition of the superconducting flow pattern of a stationary vortex, $J_0(r)$, and the transport current:

$$\mathbf{J}=\mathbf{J}_T+\mathbf{J}_0(\mathbf{r}-\mathbf{v}_Lt). \tag{1.1}$$

This is true for all fields between H_{c1} and H_{c2}. Part of $\mathbf{J}_T$ comes from $\mathbf{J}_n$ and part from $\mathbf{J}_s$. Electric fields set up by the motion drive the transport current through the normal core. If the vortex were pinned, the transport current would flow around the core and there would be no dissipation.

When the Hall effect is taken into account, the direction of the electric field in the core is at the Hall angle α of the normal metal relative to the transport current. The motion of the vortex is in a direction perpendicular to the electric field, and is thus not in the direction of the force ($\perp$ to $\mathbf{J}_T$) but, as illustrated in Fig. 1, is at the Hall angle relative to this direction. The theory indicates that for our model the component of $\mathbf{v}_L$ parallel to $\mathbf{v}_T$ is independent of the electron-lattice relaxation time τ and is equal to $(H/H_{c2})\mathbf{v}_T$, where H is the field in the core.[12]

de Gennes and Nozières[12] have suggested that in sufficiently pure type II superconductors vortex lines should move with the fluid according to the Magnus force concept. This would imply that as $\tau\rightarrow\infty$, the velocity $\mathbf{v}_L$ should approach $\mathbf{v}_T$ rather than $(H/H_{c2})\mathbf{v}_T$ as predicted by the present theory. We believe that the difference comes in assumptions as to how relaxation to steady-state motion is brought about. As we have mentioned, our theory should apply if the dominant

[10] C. Caroli, P. G. de Gennes, and J. Matricon, Phys. Letters **9**, 307 (1964).

[11] P. G. de Gennes and J. Matricon, Rev. Mod. Phys. **36**, 45 (1964).

[12] The effect of Hall fields on the motion of flux bundles in type I superconductors has been observed by W. De Sorbo, Phil. Mag. **112**, 853 (1965). For a discussion of the theory, see P. G. de Gennes and P. Nozières, Phys. Letters **15**, 216 (1965).

relaxation process is electron-lattice scattering but their theory would apply if electron-electron scattering were more important.

The pattern of circulation outside of the core can be described in terms of a momentum $p_\theta = -\hbar/2r$ for unit flux quantum, where the factor of 2 in the denominator comes from pairing. As the vortex moves, momentum is lost at the core boundary at a rate[13]

$$\pi r n p_\theta v_L = -\tfrac{1}{2}\pi n \hbar v_L\,, \tag{1.2}$$

where n is the electron density. There is an equal loss of momentum from current flow in the core. The total corresponds to the Magnus force

$$\mathbf{F}_m = -(ne/c)(\mathbf{v}_L \times \boldsymbol{\varphi}_0) \tag{1.3}$$

which must act somewhere in the system to supply this rate of loss of momentum. In a superconductor, the momentum may be supplied from a remote part of the system.

Just how the momentum is supplied depends on the circumstances; it may come from a decrease in supercurrent flow in part of the system, or under steady-state conditions when an array of vortices is moving, it may be supplied by an electric field maintained by an external source. In this latter case, the electric field is just what one would calculate as generated from moving flux lines by induction, $\nabla \times \mathbf{E} = -c^{-1}\partial\mathbf{B}/\partial t$. In steady-state flow, vortices enter on one side of the specimen and leave on the other, but when the current is maintained by a battery, there is no net change of total flux. For this reason, objections have been raised against calling the field an induction field,[14] but nevertheless the result is correct. If flux were leaking out of a persistent current ring, the current flowing around the ring would decrease in time, and one could then call the electric field a true induction field.

Kim and co-workers[2] have shown that the flow resistivity ρ_f of alloys and metals studied follows an approximate law of corresponding states. When expressed in terms of the reduced variables T/T_c and H/H_{c2}, it is given by

$$\rho_f/\rho_n = [H/H_{c2}(0)] f(T/T_c)\,, \tag{1.4}$$

where $f(T/T_c)$ is a slowly varying function which reduces to unity as $T \to 0$. For $f(T/T_c)=1$, this corresponds to a viscous drag ηv_L opposing the Lorentz force $J_T \varphi_0/c$ on a vortex, with

$$\eta_{\exp} = \varphi_0 H_{c2}(0)\sigma_n/c^2 = \pi\hbar n \omega_{c2}\tau\,, \tag{1.5}$$

where σ_n is the conductivity of the normal metal and $\omega_{c2} = eH_{c2}/(mc)$ is the cyclotron frequency in the field H_{c2}.

About half of the dissipation occurs within the normal core and about half in the transition region just outside the core. In this latter region, there is a rapid decrease in pairing indicated by a decrease in the gap parameter Δ as one approaches the core boundary. A dissipation associated with a time rate of change of Δ as the vortex moves by has been suggested earlier by Tinkham.[15] In our local model, Δ goes to zero at the core boundary, rather than at the axis, as would be the case in the true nonlocal theory. Similarly, about half of the effective mass of the moving vortex comes from the core and about half from the transition region. The latter contribution has been considered by Suhl.[16] The time required for relaxation of the vortex motion to steady-state flow is of the order of the electron relaxation time τ.

The criterion for the validity of the theory is that $v_L\tau \ll a$; meaning that the distance the line moves in a relaxation time be small compared with the core radius. An equivalent statement is that the normal currents be small compared with the circulating supercurrents just outside of the core. This should be true under nearly all conditions. We believe that the theory should be valid even when the mean free path of the electrons is large in comparison with the core diameter, as it may be in pure metals. Even though the probability that a given electron be scattered as it goes through the core may be small, considering all of the electrons, a large number of scatterings will take place in the core within a relaxation time. One may regard the core as similar to a small region in the interior of a pure metal. Voltage gradients and current densities are well defined even though the size of the region is small compared with the mean free path (m.f.p.). The response of the electrons to electric and magnetic fields between scatterings is taken into account in the theory.

In Sec. II we give the basis for the local model. The theory of the flow pattern associated with a moving vortex line for a local model near $T=0$°K is discussed in Sec. III. Expressions for the dissipation and viscosity coefficient for this model are derived in Sec. IV. The Hall effect, which gives a component of motion parallel to the transport current $\mathbf{v}_T$ is discussed in Sec. V, in which we also consider problems associated with effective mass and backflow. Modifications of the theory required for finite temperatures and for the case of a small m.f.p. from impurity scattering, such that $\rho_s < \rho$, are discussed in Sec. VI. Finally, in Sec. VII we consider possible extensions of the theory and modifications that may be required to take nonlocal effects into account.

II. LOCAL MODEL OF A SUPERCONDUCTOR

Since superconductivity theory is nonlocal, a correct mathematical description of a vortex is difficult. The

[13] W. F. Vinen and P. Nozières, by somewhat independent methods, have analyzed vortex motion in terms of momentum and energy fluxes, giving insight into the connection with Magnus force concepts (private communcations). Equation (1.2) is one of the relations they derive.

