TEMPERATURE DISTRIBUTION IN SOLIDS DURING HEATING OR COOLING.

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Synopsis.

Temperature distribution in solids; surface heated at uniform rate. Equations are derived for the following typical shapes: (1) Rectangular parallelopiped. (1a) Long rectangular rod. (1b) Very thin slab. (2) Cylinder. (2a) Long cylindrical rod. (3) Sphere. (4a) Cylindrical tube heated only outside. (4b) Cylindrical tube heated both inside and outside. (5) Spherical shell heated only outside. Results calculated from these equations are tabulated and in some cases shown graphically. These numerical results may be readily applied to the case of similar solids of any size and diffusivity.

Temperature distribution in solids; surface suddenly cooled or heated. Equations are derived for the following shapes: (1) Rectangular parallelopiped. (1a) Long rectangular rod. (1b) Very thin slab. (2) Cylinder. (2a) Long cylindrical rod. (3) Sphere. The distribution inside a sphere at various instants is computed and also the temperature at the center of a slab, square bar, cube, long cylinder, and sphere as a function of the time. These numerical results may be readily applied to the case of similar solids of any size and diffusivity.

Thermal diffusivity; method of measurement involving determination of temperature-time relation at the center of a symmetrical solid whose surface is heated either at a uniform rate or very suddenly. The convenience of this method is pointed out, but practical details are not considered. The equations given are in convenient form for such uses.

In deciding on the best methods of carrying out various operations in the manufacture of optical glass we found it necessary to have some idea of the temperature gradients in the pieces during heat treatment. While great precision in absolute magnitudes is generally of minor importance in such cases, the only way to gain insight into the question of the variation of the temperature differences with the shape and dimensions of the blocks and the method of heating is actually to work out numerical examples.

While the authors' main interest at the time was in the application to glass manufacture, the equations are perfectly general, as are also all the qualitative deductions made. The numerical computations can be applied to other cases with a very little manipulation owing to the fact that the only physical constant used (κ, the so-called diffusivity constant), occurs, at least when it occurs in any complicated term, multiplied by the time. Hence the values of these terms for different substances are the same at equal values of the product of the time by the diffusivity constant.

The formulas given also indicate methods for the accurate determination of the heat conductivity of solids. For instance, anticipating a little, one of the last equations is for calculating the temperature at the center of a cylinder which is initially at a uniform temperature and is plunged into a well-stirred constant temperature bath. The temperature would not be appreciably disturbed by the presence of a small axial hole as the temperature gradient across the axis is zero. A thermocouple may therefore be introduced and the temperature measurements made which serve to compute the constant required.

The ordinary text-books[1] dealing with heat conduction indicate the necessary mathematical transformations for the discussion of the problems, but there is a lack of actual detailed results available for practical reference. In the following pages will be found a synopsis of what we have found of actual use to us.

Solids of the following shapes are considered: (1) Brick (rectangular parallelopiped). (1*a*) Rod of rectangular section, or brick with two opposite faces insulated. (1*b*) Infinite slab, thin plate, or brick with two pairs of opposite faces insulated. (2) Cylinder. (2*a*) Rod of cylindrical section or cylinder with the flat ends thermally insulated. (3) Sphere.

Two modes of heating are considered: (A) Heating of the surface at a uniform rate. (B) Sudden change of surface temperature, as, for instance, by plunging the block into a constant temperature bath.

In both cases the initial temperature is taken as being uniform throughout.

Mathematical Discussion.

Nomenclature.

θ° = difference, from the initial temperature, of the point (x, y, z) at time t seconds from start.

κ = diffusivity constant in cm.^{2} per second.

h = number of degrees that the surface changes per second.

t = time in seconds.

The forms[2] of the differential equations for heat conduction suggest

[1] *E. g.*, Ingersoll and Zobel, Mathematical Theory of Heat Conduction, Ginn & Co. Byerly, Fourier's Series and Spherical Harmonics, Ginn & Co. Carslaw, Introduction to Fourier's Series and Integrals, Macmillan Co.

[2] Case (1)
$$\frac{\partial\theta}{\partial t} = \kappa\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2}\right) \tag{1}$$

Case (2)
$$\frac{\partial\theta}{\partial t} = \kappa\left(\frac{\partial^2\theta}{\partial x^2} + \frac{1}{x}\frac{\partial\theta}{\partial x} + \frac{\partial^2\theta}{\partial z^2}\right), \quad [x \text{ radial, } z \text{ axial}] \tag{2}$$

Case (3)
$$\frac{\partial(x\theta)}{\partial t} = \kappa\frac{\partial^2(x\theta)}{\partial x^2} \quad [x \text{ radial}] \tag{3}$$

that a likely solution may be made up of terms which are the product of an exponential function of the time and a trigonometrical function of the coördinates except in case (2) which is likely to require Bessel's functions. The values of such functions are, however, tabulated in readily accessible form and therefore it is a simple matter to compute particular examples when the equations are known.

