between grid and cathode was gratifying since it served to validate the longer lifetimes reported previously[2] using only the triode. Such a reinforcement in confidence in the triode is desirable since it is so much more flexible a device, and we expect to use it solely in future work.

Above the saturation state expected when full blockading of the 3^1P-1^1S transition sets in, a decline from saturation occurs as density is further increased. This may perhaps be explained by the hypothetical process of molecular formation in collisions between 1^1S states and the 3^1P states.

The cross section for this process would have needed to be 2.0×10^{-16} cm^2 for formation from the 3^1P state compared with 2.7×10^{-16} cm^2 as found by Fowler and Duffendack[5] (their Table II contains a computational error) and with a value of 2.0×10^{-16} deducible from Hornbeck's measurements[6] on the assumption that the formation cross section is the same for all states.

In the course of these measurements, additional high-pressure data were obtained on two other transitions: 3^3D-2^3P at 5876 Å and 4^1D-2^1P at 4921 Å. At 20 mm Hg these lifetimes were 56 nsec and 42 nsec, respectively, and at 44 mm Hg they were 46 nsec and 34 nsec. The 5876 Å is down considerably from the 200-nsec value obtained at low pressures, but the 4921-Å lifetime is within the experimental error of the 35-nsec value observed in a low pressure.

[5] R. G. Fowler and O. S. Duffendack, Phys. Rev. **76**, 81 (1949).
[6] J. H. Hornbeck, Phys. Rev. **84**, 1072 (1951).

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Stimulated Emission of Radiation in a Single Mode

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Recently Glauber has described the properties of coherent radiation fields, and has constructed the density matrix of the field in two simple cases: (1) The radiating system is a classical radiator and no reaction is considered; (2) the central-limit theorem applies to a collection of radiators. This paper investigates other simple "almost" exactly soluble problems, in which a quantum-mechanical two-level system interacts with a quantized electromagnetic field originally in a pure coherent state in a single mode. The first-order correlation function $G^{(1)}=\langle E^-E^+\rangle$ is compared with $\langle E^-\rangle\langle E^+\rangle$ at resonance when the stimulating field is initially a pure coherent state and the two-level system is initially in its excited state. The corresponding quantities are also computed for a field whose initial density matrix is a Gaussian superposition of coherent states (e.g., blackbody radiation), as well as for a field which is initially described as having a given number of photons.

I. INTRODUCTION

THE concept of coherence of an electromagnetic field has been introduced by Glauber[1] in terms of an nth-order correlation function

$$G_{\mu_1\cdots\mu_{2n}}{}^{(n)}=\langle E_{\mu_1}{}^-(x_1)E_{\mu_2}{}^-(x_2)\cdots E_{\mu_n}{}^-(x_n)\times E_{\mu_{n+1}}{}^+(x_{n+1})\cdots E_{\mu_{2n}}{}^+(x_{2n})\rangle\,, \quad \text{(I1)}$$

where $\langle(\ )\rangle$ stands for trace $\rho(\ )$, $x_n\equiv(\mathbf{x}_n,t_n)$, and the μ's denote the polarization. The electric field operator E is written as a sum of positive- and negative-frequency parts, $E=E^++E^-$. A "pure coherent" state is one for which $G^{(n)}$ factors into the product

$$G_{\mu_1\cdots\mu_{2n}}{}^{(n)}=\mathcal{E}_{\mu_1}{}^*(x_1)\,\mathcal{E}_{\mu_2}{}^*(x_2)\cdots\mathcal{E}_{\mu_n}{}^*(x_n)\times\mathcal{E}_{\mu_{n+1}}(x_{n+1})\cdots\mathcal{E}_{\mu_{2n}}(x_{2n}) \quad \text{(I2)}$$

for all n. We are led to a definition of "nth-order coherence" as factorization through order n of $G^{(n)}$. This definition is the quantum-mechanical generalization of previous ones[2] and includes nonstationary processes, an example of which is investigated in this paper. In two elegant papers,[1,3] Glauber defines nth-order coherence and describes the properties of coherent states of the electromagnetic (e.m.) field. Two examples are given, one in which the field is produced by a classical radiator (which always produces pure coherent states) and in the other by a chaotic source (e.g., discharge lamp) for which the density matrix of the field is seen from the central-limit theorem to be a Gaussian superposition of coherent density operators in each normal mode. The problem which we shall investigate here is the following: An atom (molecule, spin) is initially in an excited state at $t=0$, at which time it comes into interaction with a quantum-e.m. field in a pure coherent state in a single mode at the resonance frequency of the two-level system (TLS).[4] The question arises: "To what extent will the field produced by stimulation also be coherent?" It is also interesting to compare the resulting field, in which the stimulating

[1] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
[2] M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, Ltd., London, 1959), Chap. X.
[3] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[4] For the semiclassical solution for the long-time transition probabilities, cf., F. W. Cummings, Am. J. Phys. **30**, 898 (1962).