[14] R. G. Jones, E. H. Rhoderick, and A. C. Rose-Innes, Phys. Letters **15**, 214 (1965) and J. Pearl, *ibid.* **17**, 12 (1965). Also see B. D. Josephson, Phys. Letters **16**, 242 (1965) and H. B. G. Casimir, *ibid.* **17**, 177 (1964) who have shown that the flux cutting argument gives the correct result.

[15] M. Tinkham, Phys. Rev. Letters **13**, 804 (1964).

[16] H. Suhl, Phys. Rev. Letters **14**, 226 (1965).

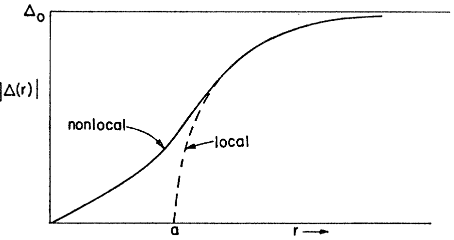

FIG. 2. Variation of the gap parameter Δ as a function of the radial distance r from the axis of the vortex in the local and nonlocal theories. The core radius a is where Δ goes to zero in the local theory and is roughly the point of inflection in the nonlocal theory.

current density at a given point depends on the fields in a region surrounding the point extending over distances of the order of the coherence distance ξ. Qualitatively, it is known that for unit flux quantum the complex gap parameter $\Delta(r)$ which measures the pairing, is of the form $f(r)e^{i\theta}$ where $f(r)$ vanishes for $r\rightarrow 0$, increases linearly for small r, and approaches the normal value Δ_0 for large r, as indicated in Fig. 2. The point of inflection, which occurs at $r\simeq\xi$, is a measure of the size of the core. The inside of the core is a region of gapless superconductivity; the energy required to create quasiparticle excitations is negligibly small. By solving the appropriate nonlocal Gor'kov equations for a pure superconductor near $T=0$°K, Caroli *et al.*[10] have shown that the density of quasiparticle states in the core is very nearly that of a normal region of radius ξ.

In order to simplify the mathematics, we replace the true nonlocal superconductor by an ideal local model in which the supercurrent density $\mathbf{J}_s(\mathbf{r})$ depends only on the kinetic momentum $\mathbf{P}$ of the paired electrons in the ground state. When there is a magnetic field described by a vector potential $\mathbf{A}$, $\mathbf{P}=\mathbf{p}_s-(e/c)\mathbf{A}$. In place of the current density, it is convenient to introduce the velocity $\mathbf{v}_s$ of the superfluid, defined so that the density of mass flow is $\mathbf{J}_s=\rho_s\mathbf{v}_s$, where ρ_s is the superfluid density for $\mathbf{P}=\mathbf{v}_s=0$.[17,18] One may express v_s as a derivative of the local free energy per electron, $F(P)$:

$$v_s=\partial F/\partial P. \tag{2.1}$$

A core may be introduced by taking a model such that $v_s\rightarrow 0$ when the momentum reaches a critical value $P=P_c$, as shown in Fig. 3. The core radius a is that for which $P\rightarrow P_c$ and $\Delta\rightarrow 0$. Thus for $H\ll H_{c2}$, $P=p_s=\hbar/2r$ and $a=\hbar/2P_c$. A schematic plot of $\Delta(r)$ for the local model is shown in Fig. 2 along with that for the nonlocal model.

The free energy $F(P_c)$ for $P=P_c$ is equal to the normal-superconducting free energy difference per electron. To get the free-energy difference per unit volume, one must multiply by n the electron density, and the superfluid density ratio ρ_s/ρ:

$$F_n-F_s=H_c^2/8\pi=n(\rho_s/\rho)F(P_c). \tag{2.2}$$

Here H_c is the critical field at temperature T, and $\rho_s/\rho=\lambda_L^2/\lambda^2$ when expressed in terms of the penetration depths.

The Ginzburg-Landau theory is an example of this sort. Here $F(P)$ is of the form

$$F(P)=(P^2/2m)[1-\tfrac{1}{2}(P/P_c)^2], \tag{2.3}$$

which gives

$$v_s=(P/m)[1-(P/P_c)^2]. \tag{2.4}$$

The parameters are evaluated as follows:

$$F(P_c)=P_c^2/4m=H_c^2\lambda^2/8\pi n\lambda_L^2. \tag{2.5}$$

For $H\ll H_{c2}$, one may get a relation between the core radius $a=\hbar/2P_c$, and H_{c2}. Since for this model

$$H_{c2}=4eH_c^2\lambda^2/\hbar c, \tag{2.6}$$

we have from (2.5), with $\lambda_L^2=mc^2/(4\pi ne^2)$,

$$H_{c2}=\hbar c/(2ea^2). \tag{2.7}$$

A similar relation is found for other models. Perhaps a more realistic model for pure metals at low temperatures is to replace (2.3) by the correct relation between v_s and P for an ideal superconductor for P uniform in space, and thus to neglect nonlocal effects. This relation has been derived by Rogers.[19] At $T=0°$, $mv_s=P$ until P reaches a value P_d, such that $P_dv_F=\Delta_0$, above which depairing can occur. Here v_F is the Fermi velocity. As P increases above P_d, Δ drops rapidly to zero. The equation which determines Δ as a function of P for $Pv_F>\Delta_0$ is

$$\ln\frac{\Delta_0}{\Delta}=\cosh^{-1}\frac{v_FP}{\Delta}-\left[1-\left(\frac{\Delta}{v_FP}\right)^2\right]^{1/2}. \tag{2.8}$$

The critical value P_c for which $\Delta=0$ is

$$P_c=\tfrac{1}{2}e_n\Delta_0/v_F=e_n\hbar/(2\pi\xi_0), \tag{2.9}$$

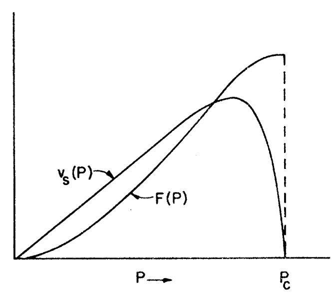

FIG. 3. Schematic variation of free energy per particle $F(P)$ and velocity $v_s=\partial F/\partial P$ in the local theory. The momentum P is that per electron of pairs in the ground state. The critical momentum is indicated by P_c.

[17] These definitions differ from those used earlier by one of the authors (Ref. 19). In the earlier article, $\mathbf{v}_s$ was used to represent the velocity of the ground-state pairs, so that $m\mathbf{v}_s$ of that article is what we now call $\mathbf{P}$. In the present article, it is more convenient to let $\mathbf{v}_s$ represent $\partial F/\partial P$, so that the current density is $\rho_s\mathbf{v}_s$, with ρ_s independent of $\mathbf{v}_s$. In the earlier article, the current density was given formally by the same expression, but this required that ρ_s be velocity-dependent.