Equations for Case of Linear Heating.

In the case of linear heating of the surface, if the suggested form apply, the time term must be of the shape $(1 - e^{-\alpha t})$ since θ must vanish for $t = 0$ and must approach a limiting steady state, as t increases, in which all parts of the body change uniformly h degrees per second.

Case 1.—Brick shape (rectangular parallelopiped).—In the discussion of this it is simplest to take the origin of coördinates[3] at the corner of the block. Let $2a$, $2b$, $2c$ be the dimensions of the block. The conditions give $\theta = 0$ when $t = 0$ and $\theta = ht$ over the surface ($x = 0$, $x = 2a$, $y = 0$, $y = 2b$, $z = 0$, $z = 2c$). The suggested form of solution is therefore

$$\theta = ht + \Sigma f(m, n, p) \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{p\pi z}{c}(1 - e^{-\alpha t}).$$

where m, n and p are any integers. However many or few terms are included under the Σ sign, the whole expression satisfies the above-mentioned conditions. It remains to be seen whether values can be found for α and $f(m, n, p)$ so that the combination of terms satisfies the differential equation

$$\frac{\partial\theta}{\partial t} = \kappa\Sigma\frac{\partial^2\theta}{\partial x^2}.$$

Equating the values of these differentials leads to

$$h + \Sigma\alpha e^{-\alpha t}f(m, n, p)\sin\frac{m\pi x}{2a}\sin\frac{n\pi y}{2b}\sin\frac{p\pi z}{2c}$$

$$\equiv -\kappa\Sigma f(m, n, p)\left(\frac{m^2\pi^2}{4a^2} + \frac{n^2\pi^2}{4b^2} + \frac{p^2\pi^2}{4c^2}\right)\sin\frac{m\pi x}{2a}\sin\frac{n\pi y}{2b}\sin\frac{p\pi z}{2c}(1 - e^{-\alpha t}).$$

On either side of this identity we have two parts, one containing $e^{-\alpha t}$

[3] If it be desired to change the origin to the center of the block the only change necessary is to put $X + a$ for x, $Y + b$ for y and $Z + c$ for z, and note that

$$\frac{\sin(2m + 1)\pi(X + a)}{2a} = (-1)^{m+1}\cos\frac{(2m + 1)\pi X}{a}.$$

and the other not. Equating these separately yields

$$\Sigma\alpha e^{-\alpha t}f(m, n, p)\sin\frac{m\pi x}{2a}\sin\frac{n\pi y}{2b}\sin\frac{p\pi z}{2c}$$

$$= \kappa\Sigma f(m, n, p)\left(\frac{m^2\pi^2}{4a^2}+\frac{n^2\pi^2}{4b^2}+\frac{p^2\pi^2}{4c^2}\right)\sin\frac{m\pi x}{2a}\sin\frac{n\pi y}{2b}\sin\frac{p\pi z}{2c}e^{-\alpha t}, \quad (4)$$

and

$$h = -\kappa\Sigma f(m, n, p)\left(\frac{m^2\pi^2}{4a^2}+\frac{n^2\pi^2}{4b^2}+\frac{p^2\pi^2}{4c^2}\right)\sin\frac{m\pi x}{2a}\sin\frac{n\pi y}{2b}\sin\frac{p\pi z}{2c}. \quad (5)$$

(4) yields

$$\alpha = \kappa\left(\frac{m^2\pi^2}{4a^2}+\frac{n^2\pi^2}{4b^2}+\frac{p^2\pi^2}{4c^2}\right). \quad (6)$$

But to evaluate (5) we need to make use of the trigonometric series

$$\frac{\pi}{4} = \sin\frac{\pi x}{2a} + \frac{1}{3}\sin\frac{3\pi x}{2a} + \frac{1}{5}\sin\frac{5\pi x}{2a} +, \text{ etc.}$$

Multiplying together three such series for x, y and z yields

$$\frac{\pi^3}{64} = \Sigma\,\frac{\sin\frac{m\pi x}{2a}\sin\frac{n\pi y}{2b}\sin\frac{p\pi z}{2c}}{mnp}.$$

where m, n and p take the values of every odd integer. Equation (5) reduces to this if $f(m, n, p)$ be put equal to

$$\frac{64h}{\pi^3}\cdot\frac{1}{mnp\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}+\frac{p^2\pi^2}{c^2}\right)}. \quad (7)$$

(6) and (7) give the required values of α and $f(m, n, p)$.