field is initially in a coherent state, to the situations where the initial density matrix of the stimulating field is that appropriate to a chaotic source in a single mode, such as a blackbody or discharge lamp, and also where it is initially known that there are n photons in the field. The problem is approached by a model employed previously by Jaynes[5] and the present author, the relevant portion of which is reviewed here for completeness. The problem can be solved "almost" exactly; the approximation involved is equivalent to the neglect of terms in the Hamiltonian which do not conserve energy in first order. Also, only the second-order correlation function, with $t_1=t_2$ is computed; we are here interested only in comparing the quantities $\langle E^-(x)E^+(x)\rangle$ and $\langle E^-(x)\rangle\langle E^+(x)\rangle$.

II. FORMULATION OF THE PROBLEM

Let the closed surface S enclose a volume V, and let $\mathbf{E}_\lambda(\mathbf{r})$ and $k^2=\omega^2/c^2$ be the eigenfunctions and eigenvalues of the boundary-value problem

$$\nabla\times(\nabla\times\mathbf{E}_\lambda)-k_\lambda{}^2\mathbf{E}_\lambda=0 \text{ in } V$$

and (II1)

$$\mathbf{n}\times\mathbf{E}_\lambda=0 \text{ on } S.$$

The $\mathbf{E}_\lambda(\mathbf{r})$ are normalized so that

$$\int_V \mathbf{E}_\lambda\cdot\mathbf{E}_{\lambda'}d^3x=\delta_{\lambda\lambda'} \quad \text{(II2)}$$

and similarly for the magnetic field $\mathbf{H}$. The electric and magnetic fields are expanded in the forms

$$\mathbf{E}(\mathbf{r},t)=-(4\pi)^{1/2}\sum_\lambda p_\lambda(t)\mathbf{E}_\lambda(\mathbf{r})$$
$$=-\sum(2\pi\hbar\omega_\lambda)^{1/2}(C_\lambda+C_\lambda{}^*)\mathbf{E}_\lambda(\mathbf{r}), \quad \text{(II3a)}$$

$$\mathbf{H}(\mathbf{r},t)=+(4\pi)^{1/2}\sum_\lambda \omega_\lambda q_\lambda(t)\mathbf{H}_\lambda(\mathbf{r})$$
$$=+\sum\left(\frac{2\pi\hbar}{\omega_\lambda}\right)^{1/2}(C_\lambda-C_\lambda{}^*)\mathbf{H}_\lambda(\mathbf{r}), \quad \text{(II3b)}$$

where $C_\lambda{}^*$ and C_λ are the usual creation and destruction operators for the λth mode. They satisfy the commutation relations

$$[C_\lambda,C_{\lambda'}{}^*]=\delta_{\lambda\lambda'}, \quad [C_\lambda,C_{\lambda'}]=[C_\lambda{}^*,C_{\lambda'}{}^*]=0 \quad \text{(II4)}$$

and have the properties, when operating on a state function of the field in the n representation,

$$C_\lambda|n_\lambda\rangle=(n_\lambda)^{1/2}|n_\lambda-1\rangle,$$
$$C_\lambda{}^*|n_\lambda\rangle=(n_\lambda+1)^{1/2}|n_\lambda+1\rangle. \quad \text{(II5)}$$

The Hamiltonian for the free field is given by

$$\mathcal{H}=\int\frac{\mathbf{E}^2+\mathbf{B}^2}{8\pi}d^3x=\sum_\lambda \tfrac{1}{2}\hbar\omega_\lambda(C_\lambda C_\lambda{}^*+C_\lambda{}^*C_\lambda), \quad \text{(II6)}$$

[5] E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).

and the matrix elements of the electric field are given by

$$\langle n_\lambda|\mathbf{E}|n_\lambda'\rangle$$
$$=-(2\pi\hbar\omega_\lambda)^{1/2}\mathbf{E}_\lambda(\mathbf{x})$$
$$\times[(n_\lambda)^{1/2}\delta_{n_\lambda,n_\lambda'+1}+(n_\lambda+1)^{1/2}\delta_{n_\lambda+1,n_\lambda'}]. \quad \text{(II7)}$$

If the TLS moves along the axis of a cylindrical cavity so that only the lowest TM mode is excited (as in the ammonia-beam maser) then

$$\langle n|\mathbf{E}|n'\rangle=-(2\pi\omega/J_1{}^2V)^{1/2}$$
$$\times[n^{1/2}\delta_{n,n'+1}+(n+1)^{1/2}\delta_{n+1,n'}]\boldsymbol{\varepsilon}_3, \quad \text{(II8)}$$

where $J_1=J_1$, $(u)=0.5191$ and $u=2.405$, the first root of $J_0(u)=0$, V is the volume of the cavity, and we have set $\hbar=1$ and dropped the subscript λ, since we are concerned hereafter with only one mode. Suppose now that a single TLS having only two possible energy levels enters the cavity via a small hole in the end. With the TLS field interaction in the usual form $\sim(\mathbf{J}\cdot\mathbf{A})$, even this simple problem cannot be solved exactly, but it is possible to find stationary states of the system (TLS +field) to an accuracy of about one part in 10^7 for radiation densities up to the order of those encountered in the ammonia maser, for example.

