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Relativistic Shock Structure*

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The equations for a relativistic fluid with viscous and thermal dissipation are discussed and are applied to a plane shock wave in ionized hydrogen. It is shown that if the shock velocity is greater than a critical value ($v/c>\sqrt{\frac{2}{3}}$ at low temperatures), the shock profiles show an increase in velocity or a decrease in the heat-transfer parameter γT at the upstream end of the shock layer. This qualitative change from the non-relativistic case is due to the relativistic interaction of heat transfer and momentum transfer.

1. INTRODUCTION

THE study of simple problems often helps to bring into focus the new effects predicted by a physical theory. The shock structure problem plays such a role in relativistic gas dynamics because the relativisitic interaction of heat transfer and momentum transfer brings about, for sufficiently strong shocks, shock curves which are qualitatively different from those predicted by the nonrelativistic theory.

The formulation of the relativistic shock structure problem is made possible by recent studies[1–3] of relativistic kinetic theory, which have in turn shed light on the form of the equations of relativistic fluid dynamics with dissipation, first discussed by Eckart[4] and later by Landau and Lifshitz.[5] Among the results of these studies[1] is a recognition that some arbitrariness exists in the definition of the velocity four-vector $\lambda_i(i=0,1,2,3)$, which leads to differences in the forms of the mass (or particle) flux vector M_i and the heat flux vector q_i. If λ_i is chosen in the direction of mean particle motion, as is done by Eckart[4] and Kelly,[2] M_i keeps its nondissipative form, and q_i consists of two terms, one proportional to the negative temperature gradient and the second to the fluid deceleration along a streamline. If λ_i is chosen in the direction of mean mass-energy transport, as is done by Landau and Lifshitz,[5] M_i changes by a term proportional to the heat flux, and q_i is proportional to the gradient of $(1+\mu)/T$, where μ is the relativistic chemical potential and T is the temperature. It can be shown that the two approaches are equivalent to first order in $q/c(\omega+p)$. For this problem, the former has been found more convenient, and the formulation of the problem in the following section is adapted from the results of Kelly.[2]

2. FORMULATION OF THE PROBLEM

The coordinate system used here is

$$x_i=(ict,x,y,z)\,,\quad i=0,1,2,3$$

for which the metric tensor is

$$g_{ij}=\delta_{ij}\,.$$

For a plane shock wave, with fluid motion in the X direction only, the velocity four-vector λ_i is

$$\lambda_i=(i\gamma,\beta\gamma 0,0)\,,$$

where

$$\beta=v/c\,,\quad \gamma=(1-\beta^2)^{-1/2}\,.$$

v is the speed of the fluid relative to the shock, and c is the speed of light.

The heat flux vector and the viscous stress tensor are, respectively,[2]

$$q_i=\frac{\kappa mc^2}{\sigma^2}s_{ik}\left(\frac{\partial\sigma}{\partial x_k}-\sigma\lambda_j\frac{\partial\lambda_k}{\partial x_j}\right),\tag{2.1}$$

$$\tau_{ij}=-c\eta s_{im}s_{jn}\left(\frac{\partial\lambda_m}{\partial x_n}+\frac{\partial\lambda_n}{\partial x_m}-\tfrac{2}{3}s_{mn}\frac{\partial\lambda_k}{\partial x_k}\right)-\zeta s_{ij}\frac{\partial\lambda_k}{\partial x_k},\tag{2.2}$$

where κ is the thermal conductivity, σ is the reciprocal effective temperature:

$$\sigma=mc^2/kT\,,\tag{2.3}$$

m is the rest mass of the fluid particle, k is Boltzmann's constant, s_{ij} is the "projection tensor"

$$s_{ij}=g_{ij}+\lambda_i\lambda_j\,,\tag{2.4}$$

η is the coefficient of viscosity, ζ is the bulk or second viscosity.