Substitution of these values in the suggested form of solution gives

$$\theta = ht - \frac{64h}{\kappa\pi^3}\sum_{\substack{m=1\\n=1\\p=1}}^{\substack{p=\infty\\n=\infty\\m=\infty}}\frac{\sin\frac{(2m-1)\pi x}{2a}\sin\frac{(2n-1)\pi y}{2b}\sin\frac{(2p-1)\pi z}{2c}}{(2m-1)(2n-1)(2p-1)\left(\frac{(2m-1)^2\pi^2}{4a^2}+\frac{(2n-1)^2\pi^2}{4b^2}+\frac{(2p-1)^2\pi^2}{4c^2}\right)}\times\left[1-e^{-\kappa\left\{\frac{(2m-1)^2\pi^2}{4a^2}+\frac{(2n-1)^2\pi^2}{4b^2}+\frac{(2p-1)^2\pi^2}{4c^2}\right\}t}\right] \quad (8)$$

the form of the constants having been so changed that m, n and p take all integral values from 1 to ∞.

Case 1a (rectangular rod).—This may be treated separately or deduced from the above by assuming c large as compared with a and b.

$$\theta = ht - \frac{16h}{\kappa\pi^2}\sum_{m,n=1}^{m,n=\infty} \frac{\sin\frac{(2m-1)\pi x}{2a}\sin\frac{(2n-1)\pi y}{2b}}{(2m-1)(2n-1)\left(\frac{(2m-1)^2\pi^2}{4a^2}+\frac{(2n-1)^2\pi^2}{4b^2}\right)} \times\left[1 - e^{-\kappa\left\{\frac{(2m-1)^2\pi^2}{4a^2}+\frac{(2n-1)^2\pi^2}{4b^2}\right\}t}\right]. \quad (9)$$

Case 1b (thin slab).—This may also be treated separately or deduced from the above by assuming b and c large as compared with a, giving

$$\theta = ht - \frac{4h}{\kappa\pi}\sum_{m=1}^{m=\infty}\frac{\sin\frac{(2m-1)\pi x}{2a}}{(2m-1)\frac{(2m-1)^2\pi^2}{4a^2}}\left\{1 - e^{-\frac{\kappa(2m-1)^2\pi^2}{4a^2}t}\right\}. \quad (10)$$

When t becomes sufficiently large (the exact time will be considered later) the exponential term may be dropped out of any of these equations. When this state is reached differentiation yields

$$\frac{\partial\theta}{\partial t} = h \quad \text{and} \quad \Sigma\frac{\partial^2\theta}{\partial x^2} = h.$$

Every portion is heating uniformly at h degrees per second (steady state).

These simplified differential equations in the case of the slab have as solution

$$\theta = ht - \frac{h}{\kappa}\left(ax - \frac{x^2}{2}\right) \quad (11)$$

$$= ht - \frac{ha^2}{\kappa}\left(\frac{x}{a} - \frac{x^2}{2a^2}\right),$$

as can be immediately verified by differentiation.

Those familiar with Fourier's method of expansion will see the identity of the forms; otherwise this may serve as a proof of the expansion,

$$ax - \frac{x^2}{2} = \frac{16a^2}{\pi^3}\sum_{m=1}^{m=\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{2a}$$

between $x = 0$ and $x = a$.

Case 2 (cylinder).—Let a = radius of cylinder and $2b$ = length of axis. Let the origin of coördinates be at the center. Reasoning similar to the above leads to the equation

$$\theta = ht - \frac{8h}{\pi\kappa}\sum_{m,\,n=1}^{m,\,n=\infty} \frac{1}{(2m-1)R_n\left(\frac{R_n^2}{a^2}+\frac{(2m-1)^2\pi^2}{4b^2}\right)J_1(R_n)}$$
$$\times J_0\left(\frac{R_n x}{a}\right)\sin\frac{(2m-1)\pi(z+b)}{2b}\left\{1-e^{-\kappa\left[\frac{R_n^2}{a^2}+\left(\frac{(2m-1)^2\pi^2}{4b^2}\right)\right]t}\right\}. \quad (12)$$

In this the R_n's are the roots of $J_0(x) = 0$ and the values are given in a table in Byerly. Tables of the values of $J_0(x)$ and $J_1(x)$ are also given there, and in Gray and Matthew, Treatise on Bessel's Functions.