Let the two possible energy levels of the TLS be denoted by E_m and the corresponding states by $|m\rangle(m=1,2)$. Similarly the number of quanta in the field will be n and the corresponding state of the field denoted by $|n\rangle$. The state vectors $|m\rangle|n\rangle$ then form a basis for the system (TLS+field). In this representation the total Hamiltonian is

$$\langle mn|H|m'n'\rangle=(E_m+n\omega)\delta_{mm'}\delta_{nn'}$$
$$+\langle mn|H_{\text{int}}|m'n'\rangle. \quad \text{(II9)}$$

The interaction between TLS and field is taken as

$$H_{\text{int}}=-\boldsymbol{\mu}\cdot\mathbf{E}, \quad \text{(II10)}$$

where $\boldsymbol{\mu}$ is the electric-dipole-moment operator of the TLS, whose component along $\mathbf{E}$ shall have the matrix elements

$$\langle mn|\mu_z|m'n'\rangle=\mu(1-\delta_{mm'})\delta_{nn'}. \quad \text{(II11)}$$

Thus, the matrix elements for the interaction energy are

$$\langle mn|H_{\text{int}}|m'n'\rangle$$
$$=\gamma(1-\delta_{mm'})[n^{1/2}\delta_{n,n'+1}+(n+1)^{1/2}\delta_{n+1,n'}], \quad \text{(II12)}$$

where

$$\gamma=(\mu/J_1)(2\pi\omega/V)^{1/2} \quad \text{(II13)}$$

is the interaction constant (and has the value 5 cps for a typical gas maser). The interaction Hamiltonian has matrix elements of two different types:

$$H_{\text{int}}=V+W, \quad \text{(II14)}$$

where

$$V_n\equiv\langle 1,n+1|V|2,n\rangle$$
$$=\langle 2,n|V|1,n+1\rangle=\gamma(n+1)^{1/2}, \quad \text{(II15)}$$

$$W_n \equiv \langle 1, n | W | 2, n+1 \rangle = \langle 2, n+1 | W | 1, n \rangle = \gamma(n+1)^{1/2}, \quad \text{(II16)}$$

with all other elements zero. The term V has matrix elements connecting "unperturbed" states with an energy separation $(E_2 - E_1 - \omega)$ which goes through zero as the cavity is tuned exactly to the TLS natural frequency. Elements of W, however, connect states with unperturbed-energy separation $(E_2 - E_1 + \omega) \sim 2\omega$. If we diagonalize the Hamiltonian to order $(n^{1/2}\gamma/\omega)^2$, we neglect W entirely and it is this approximation made here, which breaks down for extremely intense fields.

The resulting Hamiltonian can then be diagonalized exactly, since it now has a "block" form consisting of (2×2) matrices along the main diagonal. It is at this point that the difficulty involved in solving the multi-TSL problem becomes apparent. For two TLS we must diagonalize a 4×4, and for N two-level systems we must diagonalize matrices whose dimensionality is as high as

$$\sum_{j=0}^{N} N!/j!(N-j)! \quad \text{(II17)}$$

and thus the complexity of the problem increases rapidly with N. Physically this arises because each TLS "sees" all the other $(N-1)$ TLS via the e.m. field.

The eigenvalues and eigenfunctions of H_0, defined by neglecting W, which satisfy

$$H_0 \Phi_n^{(\pm)} = E_n^{(\pm)} \Phi_n^{(\pm)} \quad \text{(II18)}$$

are the ground-state

$$E_0 = E_1 = \omega_0, \quad \Phi_0 = |m=1\rangle |n=0\rangle \quad \text{(II19)}$$

and for $n > 0$

$$E_n^{(\pm)} = \omega_n^{(\pm)} = (n - \tfrac{1}{2})\omega \pm \tfrac{1}{2}[(\omega - \Omega)^2 + 4n\gamma^2]^{1/2}, \quad \text{(II20)}$$

where we have defined the zero of energy of the TLS as midway between E_1 and $E_2 = 0$ and $E_2 - E_1 = \Omega$. Now

$$\Phi_n^{(+)} = |2\rangle |n-1\rangle \cos\theta_n + |1\rangle |n\rangle \sin\theta_n, \quad \text{(II21)}$$

$$\Phi_n^{(-)} = -|2\rangle |n-1\rangle \sin\theta_n + |1\rangle |n\rangle \cos\theta_n, \quad \text{(II22)}$$