[18] V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. **20**, 1064 (1950); A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. **32**, 1442 (1957) [English transl.: Soviet Phys.—JETP **5**, 1174 (1957)]; L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **37**, 1407 (1959) [English transl.: Soviet Phys.—JETP **10**, 998 (1960)].

[19] K. T. Rogers, Ph.D. thesis, University of Illinois, 1960 (unpublished); see J. Bardeen, Rev. Mod. Phys. **34**, 667 (1962).

where $e_n=2.178$ and $\xi_0=\hbar v_F/\pi\Delta_0$ is the coherence distance. The core radius a, determined from $P_c=\frac{1}{2}\hbar/a$, is

$$a=\hbar/2P_c=(\pi/e_n)\xi_0=1.16\xi_0. \tag{2.10}$$

Gor'kov's expression[20] for H_{c2} at $T=0°\mathrm{K}$ may be written

$$H_{c2}\simeq 1.5\hbar c/(4e\xi_0^2)\simeq \hbar c/(2ea^2). \tag{2.11}$$

Finally, for the impure case such that $l\ll\xi_0$, the critical momentum for which $\Delta\rightarrow 0$ as given by Maki's theory[21] is

$$P_c^2=\hbar^2/4a^2=3\hbar^2/4\pi l\xi_0, \tag{2.12}$$

which gives

$$a^2=(\pi/3)l\xi_0. \tag{2.13}$$

The expression for H_{c2} in terms of a is again

$$H_{c2}=3\hbar c/2\pi el\xi_0=\hbar c/2ea^2. \tag{2.14}$$

Thus this appears to be a universal relation. We shall show that for a rather general local model the vortex motion does not depend on the form of the function $F(P)$, but only on P_c. When the core radius is expressed in terms of H_{c2}, we are led to a law of corresponding states for the flow resistance similar to that proposed by Kim *et al.*

According to the local model, all of the momentum loss associated with the vortex motion occurs at the boundary of the core, even though v_s goes gradually to zero as r approaches a. This is an unrealistic feature of the model. In the nonlocal model, the total rate of loss of momentum will be the same, but it will be spread out over a region of $\sim\xi$ about the axis.

III. FLOW PATTERN FOR A LOCAL MODEL

We shall present in this section the theory of motion of an isolated vortex for a local model of a relatively pure superconductor near $T=0°\mathrm{K}$. It is assumed that the m.f.p. $l>\xi_0$, so that the superfluid density $\rho_s=\rho=nm$. We also assume that the normal current densities generated by the motion, as well as the transport current density $\mathbf{J}_T$ which produces the force on the vortex, are small in comparison with the superfluid flow just outside the core. Since the latter are of the order of 10^6 or 10^7 A/cm^2, this should be the case in actual superconductors except under very extreme conditions. We shall defer consideration of Hall fields to a later section, so that here we are concerned only with the component of the vortex motion perpendicular to $\mathbf{J}_T$ and parallel to the force produced by $\mathbf{J}_T$. In accordance with the notation of Fig. 1, $\mathbf{J}_T$ is in the x direction and $\mathbf{v}_L$ in the y direction. As discussed in Sec. II, our model is a generalization of the London theory.

With this model there is no normal current flow except when P exceeds the value $P_d=\Delta_0/v_F$ above which depairing is possible. If the electrons are accelerated by the field in the depairing region, relaxation will occur to the supercurrent flow $J_s(P)$. We assume a simple relaxation time τ for this relaxation process, with τ equal to the collision relaxation time in the normal state.

In the local London theory there is a linear relation between the velocity and the momentum

$$m\mathbf{v}_s=\mathbf{P}=\mathbf{p}_s-(e/c)\mathbf{A}. \tag{3.1}$$

The equation of motion of the superfluid is

$$\frac{d\mathbf{v}_s}{dt}=-\nabla\mu_0+\frac{e}{m}\left[\mathbf{E}+\frac{1}{c}(\mathbf{v}_s\times\mathbf{H})\right], \tag{3.2}$$

where μ_0 is the chemical potential per unit mass in the absence of currents or fields. All currents will be measured in the lattice frame of reference. This equation can be simplified by use of

$$\frac{d\mathbf{v}_s}{dt}=\frac{\partial\mathbf{v}_s}{\partial t}+\mathbf{v}_s\cdot\nabla\mathbf{v}_s=\frac{\partial\mathbf{v}_s}{\partial t}-\mathbf{v}_s\times(\nabla\times\mathbf{v}_s)+\tfrac{1}{2}\nabla v_s^2,$$

and the London equation

$$\mathrm{curl}\,\mathbf{v}_s=-(e/mc)\mathbf{H}.$$

After cancellation of some common terms we have

$$\frac{\partial\mathbf{v}_s}{\partial t}=-\nabla(\mu_0+\tfrac{1}{2}v_s^2)+\frac{e}{m}\mathbf{E}.$$

Alternatively, this equation can be written, with use of $\mathbf{E}=-(1/c)(\partial\mathbf{A}/\partial t)-\mathbf{\Delta}\varphi$, where A and φ are the vector and scalar potentials,

$$\frac{\partial\mathbf{v}_s}{\partial t}=-\nabla\mu-\frac{e}{mc}\frac{\partial\mathbf{A}}{\partial t}. \tag{3.3}$$

Here μ is the chemical potential per unit mass:

$$\mu=\mu_0+\tfrac{1}{2}v_s^2+(e/m)\varphi. \tag{3.4}$$

The driving force on the electrons consists of the two terms on the right-hand side of (3.3).

In the case where $\mathbf{v}_s$ is no longer a linear function of $\mathbf{P}$, as discussed in Sec. II, this equation should be modified to

$$\frac{\partial\mathbf{P}}{\partial t}=-\nabla[F(\mathbf{P})+e\varphi]-\frac{e}{c}\frac{\partial\mathbf{A}}{\partial t}=\mathbf{f}, \tag{3.5}$$

where $\mathbf{f}$ is the total force. Multiplying this relation by

$$\mathbf{v}_s=\frac{\partial}{\partial\mathbf{P}}F(P),$$

we get

$$\frac{\partial F}{\partial t}=\mathbf{v}_s\cdot\mathbf{f} \tag{3.6}$$

which is the expression for energy conservation.

[20] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. 37, 1407 (1959) [English transl.: Soviet Phys.—JETP 10, 998 (1960)].

[21] K. Maki, Physics 1, 21 (1964); 1, 127 (1964).

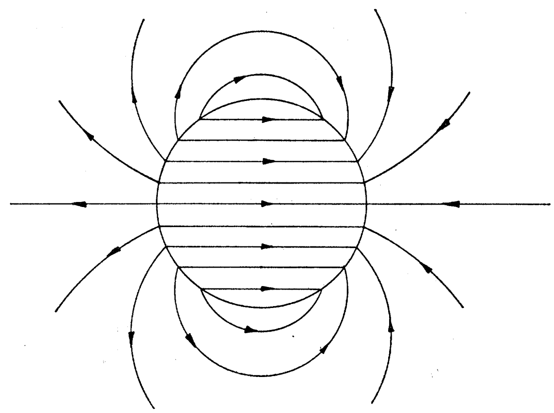

FIG. 4. Schematic diagram of the force field in the vicinity of the core from the free energy gradient, $-\text{grad}\mu$, resulting from motion in the vertical direction.