For those who wish to verify the formula it may be mentioned that only the following very elementary properties of Bessel's functions are required

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\times 4^2} - \frac{x^6}{2^2\times 4^2\times 6^2} +, \text{ etc.}$$

$$\frac{dJ_0(x)}{dx} = -J_1(x), \qquad \frac{dJ_1(x)}{dx} = J_0(x) - \frac{1}{x}J_1(x),$$

and

$$\sum_{n=1}^{n=\infty}\frac{2}{R_nJ_1(R_n)}J_0\left(R_n\frac{x}{a}\right) = 1.$$

Case 2a (cylindrical rod).—When b becomes large compared with a the above equation reduces to

$$\theta = ht - \frac{2ha^2}{\kappa}\sum_{n=1}^{n=\infty}\frac{1}{R_n^3J_1(R_n)}J_0\left(\frac{R_nx}{a}\right)\left(1-e^{-\frac{\kappa R_n^2}{a^2}t}\right),$$

which simplifies to

$$\theta = ht\frac{h}{4\kappa}(a^2-x^2) + \frac{2ha^2}{\kappa}\sum_{n=1}^{n=\infty}\frac{1}{R_n^3J_1(R_n)}J_0\left(\frac{R_nx}{a}\right)e^{-\frac{R_n^2}{a^2}t} \quad (13)$$

The same remarks as regards this simplification apply (*mutatis mutandis*) as in the case 1*b*.

Case 3 (sphere).—Let a be the radius of the sphere and let the origin be at the center. The same method gives

$$\theta = ht - \frac{2ha^2}{\kappa}\sum_{m=1}^{m=\infty}\frac{1}{m^3\pi^3(-1)^{m+1}}\left(1-e^{-\frac{\kappa m^2\pi^2}{a^2}t}\right)\frac{a}{x}\sin\frac{m\pi x}{a}.$$

The only difference in proof from Case (1) is that $x\theta$ takes the place of θ.

Simplification yields

$$\theta = ht - \frac{h}{6\kappa}(a^2-x^2) + \frac{2ha^2}{\kappa}\sum_{m=1}^{m=\infty}\frac{1}{m^3\pi^3(-1)^{m+1}}e^{-\frac{\kappa m^2\pi^2}{a^2}t}\frac{a}{x}\sin\frac{m\pi x}{a}. \quad (14)$$

The similarity of the equations for cases 1*b*, 2*a* and 3 cannot have escaped the reader's attention. For the sake of comparison we move the origin of coördinates to the center of the slab for case 1*b* and write the equations in the same form:

Inf. Slab. $\theta = ht - \frac{h}{2\kappa}(a^2 - x^2)$

$$+ \frac{2ha^2}{\kappa} \sum_{m=1}^{m=8} \frac{1}{Q_m^3(-1)^{m+1}} e^{-\frac{\kappa Q_m^2 t}{a^2}} \cos\left(Q_m \frac{x}{a}\right). \quad (15)$$

Inf. Cylinder. $\theta = ht - \frac{h}{4\kappa}(a^2 - x^2)$

$$+ \frac{2ha^2}{\kappa} \sum_{m=1}^{m=\infty} \frac{1}{R_m^3 J_1(R_m)} e^{-\frac{\kappa R_m^2 t}{a^2}} J_0\left(R_m \frac{x}{a}\right). \quad (16)$$

Sphere. $\theta = ht - \frac{h}{6\kappa}(a^2 - x^2)$

$$+ \frac{2ha^2}{\kappa} \sum_{m=1}^{m=\infty} \frac{1}{S_m^3(-1)^{m+1}} e^{-\frac{\kappa S_m^2 t}{a^2}} \frac{a}{x} \sin\left(S_m \frac{x}{a}\right). \quad (17)$$

In these formulæ Q_m and S_m are written for the sake of brevity for $[(2m - 1)\pi]/2$, and $m\pi$ respectively.

In conclusion of this section, as special cases, found useful in some practical problems, the heating of a hollow tube and spherical shell will be considered briefly. The complete discussion of these problems is somewhat more difficult than of those already attempted, but for most purposes it is only necessary to consider the solution for the steady state with linear cooling. For this case the equations are, for the tube:

$$\frac{\partial \theta}{\partial t} = h \quad \text{and} \quad \kappa\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{1}{x}\frac{\partial \theta}{\partial x}\right) = h$$

and the solution is

$$\theta = ht + \frac{hx^2}{4\kappa} + C_1 \ln x + C_2$$

C_1 and C_2 being constants of integration.