where

$$\tan 2\theta_n = 2\gamma\sqrt{n}/(\omega - \Omega). \quad \text{(II23)}$$

The time-development matrix

$$U(t,t') = U(t-t') = \exp[-iH_0(t-t')] \quad \text{(II24)}$$

has the matrix elements for $n > 0$

$$\langle 2, n-1 | U | 2, n-1 \rangle = a_n = \cos^2\theta_n e^{-i\omega_n^{(+)} t} + \sin^2\theta_n e^{-i\omega_n^{(-)} t}, \quad \text{(II25a)}$$

$$\langle 2, n-1 | U | 1, n \rangle = b_n = \sin\theta_n \cos\theta_n \times [e^{-i\omega_n^{(-)} t} - e^{-i\omega_n^{(+)} t}], \quad \text{(II25b)}$$

$$\langle 1, n | U | 2, n-1 \rangle = b_n, \quad \text{(II25c)}$$

$$\langle 1,n | U | 1,n \rangle = c_n = \cos^2\theta_n e^{-i\omega_n^{(-)} t} + \sin^2\theta_n e^{-i\omega_n^{(+)} t}, \quad \text{(II25d)}$$

and for $n = 0$

$$\langle 1,0 | U | 1,0 \rangle = e^{-i\omega_0 t}, \quad \omega_0 = -\tfrac{1}{2}\Omega. \quad \text{(II26)}$$

All other elements vanish. The transition probability for emission or absorption of one photon during time t is then, neglecting terms in W

$$|b_n|^2 = \sin^2 2\theta_n \sin^2(\omega_n^{(+)} - \omega_n^{(-)})t = n\gamma^2 \sin^2\beta_n t/\beta_n^2, \quad \text{(II27)}$$

where

$$\beta_n^2 = \tfrac{1}{4}[(\omega - \Omega)^2 + 4n\gamma^2]. \quad \text{(II28)}$$

The above notation is chosen in such a way that the block form of U consists of the symmetric (2×2) unitary matrices

$$\begin{pmatrix} a_n & b_n \\ b_n & c_n \end{pmatrix} \quad n = 1, 2, \cdots \quad \text{(II29)}$$

along the main diagonal. The first row and column, however, contain only the single term $\exp(-i\omega_0 t)$.

Now consider the effect on the field of passing a single TLS through the cavity. At the instant $(t=0)$ when the TLS enters the cavity, let its state be described by the density matrix $\rho_1(0)$. The initial density matrix of the entire system is thus the direct product

$$\rho(0) = \rho_1(0) \otimes \rho_f(0) \quad \text{(II30)}$$

with matrix elements

$$\langle mn | \rho(0) | m'n' \rangle = \langle m | \rho_1(0) | m' \rangle \langle n | \rho_f(0) | n' \rangle. \quad \text{(II31)}$$

During the interaction, $\rho(t)$ undergoes a unitary transformation

$$\rho(t) = U(t,0) \rho(0) U^{-1}(t,0) \quad \text{(II32)}$$

and the density matrix $\rho_f(t)$, which describes the state of the field, only, is the projection of ρ onto the space of the field variables.

$$\langle n | \rho_f(t) | n' \rangle = \sum_{m=1}^{2} \langle mn | \rho(t) | mn' \rangle. \quad \text{(II33)}$$

The net change of the field thus consists of a linear transformation

$$\langle n | \rho_f(t) | n' \rangle = \sum_{kk'} \langle nn' | G | kk' \rangle \langle k | \rho_f(0) | k' \rangle \quad \text{(II34)}$$

or

$$\rho_f(t) = G(t) \rho_f(0),$$
$$\langle nn' | G | kk' \rangle = \sum_{mm'm''} \langle m''n | U | mk \rangle \times \langle m'k' | U^{-1} | m''n' \rangle \sigma_{mm'}, \quad \text{(II35)}$$

where we have written for brevity

$$\sigma_{mm'} = \langle m | \rho_1(0) | m' \rangle. \quad \text{(II36)}$$

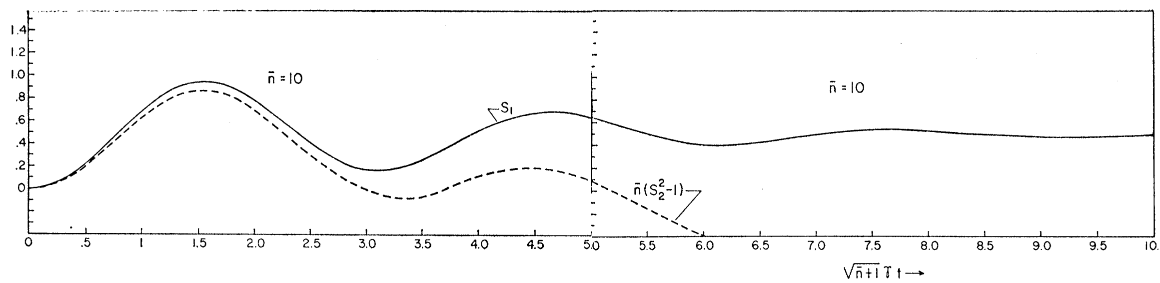

FIG. 1. $[\langle E^- E^+\rangle/(2\gamma/\mu)^2-\bar{n}]$ (solid) and $[\langle E^-\rangle\langle E^+\rangle/(2\gamma/\mu)^2-\bar{n}]$ (dashed) as functions of $(\bar{n}+1)^{1/2}\gamma t$ are compared for the case of stimulation by a pure coherent state with $\bar{n}=10$.