The momentum for the moving vortex can be written

$$\mathbf{P}(\mathbf{r}-\mathbf{v}_Lt)=\mathbf{P}_0(\mathbf{r}-\mathbf{v}_Lt)+\mathbf{P}_1(\mathbf{r}-\mathbf{v}_Lt), \tag{3.7}$$

where $P_0(\mathbf{r})$ is the momentum which gives the superfluid flow of a stationary vortex and $P_1\ll P_0$ gives the modification of the superfluid flow arising from the motion. We assume that the penetration depth $\lambda\gg\xi_0$, so that one can take the magnetic field uniform near the core. Then $\mathbf{P}_0$ will have only a θ component given by (see Fig. 1 for notation)

$$P_0=\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)_\theta=-\frac{\hbar}{2r}+\frac{e}{2c}rH, \tag{3.8}$$

where H is the magnetic field in the core (taken along the $-z$ direction if e is positive). Assuming that all quantities $(\mathbf{P},\mathbf{A},\varphi)$ vary with time like $\mathbf{P}(\mathbf{r}-\mathbf{v}_Lt)$ the driving force $\mathbf{f}$ on the electrons outside of the core is

$$\mathbf{f}=-(\mathbf{v}_L\cdot\nabla)\mathbf{P}. \tag{3.9}$$

To terms of order v_L, we may take $\mathbf{P}=\mathbf{P}_0$ in this expression. In cylindrical coordinates r, θ this force has components

$$\begin{aligned} f_r&=(\mathbf{v}_L\times\mathbf{k})_r(P_0/r),\\ f_\theta&=(\mathbf{v}_L\times\mathbf{k})_\theta(\partial P_0/\partial r), \end{aligned} \tag{3.10}$$

where $\mathbf{k}$ is a unit vector along the z direction.

In our model in which quasiparticle excitations relax to the lattice, we obtain the field in the core by continuing the force field (3.10). This leads to a uniform electric field E_c in the core given by

$$e\mathbf{E}_c=(\mathbf{v}_L\times\mathbf{k})(\partial P_0/\partial r)_a. \tag{3.11}$$

The chemical potential μ in Eq. (3.5) will be continuous across the boundary of the vortex core, corresponding to equilibrium of the core boundary in the lattice frame of reference. Since $\nabla F=0$ at the boundary, the tangential component of the electric field is continuous, but there is a discontinuity in the radial component. This arises from a surface charge density at the core boundary of magnitude $\sigma=(\hbar v_L/4\pi ea^2)\cos\theta$. This surface charge density is an unrealistic feature of the local model; in the true nonlocal theory it would be spread out over a radial distance $\sim\xi_0$. The field pattern is shown in Fig. 4.

The normal current flow in the transition region outside the core may be determined as follows. We begin by omitting Hall effects and so assume that the current is parallel with the driving field. Provided a relaxation time τ exists, the electrons may be regarded as being accelerated freely for a time τ under the force $\mathbf{f}$. The total velocity $\mathbf{v}=\mathbf{v}_s+\mathbf{v}_n$ at a time t is then

$$\mathbf{v}_s(t)+\mathbf{v}_n(t)=\mathbf{v}_s(t-\tau)+(\mathbf{f}/m)\tau. \tag{3.12}$$

Note that $\mathbf{v}_s(t-\tau)$ corresponds to the quasi-equilibrium at time $t-\tau$. The supercurrent flow is modified by the motion, and $\mathbf{v}_s$ is the total superfluid velocity

$$\mathbf{v}_s=\mathbf{v}_s(\mathbf{P}_0+\mathbf{P}_1)=\mathbf{v}_s(P_0)+\mathbf{v}_{s1}, \tag{3.13}$$

where $\mathbf{P}_1$ and the corresponding $\mathbf{v}_{s1}$ are the corrections from the transport current. When $P<P_d$, so that $m\mathbf{v}_s(P)=\mathbf{P}$, P_1 is just equal to the momentum of the transport current $\mathbf{P}_T$, and $\mathbf{v}_{s1}=\mathbf{v}_T$. However, $\mathbf{P}$ will differ from $\mathbf{P}_T$ in the transition region, $P_d<P<P_c$.

Since $\mathbf{v}_s(\mathbf{P})$ is parallel to $\mathbf{P}$, we must distinguish between adding a component $\mathbf{P}_{1\theta}$ parallel to $\mathbf{P}_0$ and a component $\mathbf{P}_{1r}$ perpendicular to $\mathbf{P}_0$. Expansions valid for $P_1\ll P_0$ are

$$\begin{aligned} \mathbf{v}_{s\theta}&=\mathbf{v}_s(\mathbf{P}_0+\mathbf{P}_{1\theta})=\mathbf{v}_s(P_0)+\mathbf{P}_{1\theta}(\partial v_s/\partial P_0),\\ \mathbf{v}_{sr}&=\mathbf{P}_{1r}(v_s(P_0)/P_0). \end{aligned} \tag{3.14}$$

Similar considerations apply to the terms in τ in (3.12). To quantities of the first order in $\mathbf{v}_T$ and $\mathbf{v}_L$, we may replace $\mathbf{P}$ by $\mathbf{P}_0$ in the difference $\mathbf{v}_s(t-\tau)-\mathbf{v}_s(t)$, and find:

$$\begin{aligned} [\mathbf{v}_s(t-\tau)-\mathbf{v}_s(t)]_\theta&=-\mathbf{f}_\theta\tau(\partial v_s(P_0)/\partial P_0),\\ [\mathbf{v}_s(t-\tau)-\mathbf{v}_s(t)]_r&=-\mathbf{f}_r\tau(v_s(P_0)/P_0). \end{aligned} \tag{3.15}$$

For simplicity we will assume that the vortex is moving in the y direction. Then

$$\mathbf{f}_\theta=-v_L\sin\theta(\partial P_0/\partial r),\quad \mathbf{f}_r=v_L\cos\theta(P_0/r). \tag{3.16}$$

Thus the normal current is given by

$$\begin{aligned} mv_{n\theta}&=-v_L\tau\sin\theta\frac{\partial P_0}{\partial r}\left(1-m\frac{\partial v_s}{\partial P_0}\right),\\ mv_{nr}&=v_L\tau\cos\theta\frac{P_0}{r}\left(1-m\frac{v_s}{P_0}\right). \end{aligned} \tag{3.17}$$