Case (4*a*) Suppose heat transfer to take place only at the outside.

This yields as conditions

$$\theta = ht \text{ at } x = a \quad \text{and} \quad \frac{\partial \theta}{\partial x} = 0 \text{ at } x = a_1,$$

where a_1 and a are the internal and external radii.

The equation becomes

$$\theta = ht + \frac{h(x^2 - a^2)}{4\kappa} + \frac{ha_1^2}{2\kappa} \ln \frac{a}{x}. \quad (18)$$

Case (4*b*).—Suppose heat transfer takes place equally outside and inside.

The conditions then are

$$\theta = ht \quad \text{at} \quad x = a \quad \text{and} \quad x = a_1.$$

The equation for this case becomes

$$\theta = ht + \frac{hx^2}{4\kappa} + h\frac{\left[a_1^2 \log\frac{x}{a} - a^2 \log\frac{x}{a_1}\right]}{4\kappa \log\frac{a}{a_1}}. \tag{19}$$

For very thin tubes either of these last two equations reduces to the case of a thin slab as given at the beginning of the paper, but for a tube whose internal bore is very small each approximates to the case of a solid cylinder.

Case (5).—Similar reasoning leads, in the case of a spherical shell heated linearly on the outer surface, to the following equation for the steady state:

$$\theta = ht + \frac{h(x^2 - a^2)}{6\kappa} + \frac{ha_1^3}{3\kappa}\left(\frac{1}{x} - \frac{1}{a}\right), \tag{20}$$

where a_1 is the internal radius.

Before making any comments on these equations we give in Tables I., II., III. and IV. the results of a number of numerical calculations from equations (15), (16) and (17).

The character of the temperature distribution in a few cases is illustrated graphically in the curves of Figs. 1 and 2.

Calculated Values[4] *of* θ.—We have taken κ equal to 0.004 cm.[1] per sec. which is an average value for glass. h in each case is taken as 0.1 deg. per sec.

TABLE I.

Showing temperature distribution in a slab of glass, 2 cm. thick, heated at a rate of 0.1 *deg. per second.* x/a is the fractional distance from the center, θ is the temperature, t is the time in seconds, and A and B are constants of equation 21. (κ = 0.004.)

x/a.	A.	$t = 50$.		$t = 100$.		$t = 200$.		$t = 500$.		$t = 1{,}000$.	
		B.	θ.	B.	θ.	B.	θ.	B.	θ.	B.	θ.
0.000	12.50	7.87	0.37	4.81	2.31	1.79	9.29	0.093	37.59	0.0008	87.50
0.333	11.11	6.82	0.71	4.16	3.05	1.55	10.44	0.080	38.97	.0007	88.89
0.500	9.38	5.57	1.19	3.40	4.02	1.27	11.89	0.066	40.69	.0005	90.62
0.667	6.95	3.94	1.99	2.40	5.45	0.90	13.95	0.046	43.10	.0004	93.05
0.800	4.50	2.43	2.93	1.49	6.99	0.55	16.05	0.029	45.53	.0002	95.50
1.000	0.00	0.00	5.00	0.00	10.00	0.00	20.00	0.000	50.00	0.0000	100.00

[1] A convenient table of values of e^{-x} will be found in Becker and Van Ostrand's "Hyperbolic Functions." (Published by the Smithsonian Institution.)

TABLE II.

Similar to Table I. but for a slab 10 cm. thick.

x/a.	A.	$t=500$.		$t=1,000$.		$t=2,000$.		$t=5,000$.		$t=10,000$.	
		B.	θ.	B.	θ.	B.	θ.	B.	θ.	B.	θ.
0	312.5	262.7	0.2	217.0	4.5	146.4	33.9	44.8	232.3	6.22	693.7
.333	277.8	229.3	1.5	188.2	10.4	126.8	49.0	38.8	261.0	5.39	727.6
.500	234.4	188.5	4.1	153.8	19.4	103.5	69.1	31.7	297.3	4.40	770.0
.667	173.6	134.3	10.7	108.9	35.3	73.2	99.6	22.4	348.8	3.11	829.5
.800	112.5	83.4	20.9	67.3	54.8	45.2	132.7	13.8	401.3	1.92	889.4
1.000	0.0	0.0	50.0	0.0	100.0	0.0	200.0	0.0	500.0	0.00	1000.0

TABLE III.