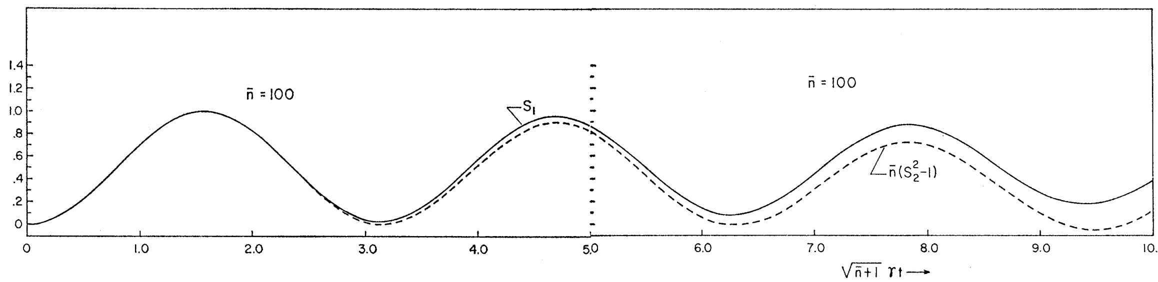

FIG. 2. Same as Fig. 1 except that $\bar{n}=100$.

The only nonvanishing elements of G are easily seen to be

$$\langle nn'|G|nn'\rangle=a_{n+1}a_{n'+1}^*\sigma_{22}+c_nc_{n'}^*\sigma_{11}, \quad \text{(II37a)}$$

$$\langle nn'|G|n+1,n'\rangle=b_{n+1}a_{n'+1}^*\sigma_{12}, \quad \text{(II37b)}$$

$$\langle nn'|G|n,n'+1\rangle=a_{n+1}b_{n'+1}^*\sigma_{12}, \quad \text{(II37c)}$$

$$\langle nn'|G|n,n'-1\rangle=c_nb_{n'}^*\sigma_{12}, \quad \text{(II37d)}$$

$$\langle nn'|G|n-1,n'\rangle=b_nc_{n'}^*\sigma_{21}, \quad \text{(II37e)}$$

$$\langle nn'|G|n+1,n'\rangle=b_{n+1}b_{n'+1}^*\sigma_{11}, \quad \text{(II37f)}$$

$$\langle nn'|G|n-1,n'-1\rangle=b_nb_{n'}^*\sigma_{22}. \quad \text{(II37g)}$$

These relations hold for all quantum numbers n if we understand that C_0 is not defined by (II25d) but by $C_0=\exp(-i\omega_0 t)$.

III. INTRODUCTION OF THE GLAUBER STATES

A pure coherent state may be defined by the property[3]

$$E_\mu^+(x)|\alpha\rangle=\mathcal{E}_\mu(x)|\alpha\rangle \quad \text{(III1)}$$

and

$$\langle\alpha|E_\mu^-(x)=\langle\alpha|\mathcal{E}_\mu^*(x). \quad \text{(III2)}$$

In the number representation, the coherent state is given by

$$|\alpha\rangle=\sum_{n=0}^{\infty}(e^{-|\alpha|^2/2}\alpha^n/\sqrt{n!})|n\rangle \quad \text{(III3)}$$

so that

$$c|\alpha\rangle=\alpha|\alpha\rangle,\quad \langle\alpha|c^*=\langle\alpha|\alpha^*. \quad \text{(III4)}$$

The density operator for a pure coherent state is then

$$\rho_f=|\alpha\rangle\langle\alpha|=\sum_{n,n'}\frac{e^{-|\alpha|^2}\alpha^n\alpha^{*n'}}{(n!n'!)^{1/2}}|n\rangle\langle n'| =\sum_{nn'}|n\rangle\langle n|\rho_f|n'\rangle\langle n'|. \quad \text{(III5)}$$

We are interested in computing the stimulated emission (or absorption) of a TLS by a field initially described by the density matrix (III5), so that initially

$$\langle E^-E^+\rangle(0)=\langle E^-\rangle(0)\langle E^+\rangle(0). \quad \text{(III6)}$$