A general form for v_s, which includes the possibility of backflow and reduces to the correct form at large distances from the core, is

$$\begin{aligned} v_{s\theta}&=v_s(P_0)-\left[v_T+v_B\frac{a^2}{r^2}-\frac{v_L\tau}{m}\frac{\partial P_0}{\partial r}\left(1-m\frac{\partial v_s(P_0)}{\partial P_0}\right)\right]\sin\theta,\\ v_{sr}&=\left[v_T-v_B\frac{a^2}{r^2}-\frac{v_L\tau}{m}\frac{P_0}{r}\left(1-m\frac{v_s(P_0)}{P_0}\right)\right]\cos\theta. \end{aligned} \tag{3.18}$$

Both (3.17) and (3.18) satisfy the condition div $\mathbf{v}=0$. We have chosen $v_{s\theta}$ and v_{sr} so that the total flow outside the core is

$$\begin{aligned} v_\theta &= v_s(P_0)-[v_T+v_B(a^2/r^2)]\sin\theta\,,\\ v_r &= \qquad\qquad [v_T-v_B(a^2/r^2)]\cos\theta\,. \end{aligned} \tag{3.19}$$

Thus at large distances from the core, $\mathbf{v}$ reduces to the transport current $\mathbf{v}_T$. With continuity of current, this corresponds to a velocity in the core

$$v_x=v_T-v_B\,.$$

There is a discontinuity in v_θ between the inside and outside of the core given by

$$v_{\theta\,\mathrm{in}}-v_{\theta\,\mathrm{out}}=2v_B\sin\theta\,.$$

We shall show that for our model, $v_B=0$. The relation between v_L and v_T is derived in the following section.

IV. DISSIPATION AND VISCOSITY

The viscosity η is determined from the total rate of energy dissipation D,

$$D=\eta v_L^2\,, \tag{4.1}$$

and the velocity v_L is determined by equating the viscous drag with the applied force $(J_T\times\varphi_0)/c$. With $\mathbf{J}_T=J_{Tx}=nev_T$ and $\varphi_0=\pi\hbar c/e$, the vortex moves in the y direction (disregarding the Hall effect) and

$$\eta v_{Ly}=n\pi\hbar v_{Tx}\,. \tag{4.2}$$

There is a uniform electric field in the core in the x direction which from (3.11) is given by

$$eE_{cx}=v_{Ly}(\partial P/\partial r)_a\,. \tag{4.3}$$

The velocity in the core is then

$$v_c=(v_L\tau/m)(\partial P/\partial r)_a\,, \tag{4.4}$$

and the dissipation inside the core is

$$D_{\mathrm{in}}=\pi a^2 ne\mathbf{v}_c\cdot\mathbf{E}_c=\pi a^2 n(v_L^2\tau/m)(\partial P_0/\partial r)_a^2\,. \tag{4.5}$$

The dissipation arising from the normal current outside the core is from (3.16) and (3.17):

$$\begin{aligned} D_{\mathrm{out}} &= n\int_a^\infty rdrd\theta\,\mathbf{v}_n\cdot\mathbf{f}\\ &= \frac{\pi n v_L^2\tau}{m}\int_a^\infty rdr\bigg[\left(\frac{\partial P_0}{\partial r}\right)^2\left(1-m\frac{\partial v_s}{\partial P_0}\right)\\ &\qquad\qquad +\left(\frac{P_0}{r}\right)^2\left(1-m\frac{v_s}{P_0}\right)\bigg]\,. \end{aligned} \tag{4.6}$$

From (3.8), $P_0=-(\hbar/2r)(1-br^2)$, where $b=eH/\hbar c$, and substituting in (4.6) we find

$$\begin{aligned} D_{\mathrm{out}}=\frac{\pi n v_L^2\tau}{m}\bigg\{&\frac{\hbar^2}{2}\int_a^{a_2} r^{-3}dr(1+b^2r^4)\\ &-m\int_a^{a_2} rdr\frac{\partial v_s}{\partial r}\frac{\partial P_0}{\partial r}-m\int_a^{a_2}dr\frac{P_0}{r}v_s\bigg\}\,, \end{aligned} \tag{4.7}$$

where a_2 is such that $mv_s=P_0$ for $r>a_2$. The first integral is simply evaluated

$$\frac{\hbar^2}{2}\int_a^{a_2} r^{-3}dr(1+b^2r^4)=\frac{\hbar^2}{4}\left[-\frac{1}{r^2}+b^2r^2\right]_a^{a_2}\,. \tag{4.8}$$

Integrating by parts, we have

$$\begin{aligned} -m\int_a^{a_2} rdr\frac{\partial v_s}{\partial r}\frac{\partial P_0}{\partial r} &= -\frac{\hbar m}{2}\left[v_s\left(\frac{1}{r}+br\right)\right]_a^{a_2}\\ +m\int_a^{a_2} v\frac{P_0}{r}dr &= \frac{\hbar^2}{4}\left(\frac{1}{a_2^2}-b^2a_2^2\right)+m\int_a^{a_2} v\frac{P_0}{r}dr\,. \end{aligned} \tag{4.9}$$

In (4.9) the first term cancels the upper limit in (4.8) and the last term the last integral in (4.7), so that we are left with

$$D_{\mathrm{out}}=\frac{\pi n v_L^2\tau\hbar^2}{4m}\left(\frac{1}{a^2}-b^2a^2\right)\,. \tag{4.10}$$

The total dissipation is

$$D=D_{\mathrm{in}}+D_{\mathrm{out}}=(2\pi na^2/m)(v_L\hbar/2a^2)^2\tau(1+ba^2)\,. \tag{4.11}$$

The viscosity coefficient η is the coefficient of v_L^2 in (4.11)

$$\eta=(2\pi na^2/m)(\hbar/2a^2)^2\tau(1+ba^2)\,. \tag{4.12}$$

Equating the viscous drag ηv_L with the applied force due to the transport current [Eq. (4.2)], we find

$$v_{Ly}=2ma^2v_{Tx}/\hbar\tau(1+ba^2)\,. \tag{4.13}$$

The velocity of the electrons in the core is, from (4.4),

$$v_{cx}=\frac{v_{Ly}\tau}{m}\left(\frac{\partial P}{\partial r}\right)_a=\frac{v_{Ly}\tau}{m}\frac{\hbar}{2a^2}(1+ba^2)=v_{Tx}\,. \tag{4.14}$$

Thus the velocity in the core is equal to the transport velocity for all fields up to H_{c2} and backflow v_B is zero. The total current density is the sum of $\mathbf{J}_T$ and $\mathbf{J}_S(P_0)$, as indicated in (1.1).

In order to compare with experiment and Kim's empirical relation (1.5) it is necessary to relate the core radius a with H_{c2}. We have seen in Sec. II that for $H\ll H_{c2}$, we should have $H_{c2}\simeq\hbar c/(2ea^2)$. Without a more detailed model, we do not know how P_c varies with H, but we do know that $P_c\to 0$ as $H\to H_{c2}$. From (3.8), this implies that near H_{c2},

$$H_{c2}\simeq\hbar c/ea^2\,.$$

If we assume a linear extrapolation, we would have

$$\hbar c/2ea^2=H_{c2}-\tfrac{1}{2}H_a\,, \tag{4.15}$$

where $H_a\simeq B$ is the applied field. When this expression is substituted into (4.12), we find exactly the empirical relation (1.5). Thus the model leads at least to a close approximation to the empirical rule.