Similar to Table I., but for an infinitely long cylinder of radius 5 cm.

x/a.	A.	$t=500$.		$t=1,000$.		$t=2,000$.		$t=5,000$.	
		B.	θ.	B.	θ.	B.	θ.	B.	θ.
0	156.3	107.1	0.8	68.6	12.3	27.2	70.9	1.70	345.4
.333	138.9	91.6	2.7	58.0	19.1	22.8	83.9	1.44	362.5
.500	117.2	73.3	6.1	45.9	28.7	18.2	101.0	1.14	383.9
.667	86.8	50.2	13.4	31.1	44.3	12.3	125.5	0.77	414.0
.800	56.3	29.8	23.5	18.4	62.1	7.3	151.0	0.46	444.2
1.000	0.0	0.0	50.0	0.0	100.0	0.0	200.0	0.00	500.0

TABLE IV.

Similar to Table I., but for a sphere 5 cm. in radius.

x/a.	A.	$t=100$.		$t=500$.		$t=1,000$.		$t=2,000$.		$t=5,000$.	
		B.	θ.	B.	θ.	B.	θ.	B.	θ.	B.	θ.
0	104.2	94.2	0	56.9	2.7	26.1	21.9	5.38	101.2	0.05	395.8
.333	92.6	82.6	0	47.1	4.5	21.6	29.0	4.45	111.9	0.04	407.4
.500	78.1	68.1	0	36.2	8.1	16.6	38.5	3.43	125.3	0.03	421.9
.667	57.9	48.1	0.2	23.5	15.6	10.8	52.9	2.23	144.3	0.02	442.1
.800	37.5	28.9	1.4	13.3	25.8	6.1	68.6	1.26	163.8	0.01	462.5
1.000	0.0	0.0	10.0	0.0	50.0	0.0	100.0	0.00	200.0	0.00	500.0

The columns headed A give the value of the quadratic term in the equation (15, 16 or 17, according to the solid referred to) and under B the values of the last term are tabulated, *i. e.*,

$$\theta = ht - A + B. \tag{21}$$

x/a denotes the fractional distance from center to surface. θ is the temperature.

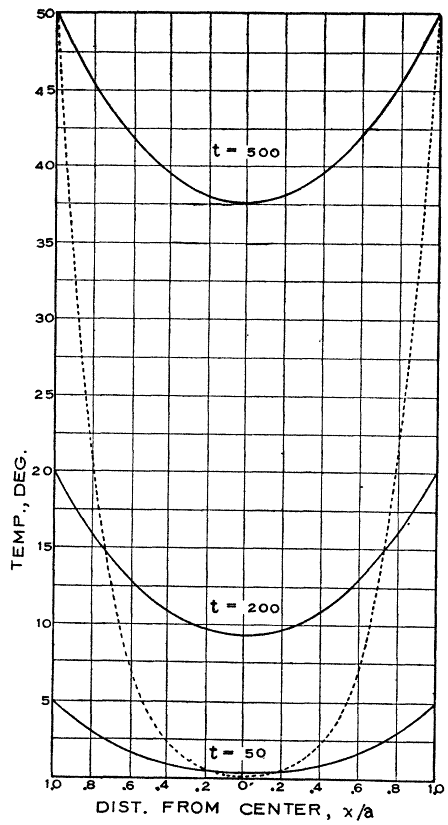

Fig. 1.

Diagram to illustrate distribution of temperature in a slab, the surfaces of which are heated at a uniform rate of 0.1 deg. per sec. The solid lines show the temperature prevailing throughout a slab 2 cm. thick, 50, 200 and 500 seconds respectively after the heating is begun. For the sake of comparison there is included the dotted curve which shows the temperature distribution at the end of 500 sec. in a slab 10 cm. thick which is heated at the same rate. The value of κ, the diffusivity, is taken as 0.004 and the slabs are assumed to be initially at a uniform temperature.

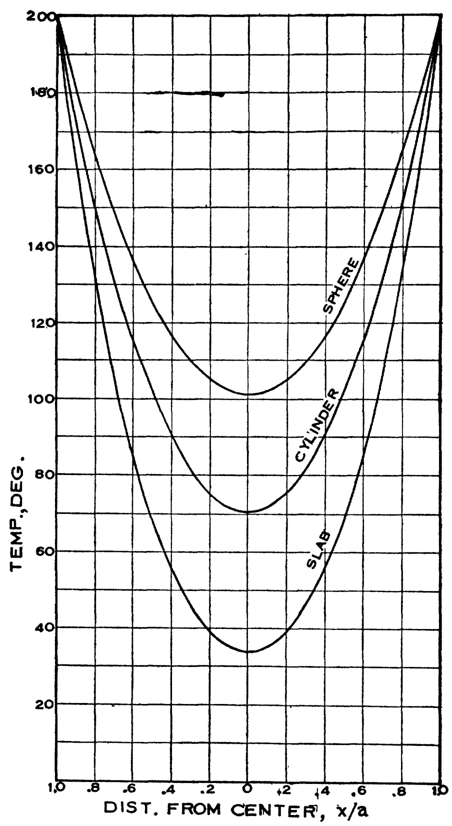

Fig. 2.