In the Schrödinger representation, from Eq. (II8)

$$\langle E^+\rangle(t)=\mathrm{tr}[\rho_f(t)E^+]=-(2\gamma/\mu)\times\textstyle\sum_n(n+1)^{1/2}\langle n|\rho_f(t)|n+1\rangle \quad \text{(III7)}$$

and

$$\langle E^-\rangle(t)=\langle E^+\rangle^*(t). \quad \text{(III8)}$$

Also

$$\langle E^-E^+\rangle(t)=(2\gamma/\mu)^2\textstyle\sum_n n\langle n|\rho_f(t)|n\rangle. \quad \text{(III9)}$$

Writing out Eq. (II34) explicitly shows that

$$\begin{aligned}\langle n|\rho_f(t)|n'\rangle &=\sigma_{11}[b_{n+1}b_{n'+1}^*\langle n+1|\rho_f(0)|n'+1\rangle\\ &\quad+c_nc_{n'}^*\langle n|\rho_f(0)|n'\rangle]\\ &\quad+\sigma_{12}[b_{n+1}a_{n'+1}^*\langle n+1|\rho_f(0)|n'\rangle\\ &\quad+c_nb_{n'}^*\langle n|\rho_f(0)|n'-1\rangle]\\ &\quad+\sigma_{21}[a_{n+1}b_{n'+1}^*\langle n|\rho_f(0)|n'+1\rangle\\ &\quad+b_nc_{n'}^*\langle n-1|\rho_f(0)|n'\rangle]\\ &\quad+\sigma_{22}[a_{n+1}a_{n'+1}^*\langle n|\rho_f(0)|n'\rangle\\ &\quad+b_nb_{n'}^*\langle n-1|\rho_f(0)|n'-1\rangle]. \end{aligned}\quad \text{(III10)}$$

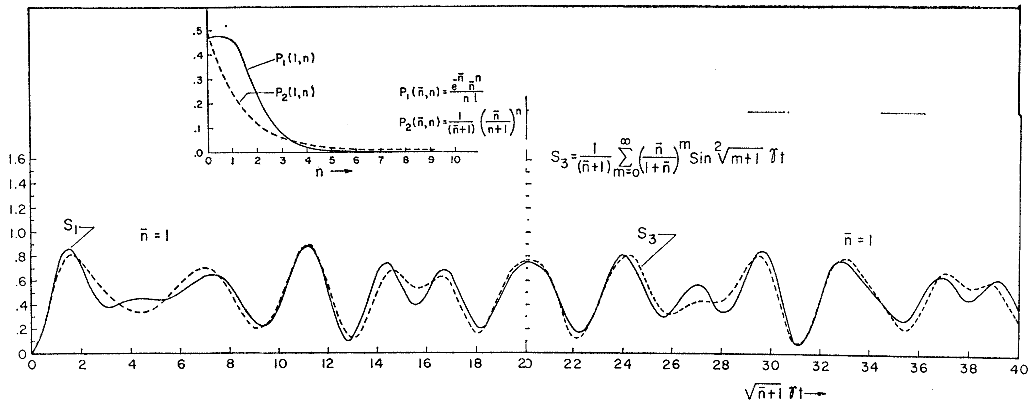

FIG. 3. $[\langle E^-E^+\rangle/(2\gamma/\mu)^2-\bar{n}]$ as a function of $(\bar{n}+1)^{1/2}\gamma t$ is plotted for the two cases of initial stimulation by a pure coherent field (solid) and by a "chaotic" field (dashed), both for $\bar{n}=1$. [The insert compares the weighting functions of the transition probability $\sin^2(n+1)^{1/2}\gamma t$ for these two cases.]

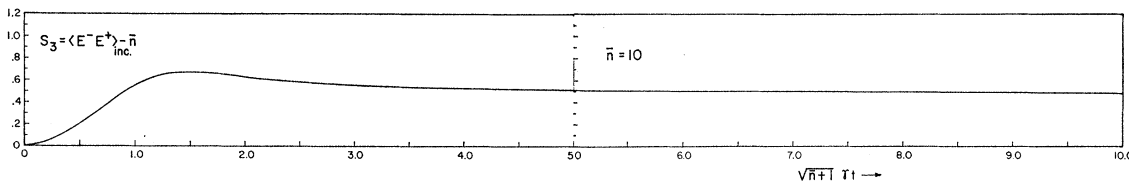

FIG. 4. $[\langle E^-E^+\rangle/(2\gamma/\mu)^2-\bar{n}]$ is shown as a function of $(\bar{n}+1)^{1/2}\gamma t$ for initial stimulation by a "chaotic" source with $\bar{n}=10$.