In determining the core radius in Sec. II from the condition $P=P_c$, we have omitted the transport current. The effect of superimposing the transport current on the flow due to the vortex is to displace the cylindrical surface at which $P\rightarrow P_c$ with respect to the center of rotation. Let $\mathbf{P}_1$ be the momentum associated with the transport current. We can identify $\mathbf{P}_1$ by comparing (3.14) with (3.18). Thus omitting v_B in (3.18), we find

$$\frac{\partial v_s(P_0)}{\partial P_0}P_{1\theta}=-\left[v_T-\frac{v_L\tau}{m}\frac{\partial P_0}{\partial r}\left(1-m\frac{\partial v_s(P_0)}{\partial P_0}\right)\right]\sin\theta. \quad (4.16)$$

With use of (4.4) and (4.14), this gives near $r=a$,

$$P_{1\theta}=-mv_T\sin\theta. \quad (4.17)$$

The expression for P_{1r} is more complicated but will not be required. When P is very close to P_c, it is no longer true that P_{1r} is small so that (3.14) requires modification.

The total superfluid momentum is (for $H\ll H_{c2}$)

$$P_\theta=-(\hbar/2r)+P_{1\theta},\quad P_r=P_{1r}. \quad (4.18)$$

We now displace the origin of coordinates a distance y_0 along the y axis, where y_0/a is assumed small. In the new coordinate system,

$$\begin{aligned}P_\theta&=-(\hbar/2r)+(\hbar y_0/2r^2)\sin\theta+P_{1\theta},\\ P_r&=(\hbar y_0/2r^2)\cos\theta+P_{1r}.\end{aligned} \quad (4.19)$$

Thus from (4.17) if we choose

$$\hbar y_0=2ma^2v_T, \quad (4.20)$$

then $P_\theta=-\hbar/2a$ at the boundary $r=a$ as before. It is interesting to note from (4.13) that y_0 corresponds to the distance the vortex will move in a time τ, i.e., $y_0=v_L\tau$. The relaxation time of the vortex is approximately τ and when this is taken into account one should take the position of the cylinder at time $t-\tau$. This displaces the cylinder back to its original position. Thus the transport current should not have any important effect in determining the core boundary in the case of small currents.

The total force exerted on the normal electrons in the core by the electric field E_c is

$$F_{x\,\mathrm{in}}=\pi a^2neE_c=\tfrac{1}{2}\pi n\hbar v_L(1+ba^2). \quad (4.21)$$

The total integrated force in the x direction on the normal electrons in the transition region is from (3.17)

$$F_{x\,\mathrm{out}}=n\pi v_L\int_a^\infty rdr\left[\frac{\partial P_0}{\partial r}\left(1-m\frac{\partial v_s}{\partial P_0}\right)+\frac{P_0}{r}\left(1-m\frac{v_s}{P_0}\right)\right]. \quad (4.22)$$

This integral may be evaluated in a similar manner to (3.23) with the result

$$F_{x\,\mathrm{out}}=\tfrac{1}{2}\pi n\hbar v_L(1-ba^2). \quad (4.23)$$

The total force is thus

$$F_x=F_{x\,\mathrm{in}}+F_{x\,\mathrm{out}}=\pi n\hbar v_L, \quad (4.24)$$

and corresponds exactly to the Magnus force. The normal electrons lose momentum at this rate to the lattice. Under the steady conditions, such as those used in measurements of resistivity, this loss of momentum is supplied by the external battery.

V. HALL EFFECT AND EFFECTIVE MASS

In the preceding sections we have considered only the component of motion of the vortex line v_{Ly} normal to the transport current $\mathbf{J}_T=\mathbf{J}_{Tx}$. As a result of Hall fields in the core, there will also be a component v_{Lx} parallel with $\mathbf{J}_T$, giving rise to an electric field within the core E_{cy} perpendicular to $\mathbf{J}_T$. As illustrated in Fig. 1, the total electric-field vector within the core is at the Hall angle relative to $\mathbf{J}_T$, and $\mathbf{v}_L$ is at the same angle relative to the force on the vortex line due to $\mathbf{J}_T$. The Hall angle α is that of the normal metal at a field equal to that in the core, thus

$$\tan\alpha=(e\tau/mc)H=\omega_c\tau, \quad (5.1)$$

where ω_c is the cyclotron frequency.

There is no force and no viscous drag associated with the component parallel with $\mathbf{v}_T$. The rate of energy dissipation in the core $\mathbf{J}_T\cdot\mathbf{E}_c$ depends only on E_{cx} and is independent of E_{cy}. In our model, the Hall angle in the transition region is the same, so that the Hall fields associated with v_{Lx} produce no currents and give no dissipation. It has been shown by Miller[22] that in the local limit, the Hall angle associated with the normal component of flow in the superconducting state is the same as that in the normal state.

When Hall terms are included, the drift velocity of the electrons in the core is given by

$$\mathbf{v}_c=(e\tau/m)\mathbf{E}_c+(e\tau/mc)(\mathbf{v}_c\times\mathbf{H}). \quad (5.2)$$

The electric field $\mathbf{E}_c$ normal to $\mathbf{v}_L$ has components

$$eE_{cx}=v_{Ly}(\partial P_0/\partial r)_a,\quad eE_{cy}=-v_{Lx}(\partial P_0/\partial r)_a. \quad (5.3)$$

In determining the normal current outside of the core we now take into account the Lorentz force acting on the normal component so that (3.12) is replaced by

$$\mathbf{v}_s(t)+\mathbf{v}_n(t)=\mathbf{v}_s(t-\tau)+(\mathbf{f}\tau/m)+(e\tau/mc)(\mathbf{v}_n\times\mathbf{H}) \quad (5.4)$$

and the normal components of the current are

$$v_{n\theta}=\frac{F_\theta+\omega_c\tau F_r}{1+\omega_c^2\tau^2},\quad v_{nr}=\frac{F_r-\omega_c\tau F_\theta}{1+\omega_c^2\tau^2}, \quad (5.5)$$

[22] P. B. Miller, Phys. Rev. **121**, 445 (1961).

where

$$F_\theta=\frac{f_\theta\tau}{m}\left(1-m\frac{\partial v(P_0)}{\partial P_0}\right),$$
$$F_r=\frac{f_r\tau}{m}\left(1-m\frac{v(P_0)}{P_0}\right),\tag{5.6}$$

and $\mathbf{f}$ is given by (3.10).