Curves showing distribution of temperature after 2000 sec. in the "unidimensional" solids (1) sphere, (2) cylinder of infinite length, and (3) slab. The heating rate is 0.1 deg. per sec., the diffusivity is 0.004, the diameter (or thickness) is 10 cm. and the solids are initially at a uniform temperature throughout.

Adaptation of these Tables to Other Numerical Values.

I. *Change of Rate of Heating.*—θ and h are directly proportional, therefore simple multiplication solves this problem. In other words, for a given solid the lag of any point behind the surface temperature is a time lag.

II. *Change of Dimensions.*—If it were not for the exponent in the exponential terms θ would also be directly proportional to a^2. This may be allowed for by changing the time so that by multiplying the values of θ by a_1^2/a^2 we get the values corresponding for the new dimensions at a time equal to $a_1^2/a^2 \cdot t$.

III. *Change of Diffusivity.*—The same remarks apply to κ save that the proportion is inversed. It is therefore necessary to multiply the θ columns by κ/κ_1 to get the values at a time equal to $\kappa/\kappa_1 \cdot t$.

In general[1] if the letters with subscripts apply to the new case

$$\theta_1 = \theta \frac{h_1 a_1^2 \kappa}{h a^2 \kappa_1} \qquad \text{at the time} \qquad t_1 = t \frac{a_1^2 \kappa}{a^2 \kappa_1}.$$

For the three simple cases the temperature differences at the steady state $h/2\kappa \cdot (a^2 - x^2)$, etc., depend on the thickness but the gradients h/κ, $h/2\kappa$, $h/3\kappa$ do not. This evidently means that the temperature distribution in the central inch of say a six-inch slab is the same as in a single one-inch slab, provided h and κ are the same for the two slabs. It follows that in a well-regulated furnace for the case of uniform heating or cooling it is futile to cover slabs of material with sand or other such material in the hope of altering the temperature gradient, as only the lag behind the furnace will be affected and not the actual distribution in the slabs.

At all save the shortest times it is only necessary to use one term of the series in calculating B of equation (21) and hence it is an easy matter to find, for instance, when the value of θ at the center will be within one per cent. of the value at the steady state.

We have only to consider the first term in B and equate it to 0.01 A. This gives in the three simple cases for the values $a = 5$, $\kappa = 0.004$

$$\text{Slab} \qquad \frac{16a^2h}{\pi^3\kappa} e^{-\frac{\kappa\pi^2 t}{4a^2}} = 0.01 \frac{ha^2}{2\kappa},$$

hence $t = 11{,}760$ secs.

[1] By multiplying both sides of equations (8) to (17) inclusive by κ/ha^2 it is readily seen that the equations may be considered as determining $\kappa\theta/ha^2$ as a function of $\kappa t/a^2$ and x/a, all of which are dimensionless quantities. To illustrate the use of Tables I, II, III, and IV for different values of h, κ, a^2 and t let us take the following example: for a slab, when $x/a = 0.5$, $t = 100$, $\kappa = 0.004$, $h = 0.1$ and $a = 1$, according to Table I, $\theta = 4.02$. Therefore when $x/a = 0.5$ $\kappa_1 = 0.008$, $h_1 = 0.2$, and $a_1 = 5$, it follows that

$$\theta = 4.02 \frac{0.2 \times 25 \times 0.004}{0.1 \times 1 \times 0.008} = 100.5$$

at the time

$$t = 100 \frac{25 \times 0.004}{1 \times 0.008} = 1250.$$

Likewise, Tables I, II, III, and IV can be converted into tables for new values of h, κ and a multiplying t and θ by the factors given in the text. The factor for θ also applies to A and B.

Similarly

Cylinder $t = 5{,}080$ secs.
Sphere $t = 3{,}040$ secs.

Comparison with the tables will show that this fits the facts.

In calculation we found that after t had a value only one hundredth of these values only one term was needed in the calculation of B to an accuracy of about one per cent.

Equations for Sudden Change of Surface Temperature.