We start the TLS system off in its upper state, and ask how the initially coherent field gets amplified. Then $\sigma_{22}=1$, $\sigma_{11}=0=\sigma_{12}$. The quantities $a_n a_{n'}^*$, $b_n b_{n'}^*$, and $c_n c_{n'}^*$ are easily worked out to give

$$a_n a_{n'}^* = e^{+i\omega(n'-n)t}[\cos\beta_n t\cos\beta_{n'}t - i\cos2\theta_n\sin\beta_n t\cos\beta_{n'}t - i\cos2\theta_{n'}\cos\beta_n t\sin\beta_{n'}t + \cos2\theta_n\cos2\theta_{n'}\sin\beta_n t\sin\beta_{n'}t], \quad \text{(III11a)}$$

$$b_n b_{n'}^* = e^{+i\omega(n'-n)t}\sin2\theta_n\sin2\theta_{n'}\sin\beta_n t\sin\beta_{n'}t, \quad \text{(III11b)}$$

$$c_n c_{n'}^* = e^{+i\omega(n'-n)t}[\cos\beta_n t\cos\beta_{n'}t + i\cos2\theta_n\sin\beta_n t\cos\beta_{n'}t + i\cos2\theta_{n'}\cos\beta_n t\sin\beta_{n'}t + \cos2\theta_n\cos2\theta_{n'}\sin\beta_n t\sin\beta_{n'}t]. \quad \text{(III11c)}$$

Using Eq. (III11) in (III10), gives for the case of resonance, $\omega=\Omega$

$$\langle E^-E^+\rangle_c(t) = (2\gamma/\mu)^2[\bar{n}+S_1(\bar{n},\gamma t)], \quad \text{(III12)}$$

$$S_1(\bar{n},\gamma t) \equiv \sum_{n=0}^{\infty}(e^{-\bar{n}}\bar{n}^n/n!)\sin^2(n+1)^{1/2}\gamma t \quad \text{(III13)}$$

and

$$\langle E^-\rangle(t) = -(2\gamma/\mu)\alpha e^{+i\omega t}S_2(\bar{n},\gamma t), \quad \text{(III14)}$$

$$S_2(\bar{n},\gamma t) \equiv \sum_n \left[\cos(n+1)^{1/2}\gamma t\cos(n+2)^{1/2}\gamma t + \left(\frac{n+2}{n+1}\right)^{1/2} \times \sin(n+1)^{1/2}\gamma t\sin(n+2)^{1/2}\gamma t\right]\frac{e^{-\bar{n}}\bar{n}^n}{n!}. \quad \text{(III15)}$$

Of interest is the comparison of the two expressions $\langle E^-E^+\rangle(t)$ and $\langle E^-\rangle\langle E^+\rangle(t)$, where

$$\langle E^-\rangle\langle E^+\rangle(t) = (2\gamma/\mu)^2[\bar{n}+\bar{n}(S_2^2-1)], \quad \text{(III16)}$$

and where $|\alpha|^2$ has been identified as $\bar{n}$, the average number of photons. Notice that there is no spontaneous emission term in $\langle E^-\rangle\langle E^+\rangle(t)$ while spontaneous emission ($\bar{n}\to0$) is predicted by

$$\langle E^-E^+\rangle(t) \underset{\bar{n}\to0}{=} \sin^2\gamma t.$$

(Notice that in general one cannot write $\langle E^-E^+\rangle = \langle E^-E^+\rangle_{\text{spont}} + \langle E^-E^+\rangle_{\text{induced}}$.) That the field remains zero even though $\langle E^-E^+\rangle$ is nonzero is a general property of the system whenever $\sigma_{12}=\sigma_{21}=0$, that is, where there is initially no coherence between the states of the TLS; there is nothing to "tell" the field what phase to have. The expression for $\langle E^-E^+\rangle$ has an intuitively simple physical content. It is the transition probability for emitting a photon into the mode given that there are n photons initially, and this is then weighted by a Poisson distribution.

It is also of interest to compare the corresponding expressions $\langle E^-E^+\rangle_{\text{inc}}$ and $\langle E^-\rangle\langle E^+\rangle_{\text{inc}}$ with the above for the situation when the density matrix of the field is initially diagonal in the n representation, the TLS again initially in its excited state and the cavity tuned to resonance. For ρ given by a Gaussian superposition[3] of density operators of pure states,

$$\rho_f(0) = \int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha = \int \frac{1}{\pi\bar{n}}e^{-|\alpha|^2/\bar{n}}|\alpha\rangle\langle\alpha|d^2\alpha = (1+\bar{n})^{-1}\sum_n(\bar{n}/(1+\bar{n}))^n|n\rangle\langle n|,$$

$$d^2\alpha \equiv d(\text{Re}\alpha)d(\text{Im}\alpha). \quad \text{(III17)}$$

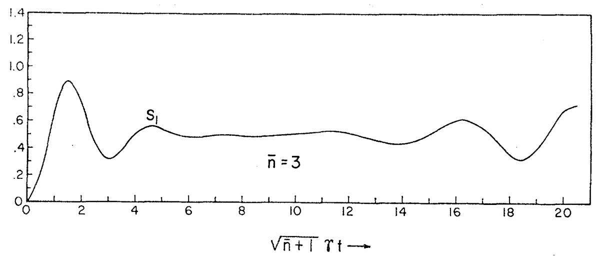

FIG. 5. $[\langle E^-E^+\rangle/(2\gamma/\mu)^2-\bar{n}]$ is shown as a function of $(\bar{n}+1)^{1/2}\gamma t$ for initial stimulation by a pure coherent state with $\bar{n}=3$.