The stress energy tensor takes the form

$$T_{ij}=T_{ij}^0+(q_i\lambda_j+q_j\lambda_i)+\tau_{ij}\,,\tag{2.5}$$

where T_{ij}^0 is the nondissipative stress energy tensor, and the mass flow vector is

$$M_i=\rho c\lambda_i\,,\tag{2.6}$$

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[2] D. C. Kelly, University of Miami (unpublished).
[3] N. A. Chernikov, Dokl. Akad. Nauk SSSR 112, 1030 (1951); Dokl. Acad. Nauk 114, 530 (1957); 133, 84 (1960); 144, 314 (1962); 144, 544 (1962) [English transls.: Soviet Phys.—Doklady 2, 248 (1957); 5, 764 (1961); 7, 414 (1962)].
[4] C. Eckart, Phys. Rev. 58, 919 (1940).
[5] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, London, 1959).

where ρ is the density of the fluid in its proper frame. The equations of relativistic fluid dynamics, $M^i{}_{;i}=0$ and $T^{ij}{}_{;i}=0$, when written in the frame of reference moving with the shock, in which x is the only independent variable, became, after a single integration:

$$M^1=\rho c\beta\gamma=\text{constant}=M\,, \tag{2.7}$$

$$T^{11}=\gamma^2 p+\beta^2\gamma^2 w+2\beta\gamma q/c+\gamma^2\tau=\text{constant}=P\,, \tag{2.8}$$

$$T^{01}=\beta\gamma^2(w+p)+\gamma(1+\beta^2)q/c+\beta\gamma^2\tau =\text{constant}=Q/c\,, \tag{2.9}$$

where

$$q=\frac{\kappa mc^2\gamma^3}{k\sigma^2}\frac{d}{dx}\frac{\sigma}{\gamma}\,, \tag{2.10}$$

$$\tau=-c(\tfrac{4}{3}\eta+\zeta)d\beta\gamma/dx\,; \tag{2.11}$$

w is the (rest+internal) fluid energy per unit proper volume, and p is the fluid pressure.

The gas is assumed to be ionized hydrogen, and radiation and "nuclear chemistry" are ignored so that the gas obeys the following equation of state, modified from the results of Synge.[6]

$$w+p=\rho c^2 H(\sigma)\,, \tag{2.12}$$

$$p=2\rho c^2/\sigma\,, \tag{2.13}$$

where

$$H(\sigma)=G(\sigma)+r_m G(r_m\sigma)\,, \tag{2.14}$$

$$G(\sigma)=K_3(\sigma)/K_2(\sigma)\,; \tag{2.15}$$

$K_n(\sigma)$ is the nth-order modified Bessel function of the second kind, and r_m is the ratio of electron to proton rest masses.

The following approximate equations of state, based on the asymptotic formulas of Synge,[6] are useful:

Region I:

$$H(\sigma)=1+r_m+5/\sigma \quad [\sigma\gtrsim 3/2r_m;\ T\lesssim 10^9\ {}^\circ\text{K}], \tag{2.16}$$

Region II:

$$H(\sigma)=1+13/2\sigma \quad [3/2r_m\gtrsim\sigma\gtrsim\tfrac{2}{3};\ 10^{13}\ {}^\circ\text{K}\gtrsim T\gtrsim 5\times 10^9\ {}^\circ\text{K}], \tag{2.17}$$

Region III:

$$H(\sigma)=8/\sigma \quad [\sigma\lesssim\tfrac{3}{2};\ T\gtrsim 10^{13}\ {}^\circ\text{K}]. \tag{2.18}$$

Region II represents the case of relativistic electrons, and in region III both the electrons and the protons are relativistic.

The above results can be combined into a pair of equations which are the relativistic analog of the usual gas dynamic shock structure equations:

$$L_H\frac{d\theta}{dx}=\frac{2(H-\gamma\gamma_0 H_0)}{B^2\gamma}-\frac{2\theta}{\gamma^2}+\frac{2\omega}{\gamma}=G(\omega,\theta)\,, \tag{2.19}$$

$$L_P\frac{d\omega}{dx}=\frac{1}{\gamma^2}\left(\omega H+\frac{\theta}{\gamma\omega}-1-\omega\gamma B^2 G(\omega,\theta)\right)=F(\omega,\theta)\,, \tag{2.20}$$

where the subscript 0 represents the upstream equilibrium state, $B=P/Mc$, $\theta=2\gamma/\sigma B^2$, $\omega=\beta\gamma/B$, $L_P=(\frac{4}{3}\eta+\zeta)/M$, $L_H=\kappa m/kM$.

Relativistic effects are represented in these equations in a number of ways:

(1) By the parameter B, which ranges from zero (nonrelativistic case) to infinity (extreme relativistic case). A heat transfer term occurs in the momentum equation (2.20) to order B^2.

(2) In the form of $H(\sigma)$, which changes the nature of the equation at high temperature.

(3) In the transport coefficients κ, η, and ζ, whose dependence on the state of the gas must be found from relativistic kinetic theory.