The energy dissipation in the core is

$$D_{\text{in}}=\pi a^2ne\mathbf{v}_c\cdot\mathbf{E}_c=\pi a^2n\frac{v_{Ly}{}^2\tau}{m}\left(\frac{\partial P_0}{\partial r}\right)_a^2,\tag{5.7}$$

which is exactly (4.5) except that v_{Ly} replaces v_L. Similarly the dissipation occurring outside the core associated with the normal current (5.5) and the total dissipation are the same as (4.10) and (4.11) with $v_L=v_{Ly}$. It is probably best to regard the viscous drag as opposing the applied force, and thus acting at an angle α relative to $\mathbf{v}_L$. Thus equating D to $\eta v_{Ly}{}^2$ we are led to the value of η given by (4.12).

If the motion of the line is produced by the force due to the transport current then v_{Ly} is again given by (4.13) and it is found that the drift velocity of the electrons in the core remains equal to $\mathbf{v}_T$. The Hall effect gives rise to a component of velocity of the vortex parallel to $\mathbf{v}_T$ which is

$$v_{Lx}=v_{Ly}\tan\alpha=\frac{2ma^2}{\hbar\tau}\frac{v_T}{1+ba^2}\left(\frac{e\tau H}{mc}\right).\tag{5.8}$$

If we use (4.15) to express a in terms of H_{c2}, we find a simple relation between v_{Lx} and v_{Tx}:

$$v_{Lx}=v_{Ly}\tan\alpha=(H/H_{c2})v_{Tx}.\tag{5.9}$$

Note that v_{Lx} is independent of τ and is equal to v_{Tx} only in the limit $H\to H_{c2}$. Near H_{c1} it is smaller in the ratio H_{c1}/H_{c2}. With increasing τ, the quantity v_{Ly} decreases while v_{Lx} remains the same.

This result implies that the Hall angle as measured in the mixed state should be equal to that of the normal metal for a field equal to the field in the core of the vortex. Experiments of Reed, Fawcett, and Kim[4] on relatively pure niobium appear to be consistent with this picture. On the other hand, measurements on alloys by Niessen and Staas[4] yield a Hall angle which increases as H drops below H_{c2} and becomes larger than that observed in the normal state for a field $H=H_{c2}$. These latter results are difficult to understand on the basis of the present model. Helicon effects should be observable in the mixed state when the Hall angle is sufficiently large.

If there is an additional force on the vortex line other than that produced by the transport current $\mathbf{J}_T$, the drift velocity $\mathbf{v}_c$ in the core will be normal to the total force and $\mathbf{v}_L$ will be at the Hall angle relative to this force. In this case, $\mathbf{v}_c$ will not in general be equal to $\mathbf{v}_T$, so that continuity of current requires a backflow outside of the core.

According to our model, the relaxation time for the vortex line is equal to that of electrons in the normal metal. This result also follows from Suhl's estimate of the effective mass of the line based on terms from $[\partial|\psi|\partial t]^2$ in the free energy. This is closely connected with Tinkham's relaxation mechanism arising from changes in the gap parameter just outside of the core. In our model, about half of the dissipation and about half of the effective mass occur inside the core and half in the transition region just outside of the core.

In order to estimate the effective mass, we first take $\mathbf{J}_T=0$ and consider the relaxation to rest of the motion of a vortex line initially moving at a velocity $\mathbf{v}_L$. When moving, there is a current through the core and an associated backflow outside of the core required by current continuity. The kinetic energy per unit length associated with the backflow is from, (3.18) with $v_B=v_c$,

$$\pi\rho\int_a^\infty v_c{}^2\left(\frac{a^2}{r^2}\right)^2rdr=\tfrac{1}{2}\pi\rho a^2v_c{}^2.\tag{5.10}$$

This is just equal to the kinetic energy within the core from flow with a velocity v_c. The total kinetic energy is

$$\text{K.E.}=\tfrac{1}{2}Mv_L{}^2=\pi\rho a^2v_c{}^2.\tag{5.11}$$

We may express v_c in terms of v_L from (4.14) with $v_{Ly}=v_L\cos\alpha$. Thus we find that M, the effective mass per unit length, is

$$M=\frac{2\pi na^2}{m}\left(\frac{\hbar}{2a^2}\right)^2\tau^2(1+ba^2)^2\cos^2\alpha$$
$$=\eta\tau\cos^2\alpha(1+ba^2).\tag{5.12}$$

Since the rate of dissipation is $\eta v_L{}^2\cos^2\alpha$, the relaxation time of the vortex line is approximately equal to τ, as expected for our model. By expressing τ in terms of $\tan\alpha$, with the use of (5.1), and by using (4.15), we find a simple form for M:

$$M=2\pi nma^2(H_{c2}/H)^2\sin^2\alpha.\tag{5.13}$$

When H is near H_{c2} and the Hall angle is near 90°, $\sin\alpha\simeq1$, and $M\to 2\pi nma^2$. In this limit the mass M is just twice that of the electrons within the core. When α is small, the effective mass is smaller in the ratio $[\omega_c(H_{c2})\tau]^2$.

When $J_T\neq0$, the relaxation will be to the steady-state motion of the vortex line described earlier. The kinetic energy is

$$nm\pi a^2(v_c-v_T)^2=\tfrac{1}{2}M\{(v_{Lx}-v_{Lx0})^2+(v_{Ly}-v_{Ly0})^2\},\tag{5.14}$$

where v_{Lx0} and v_{Ly0} are the velocities for steady-state motion of the vortex line as defined by (5.8) and (4.13). If a transport current $\mathbf{J}_T$ is applied and the vortex starts from rest, there initially will be kinetic energy from backflow, when $\mathbf{v}_c \neq \mathbf{v}_T$, which is dissipated as $\mathbf{v}_c \to \mathbf{v}_T$.

VI. IMPURITY SCATTERING, FINITE TEMPERATURE

When there is strong impurity scattering such that the m.f.p. $l < \xi_0$, or at finite temperatures, the above theory requires modification since the superfluid density ρ_s is then less than $\rho = nm$, the total density. For a given v_T, the transport current and thus the force on the vortex line is reduced by a factor ρ_s/ρ. The expression for the viscosity coefficient, when given in terms of the core radius a or H_{c2}, is substantially unchanged, so that v_L is also reduced by ρ_s/ρ. This means that the field and drift velocity within the core are reduced by ρ_s/ρ and the dissipation by $(\rho_s/\rho)^2$. The transport current will still tend to flow directly through the core with little or no backflow.

Changes in η, other than those due to changes in core radius, may occur as a result of different mechanisms for dissipation outside of the core. The field within the core will still be given by (3.11) and the corresponding dissipation by (4.5). If the impurity scattering predominates over lattice scattering, the usual situation, the relaxation time τ will be independent of temperature. At finite temperatures, however, the presence of quasiparticle excitations requires modification of the expressions used for calculating the dissipation. Further, when $l < \xi_0$, even at $T = 0°$K, the local model we have used is unsatisfactory.