Note that ρ is diagonal in the n representation whenever $P(\alpha)=P(|\alpha|)$, that is, whenever there is no information regarding the phase of the perturbing field. Equation (III17) is appropriate whenever the perturbing field is made up of a large number of independent radiators and obeys the central-limit theorem, for example, a discharge lamp, a blackbody radiator $[\bar{n}=(e^{\omega/kT}-1)^{-1}]$, or a collection of many independent lasers all at the same frequency. As expected,

$$\langle E^-E^+\rangle_{\text{inc}}=(2\gamma/\mu)^2[\bar{n}+S_3(\bar{n},\gamma t)],$$
$$S_3(\bar{n},\gamma t)=(1+\bar{n})^{-1}\times\sum_{n=0}^{\infty}(\bar{n}/(1+\bar{n}))^n\sin^2(n+1)^{1/2}\gamma t \quad \text{(III18)}$$

and

$$\langle E^-\rangle_{\text{inc}}\langle E^+\rangle_{\text{inc}}(t)\equiv 0. \quad \text{(III19)}$$

The case for $\rho_f(0)$ given by $\rho_f(0)=\delta_{\bar{n},n}$ (that is, initially the field has exactly $\bar{n}$ photons) gives

$$\langle E^-E^+\rangle_{\text{inc}}=(2\gamma/\mu)^2[\bar{n}+\sin^2(\bar{n}+1)^{1/2}\gamma t],$$
$$\langle E^-\rangle\langle E^+\rangle=0. \quad \text{(III20)}$$

This expression agrees with the semiclassical result in a more transparent way than, say, (III13).

Because of the appearance of factors like $\sin(n+1)^{1/2}\gamma t$, the sums S_1, S_2, and S_3 seem to defy expression in closed form. If $\sqrt{\bar{n}}$ is large we notice that the weighting factor in $S_{1,2}$, $P(n)=e^{-\bar{n}}\bar{n}^n/n!$ will peak at a value of $n=\bar{n}$, with a dispersion $(\langle n^2\rangle_{\text{av}}-\bar{n}^2)^{1/2}=\bar{n}^{1/2}$. Thus we can expand the square root, perform the resulting sum, and obtain

$$S_1\cong\sum\frac{e^{-\bar{n}}\bar{n}^n}{n!}\sin^2\left\{(\bar{n}+1)^{1/2}\left[1+\frac{(n-\bar{n})}{2(\bar{n}+1)}\right]\right\}\gamma t$$
$$\cong\tfrac{1}{2}-\tfrac{1}{2}\cos[(\bar{n}+1)^{1/2}2\gamma t]\times\exp[-(\gamma t)^2/2], \quad \text{(III21)}$$

which will be valid as long as $\gamma t\ll\sqrt{\bar{n}}$. Notice that S_1 approaches the value 0.5 in a manner independent of $\bar{n}$. Figures 1 and 2 show the two quantities $S_1\equiv\langle E^-E^+\rangle_c/(2\gamma/\mu)^2-\bar{n}$ and $\bar{n}(S_2^2-1)$ as a function of $(\bar{n}+1)^{1/2}\gamma t$, for the two cases $\bar{n}=10$, 10^2. Thus, a field initially in a coherent state will stimulate emission which is also coherent, at least to first order, for "times" $\gamma t\ll 1$. S_1 approaches the value 0.5 for $\gamma t\gg 1$. Figure 3 shows the two quantities $\langle E^-E^+\rangle_c/[(2\gamma/\mu)^2-\bar{n}]=S_1$ (solid) and $[\langle E^-E^+\rangle_{\text{inc}}/(2\gamma/\mu)^2-\bar{n}]=S_3$ (dashed) for the case $\bar{n}=1$. Also shown is a comparison of the two "weighting" factors $e^{-\bar{n}}\bar{n}^n/n!$ and $(1+\bar{n})^{-1}[\bar{n}/(1+\bar{n})]^n$ for $\bar{n}=1$. It is somewhat surprising to notice the very close correspondence between S_1 and S_3 in view of the difference in these two weighting factors. There is almost no difference in the effects of a coherent source and a chaotic source for $\bar{n}=1$. In Fig. 4 the function S_3 for $\bar{n}=10$ is shown. As might be expected, the function approaches the value 0.5 for $\gamma t\gg 1$, and the difference between this and Fig. 1 is apparent.

Figure 5 shows a transitional region between $\bar{n}=1$ and $\bar{n}=10$. The physical meaning is obscure.

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