(4) By the explicit and implicit occurrence of γ. In this connection it is most significant that γT, rather than T, is a naturally occurring variable.

To first order in B^2, Eqs. (2.19) and (2.20) become

$$L_H d\theta/dx=3\theta-((1-\omega)^2+A)-B^2\omega^2(3\theta+\omega)\,, \tag{2.21}$$

$$L_P d\omega/dx=\omega+\theta/\omega-1+B^2\omega[(1-\omega)-2\theta+A]\,, \tag{2.22}$$

where

$$A=2(\gamma_0 H_0-(1+r_m))/B^2-1\,.$$

The zeroth order terms in Eqs. (2.21) and (2.22) are identical to the nonrelativistic shock structure equations for a monatomic gas.

3. ANALYSIS OF THE SHOCK CURVE

The qualitative changes in the shock structure curve brought about by relativistic effects are best demonstrated by study of Eqs. (2.19) and (2.20) in the $\theta-\omega$ "phase plane," and application of standard shock structure methods[7] to the problem.

The nature of the shock curve is determined by the functions $F(\omega,\theta)$ and $G(\omega,\theta)$, whose derivatives with respect to their arguments can be written as follows:

$$F_\omega=[(H-4/\sigma)-\beta^2(\sigma H_\sigma+2/\sigma)-2/\sigma\beta^2]/\gamma^2\,, \quad \text{for } F=0 \tag{3.1}$$

$$F_\theta=[1+\beta^2(\sigma^2 H_\sigma+2)/2]/\gamma\omega\,, \tag{3.2}$$

$$G_\omega=2\omega[(\sigma H_\sigma+2/\sigma)+2/\sigma\beta^2]/\gamma^3\,, \tag{3.3}$$

$$G_\theta=-(\sigma^2 H_\sigma+2)/\gamma^2\,, \tag{3.4}$$

where

$$H_\sigma=dH/d\sigma\,.$$

The nature of the singularities at upstream point 0 and downstream point 1 depends on the sign of $(F_\omega G_\theta-F_\theta G_\omega)$ at these points, and from Eqs. (3.1)–(3.4) and the definitions of F and G it follows that

$$F_\omega G_\theta-F_\theta G_\omega=[2\sigma H_\sigma-\beta^2 H(\sigma^2 H_\sigma+2)]/\beta^2\gamma^4\,. \tag{3.5}$$

It can be shown, following the procedure of Synge,[6]

[6] J. L. Synge, *The Relativistic Gas* (North-Holland Publishing Company, Amsterdam, 1957).

[7] D. Gilbarg and D. Paolucci, J. Rat. Mech. Anal. **2**, 617 (1953).

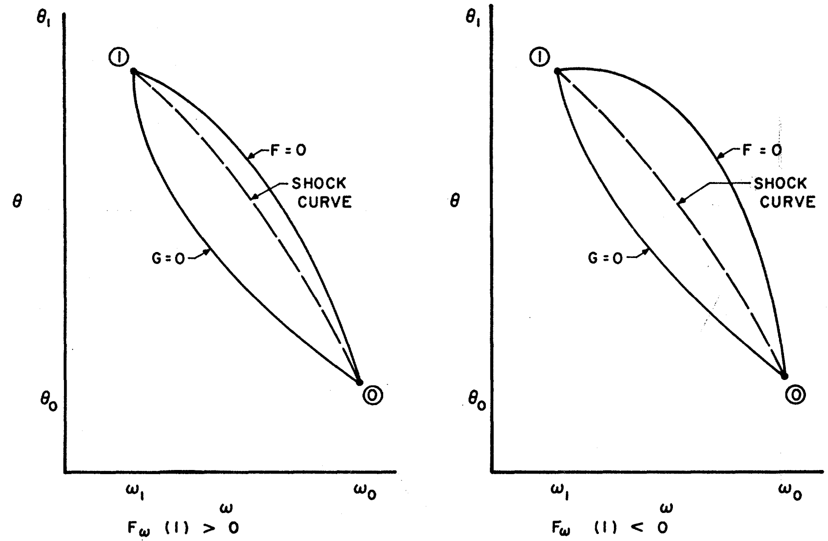

FIG. 1. Bounding curves—case A.

that the signal speed in an ionized relativistic gas is

$$\beta_s^2 = 2\sigma H_\sigma / H(\sigma^2 H_\sigma + 2)\,, \tag{3.6}$$

so that Eq. (3.5) can be written as

$$F_\omega G_\theta - F_\theta G_\omega = 2\sigma H_\sigma (1 - N_M^2)/\beta^2\gamma^4\,, \tag{3.7}$$

where N_M is the Mach number, β/β_s. Therefore, since H_σ is negative, this expression is positive at point 0, for which $N_M > 1$, and negative at point 1, for which $N_M < 1$. Thus, point 0 is a node and point 1 a saddle point. This means that a shock curve exists, and that for the sake of numerical determination of this curve its slope at point 1 is known.