A limiting case, valid for all values of l, which can be treated without difficulty is $T \to T_c$, for then the conductivity of the superfluid will approach that of the normal metal. The total dissipation outside of the core is then

$$\pi \sigma_n v_{Ly}^2 \int_a^{a_2} \left\{ \left(\frac{\partial P_0}{\partial r} \right)^2 + \left(\frac{P_0}{r} \right)^2 \right\} r dr. \tag{6.1}$$

If a uniform magnetic field is assumed, the upper limit of integration must be chosen so that the total flux enclosed is one quantum, or

$$\pi a_2^2 H = \pi h c / e. \tag{6.2}$$

When the dissipation outside of the core is added to that inside, and a is expressed in terms of $H_{c2}(T)$ by means of (4.15), one again finds a result for η very close to Kim's empirical expression (1.5), but now with $H_{c2}(T)$ in place of $H_{c2}(0)$.

The reason that the local model we have used can not be applied directly to the impure case, $l \ll \xi_0$, is that changes in the gap parameter Δ occur with increasing momentum P long before one reaches the gapless region. In our model it is assumed that $mv_s = P$ for $P < P_d$, when no depairing can take place, and it was reasonable to assume a simple relaxation time for the quasiparticles for $P > P_d$. For $T = 0°$K, in the local limit of the impure case, there is an energy gap and thus no dissipation until P reaches the value for gapless superconductivity. This occurs for P very close to P_c. The dissipation outside the core calculated on the basis of the local model is quite small, less than 10% of that inside the core.[23] While the dissipation in the core is unchanged, this would lead to a value of η not much more than half the empirical one, and the current density in the core would be almost twice the transport current density J_T, implying that there would be backflow.

The correct nonlocal model would very likely lead to an additional dissipation associated with changes of Δ in time, as suggested by Tinkham, and in regard to effective mass, by Suhl. Tinkham shows that if the relaxation time for changes in Δ were the order of $\tau_{\text{eff}} = \xi_0 / v_F$, one would get the right order of magnitude. Note that for the impure case τ_{eff} is longer than the collision relaxation time $\tau_{\text{coll}} = l / v_F$. By use of a time-dependent version of the Ginzburg-Landau equations, derived by Suhl and Stephen, Suhl has estimated the effective mass M of the moving vortex from terms involving $[\partial |\psi| / \partial t]^2$. When combined with the empirical expression for the viscosity, a relaxation time of $\sim \tau_{\text{col}}$ is obtained. The latter is more reasonable than Tinkham's estimate. It would be desirable to have a complete time-dependent version of the Ginzburg-Landau equations for the impure case.

VII. CONCLUSIONS

We have seen that a local but otherwise rather general model of a relatively pure superconductor ($l > \xi_0$) leads to a flow pattern of a moving vortex which is a superposition of a uniform transport current density $\mathbf{J}_T$ and the adiabatic motion of the flow pattern of a stationary vortex $\mathbf{J}_0(r - v_L t)$. Part of $\mathbf{J}_T$ represents supercurrent flow, part normal current flow. Electric fields set up by the motion drive the current through the normal core. The force on the vortex is normal to $\mathbf{J}_T$, but as a result of the Hall effect of the normal component in the region of the core, the motion is at the Hall angle α relative to the force. The component of velocity in the direction of $\mathbf{J}_T$ is $(H/H_{c2}) v_T$, independent of the relaxation time τ. This component leads to a Hall field in the mixed state with a Hall angle α. The model leads to an expression for bulk resistivity equivalent to the one Kim and coworkers derived empirically.

One might hope that these conclusions would remain essentially the same in the nonlocal theory. There would no longer be a sharp boundary to the core; the driving force from $-\text{grad}\mu$ would be a superposition of a distribution of fields of type shown in Fig. 4 with different core radii. The space integral of this force, however, would be half of the Magnus force, $-(ne/c)(\mathbf{v}_L \times \boldsymbol{\varphi}_0)$, as in the local model, the other half coming from the field $-(1/c)\partial \mathbf{A}/\partial t$. It would be desirable to show more generally, if indeed it is true in the nonlocal theory,

[23] V. Galaiko (private communication).

that the transport current flows directly through the core.

Helicon effects should be observable when the Hall angle is sufficiently large. While macroscopically they are related to the Hall effect, as in normal metals, in the mixed state they are associated with helical waves on vortex lines, such as those which have been observed in superfluid helium. The dispersion relation for such a wave on an isolated vortex for $k\lambda \gg 1$ is

$$\omega = \frac{H}{H_{c2}} \frac{\hbar k^2}{4m} \ln\frac{\lambda}{a}, \qquad (7.1)$$

where the factor H/H_{c2} comes from the ratio v_{Lx}/v_T [Eq. (5.9)]. This factor is missing in the expression of de Gennes and Matricon[11] based on Magnus force concepts.

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g Factor of Conduction Electrons in Metallic Lithium*

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The electron spin resonance of metallic lithium particles immersed in mineral oil has been carefully examined in order to determine the g shift of the conduction electrons with respect to the free-electron value. By comparing the ESR frequency with the proton nuclear-spin frequency of the mineral oil, and extending Dyson's line-shape theory to apply to intermediate-size metallic particles, the shift at room temperature was found to be $\Delta g = -2\times10^{-6}$. This small g shift indicates that the inherent linewidth of the ESR in metallic lithium may well be only 4 mG in a 3000-G field. In addition, an Overhauser enhancement of over 750 was measured when the lithium NMR was monitored and the ESR saturated. The shift in the lithium NMR arising from ESR saturation demonstrated that at least 75% of the measured Knight shift is due to electron-gas paramagnetism.

I. INTRODUCTION

THE shift of the alkali-metal conduction-electron g value from that of the free electron is primarily due to the spin-orbit interaction and as such is a measure of the $\Delta l > 0$ terms in the expansion of the wave function of the conduction electrons. This theoretical g shift for the alkali metals has been calculated by Bienenstock and Brooks[1] and found to agree with experimental data for sodium,[2] the alkali most amenable to measurement. The theory does not give good numerical values for lithium, the metal of interest in this work, but predicts a small shift of parts in 10^{-6}. An earlier theory by Elliot[3] also relates the g shift to inherent linewidth without predicting the absolute value of either.

The experimental measurement of the lithium's g shift is hampered by several factors. Ideally, a sample of metallic lithium should have as narrow an ESR line as possible in order to achieve good signal-to-noise conditions and hopefully approach the inherent linewidth. Experimental data[2] from bulk metallic samples do not give sufficiently good signal-to-noise ratios to permit a precise g-shift measurement, because bulk samples have not been prepared pure enough to narrow the impurity-limited linewidth and because the volume of lithium exposed to the microwaves is limited by the electromagnetic skin depth. The narrowest linewidths observed[4] (0.1 G in a 3000-G field), arise from lithium-particle conglomerates formed by ultraviolet irradiation of LiH. However, with these submicron-sized conglomerates, one cannot be certain that surface properties do not distort the metallic crystalline structure.

In this work clusters of intermediate-size particles were used as samples. This has the advantages of guaranteeing that the sample is indeed metallic, allowing purification techniques that cause an impurity-broadened line to sharpen considerably, and also increasing the ratio of surface area to volume so that the microwaves can penetrate a large fraction of the sample and give a very large signal-to-noise ratio. However, in

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