The equation of the shock curve in the phase plane is

$$\frac{d\theta}{d\omega} = \frac{L_P}{L_H}\frac{G(\omega,\theta)}{F(\omega,\theta)}\,. \tag{3.8}$$

The progress of the curve from point 1 to point 0 is determined by the shapes of the curves $F(\omega,\theta)=0$ and $G(\omega,\theta)=0$, whose respective slopes are given by

$$d\theta/d\omega = -F_\omega/F_\theta\,, \tag{3.9}$$

$$d\theta/d\omega = -G_\omega/G_\theta\,. \tag{3.10}$$

There are two distinct cases, one of which is qualitatively similar to nonrelativistic shocks, and the other of which is quite different. The dividing line between these cases is a shock speed of

$$\beta_0^2 = \beta_c^2 = -2/(\sigma^2 H_\sigma + 2)\,, \tag{3.11}$$

which corresponds to

$$N_{M0}^2 = H/(-\sigma H_\sigma)\,. \tag{3.12}$$

From Eqs. (3.2) and (3.3), F_θ and G_ω are greater than 0 for $\beta < \beta_c$ and less than 0 for $\beta > \beta_c$. Since

$$H \geqslant -\sigma H_\theta\,, \tag{3.13}$$

it is clear from Eq. (3.12) that at point 1, the subsonic state, F_θ and G_ω, are both greater than 0. At point 0, however, their sign is determined by the ratio β_0/β_c:

$$\beta_0 < \beta_c \quad (F_\theta > 0, G_\omega > 0) \quad \text{case A}\,, \tag{3.14}$$

$$\beta_0 > \beta_c \quad (F_\theta < 0, G_\omega < 0) \quad \text{case B}\,. \tag{3.15}$$

It can also be shown that $F_\omega > 0$ at point 0, and $G_\theta > 0$ always, so that the slopes of both $F=0$ and $G=0$ at point 0 are negative in case A and positive in case B. Furthermore, the sign of F_ω at point 1 can be either positive or negative, as in the nonrelativistic case, but is always negative in case B.

The shapes of the bounding curves in case A are thus qualitatively similar to those for nonrelativistic shocks, as shown in Fig. 1, and by the argument of Gilbarg and Paolucci,[7] the shock curve remains within the region between the two curves as integration proceeds from point 1 to point 0.

The situation in case B, however, is more complicated, particularly for the curve $F=0$. This curve, shown in Fig. 2, has a singularity when $\beta=\beta_c$. At that point both F_ω and F_θ change sign and analysis shows that on the left leg of the curve $\theta\to\infty$. The curve $G=0$ has a horizontal slope at $\beta=\beta_c$. Qualitatively, there are two possible paths along which the shock curve can reach point 0, as shown by the dashed lines in Fig. 2. One of these (case B1) crosses $F=0$ with a vertical tangent ($d\omega/dx=0$), and the second (case B2) crosses $G=0$ with a horizontal tangent ($d\theta/dx=0$). The particular path will be determined by the shock conditions and the ratio L_P/L_H, which in turn depends on the relativistic transport properties η, ζ, and κ which are as yet unknown. The shapes of the shock curve (γT and v versus x) for cases B1 and B2 are shown in Fig. 3. In case B1 the fluid velocity relative to the shock increases before it decreases, and in case B2 the product γT within the shock decreases before it increases.

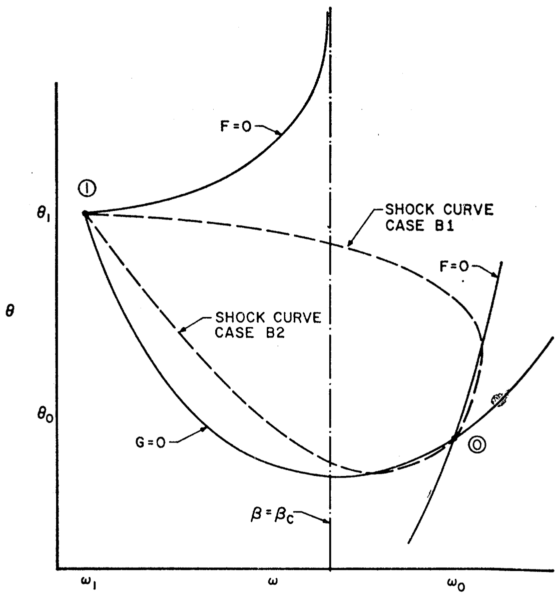

FIG. 2. Bounding curves—case B.

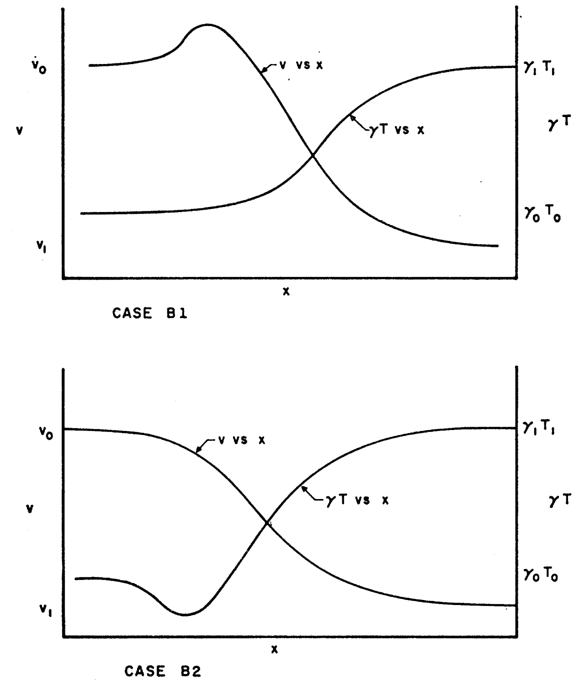

FIG. 3. Shock curves—case B.

This phenomenon of the velocity rise or γT drop in a relativistic shock, for $\beta>\beta_c$ is a striking example of what Eckart calls "the momentum of heat." That is, momentum transfer and heat transfer within the upstream end of the shock layer interact in such a way that γT cannot rise monotonically unless the resulting heat transfer imparts some extra momentum to the fluid, or alternatively a monotonic decrease in the fluid momentum can only be brought about by a drop in γT and the resultant heat transfer.

It is of interest to investigate the circumstances under which the drop in γT is accompanied by a decrease in temperature itself. Since the minimum in θ/γ occurs at $G=0$, it is sufficient for this purpose to find the sign of $d(\theta/\gamma)/d\omega$ along this curve:

$$\frac{d}{d\omega}\left(\frac{\theta}{\gamma}\right)=\left(\frac{1}{\gamma}\frac{d\theta}{d\omega}\right)_{G=0}-\frac{\theta}{\gamma^2}\frac{d\gamma}{d\omega}=\frac{\beta\theta B}{\gamma^2}\left[\gamma\left(1-\frac{\beta_c^2}{\beta^2}\right)-1\right]. \quad (3.16)$$

The expression in brackets in Eq. (3.16) is positive for $\beta>\beta_T>\beta_c$, where

$$\beta_T^6-2\beta_c^2\beta_T^2+\beta_c^4=0. \quad (3.17)$$

Thus, if $\beta_0>\beta_T$, the temperature decreases with the velocity in the upstream end of the shock layer.

The value of β_c can be found from the approxima-

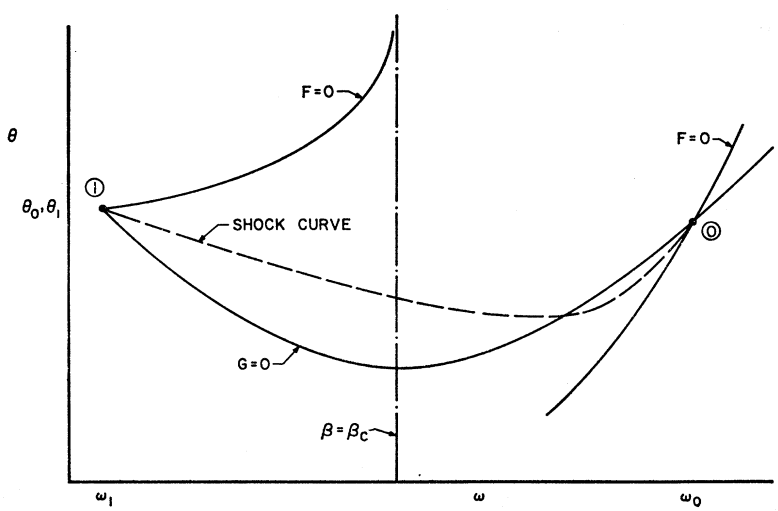

FIG. 4. Bounding curves—Shock into region III.

tions to the state function $H(\sigma)$, depending on the state of the preshock gas:

$$\beta_c^2=\tfrac{2}{3}, \quad \text{region I} \tag{3.18}$$

$$\beta_c^2=4/9, \quad \text{region II} \tag{3.19}$$

$$\beta_c^2=\tfrac{1}{3}, \quad \text{region III.} \tag{3.20}$$

Since the signal speed for a gas in region III is $\beta_s^2=\frac{1}{3}$, all shocks into a relativistic gas are above the critical speed.

This special case is also characterized by the fact that $\theta_0=\theta_1$. This is most readily seen from combination of Eqs. (2.7), (2.9), and (2.12) at the equilibrium points, which gives the jump equation

$$\gamma_0 H_0=\gamma_1 H_1. \tag{3.21}$$

The phase-plane plot for this case is shown in Fig. 4. It is clear that case B2 applies here, since $d\theta/dx$ must be 0 at some point in the shock.

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Calculation of the Yang-Lee Distribution of Zeros

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The connection between the zeros of the grand partition function Z and the function $\chi=\lim V^{-1}\ln Z$ at infinite volume is discussed. Assuming a functional equation for χ given, conditions for the lines containing zeros and an expression for the density of zeros are derived. Explicit results are given in a special case with an attractive interaction potential.

INTRODUCTION

It is known that Yang and Lee based their phase-transition theory[1] on the observation that, if the zeros of the grand partition function $Z(z)$ in the limit of infinite volume have an accumulation point at a positive real value z_0 of the fugacity, some thermodynamic functions are discontinuous at z_0. For all thermodynamic systems which so far have yielded to an explicit solution,[2] the zeros fall on definite lines C in the complex z plane and become dense on C at $V\to\infty$. If this turns out to be a general feature of thermodynamic systems, phase-transition points z_0 are determined as the points of intersection of C and the positive real z axis. The nature of a transition is governed by the density of zeros on C near z_0.

Only in trivial cases is it possible to find the roots of $Z(z)=0$ directly, because when the number of particles N becomes large, $Z(z)$ is an extremely complicated polynomial of degree of the order of N. In certain cases, however, it is possible to find a functional equation, the solution of which is the limit of $V^{-1}\ln Z$ for $V\to\infty$. In this paper we consider the relation between the distribution of zeros and the solutions of an equation of this type. It is found that under certain conditions the lines of zeros are uniquely determined. An expression is derived for the density of zeros in terms of the limit function.

[1] C. N. Yang and T. D. Lee, Phys. Rev. **87**, 404 (1952).

[2] T. D. Lee and C. N. Yang, Phys. Rev. **87**, 410 (1952); E. H. Hauge and P. C. Hemmer, Physica **29**, 1338 (1963); P. C. Hemmer and E. H. Hauge, Phys. Rev. **133**, A1010 (1964).

As an application of the method the distribution has been explicitly solved in the case of a linear model with potential consisting of hard core and square-well attraction. The main features of the results are shown graphically.

ANALYTIC CONTINUATION OF $\chi'(z)$

We assume that in the limit $V\to\infty$ all the zeros of $Z(z,V,T)$ fall on definite smooth lines C in the complex z plane, and are dense on C. Denoting the number of zeros on the line element ds at $z=z(s)$ by $Vg(s)ds$, s being a real parameter taken along C, we have

$$\chi(z)=\lim_{V\to\infty}\frac{1}{V}\ln Z(z,V)=\int_C ds g(s)\ln\left(1-\frac{z}{z(s)}\right). \tag{1}$$

The derivative of $\chi(z)$ is

$$\chi'(z)=\int_C ds\frac{g(s)}{z-z(s)}. \tag{2}$$

Along the curve C we have $dz=dse^{i\phi(s)}$, where $\phi(s)$ is the argument of the tangent of C at the point $z(s)$. Define

$$G(z(s))=g(s)e^{-i\phi(s)}.$$

The real variable s can now be replaced by the complex