The other method of heating considered is by sudden change of the surface from one temperature to another of a solid originally at a uniform temperature. This approximates to a number of physical problems, such as the cooling of the earth, or the cooling of a solid which is plunged into an ice bath.

The equations for such a case can be deduced by the use of exactly the same relations as in the previous discussion. We therefore merely give the equations without comment. The verification is simple by the methods already shown.

In the equations θ is used as before for the temperature at any point. θ_0 is the original uniform temperature and θ_1 is the new temperature of the surface. In all cases the origin is at the center.

(1) Brick shape

$$\frac{\theta-\theta_1}{\theta_0-\theta_1}=\frac{64}{\pi^3}\sum_{m,n,p=1}^{m,n,p=\infty}\frac{\cos\frac{(2m-1)\pi x}{2a}\cos\frac{(2n-1)\pi y}{2b}\cos\frac{(2p-1)\pi z}{2c}}{(2m-1)(2n-1)(2p-1)(-1)^{m+n+p+1}} \times e^{-\kappa\left(\frac{(2m-1)^2\pi^2}{4a^2}+\frac{(2n-1)^2\pi^2}{4b^2}+\frac{(2p-1)^2\pi^2}{4c^2}\right)t}, \tag{22}$$

(1a) Rectangular rod.

$$\frac{\theta-\theta_1}{\theta_0-\theta_1}=\frac{16}{\pi^2}\sum_{m,n=1}^{m,n=\infty}\frac{\cos\frac{(2m-1)\pi x}{2a}\cos\frac{(2n-1)\pi y}{2b}}{(2m-1)(2n-1)(-1)^{m+n+1}} \times e^{-\kappa\left(\frac{(2m-1)^2\pi^2}{4a^2}+\frac{(2n-1)^2\pi^2}{4b^2}\right)t}, \tag{23}$$

(1b) Slab.

$$\frac{\theta-\theta_1}{\theta_0-\theta_1}=\frac{4}{\pi}\sum_{m=1}^{m=\infty}\frac{\cos\frac{(2m-1)\pi x}{2a}}{(2m-1)(-1)^{m+1}}e^{-\kappa\left(\frac{(2m-1)^2\pi^2}{4a^2}\right)t}, \tag{24}$$

(2) Cylinder.

$$\frac{\theta-\theta_1}{\theta_0-\theta_1}=\frac{8}{\pi}\sum_{m,n=1}^{m,n=\infty}\frac{(-1)^{m+1}}{2m-1}\frac{J_0(R_nx/a)}{R_nJ_1(R_n)}\cos\frac{(2m-1)\pi y}{2a} \times e^{-\kappa\left(\frac{R_n^2}{a^2}+\frac{(2m-1)^2\pi^2}{b^2}\right)t}, \tag{25}$$

(2a) Cylindrical rod.

$$\frac{\theta-\theta_1}{\theta_0-\theta_1}=2\sum_{m=1}^{m=\infty}\frac{J_0\left(R_m\frac{x}{a}\right)}{R_mJ_1(R_m)}e^{-\kappa\frac{R_m^2}{a^2}t},\tag{26}$$

(3) Sphere.[1]

$$\frac{\theta-\theta_1}{\theta_0-\theta_1}=\frac{2}{\pi}\sum_{m=1}^{m=\infty}\frac{\frac{a}{x}\sin\frac{m\pi x}{a}}{m(-1)^{m+1}}e^{-\kappa\frac{m^2\pi^2}{a^2}t},\tag{27}$$

The last equation is given in Byerly (p. 116, ex. 2).

The next table gives the values of $(\theta-\theta_1)/(\theta_0-\theta_1)$ $(=F)$ for a sphere for different distances from the center and for different values of $\kappa t/a^2$.

TABLE V.

Table for calculating the temperature at any time of any point in a sphere originally at a uniform temperature the surface of which is suddenly changed to another temperature.

Values of $F\left(=\frac{\theta-\theta_1}{\theta_0-\theta_1}\right)$.

x/a.	$\kappa t/a^2$ =.000.	$\kappa t/a^2$ =.0040.	$\kappa t/a^2$ =.0160.	$\kappa t/a^2$ =.0360.	$\kappa t/a^2$ =.0640.	$\kappa t/a^2$ =.1000.	$\kappa t/a^2$ =.1960.	$\kappa t/a^2$ =.2560.	$\kappa t/a^2$ =.4000.	$\kappa t/a^2$ =∞.
.0000	1.0000	1.0000	1.0000	.9943	.9103	.7071	.2881	.1598	.0386	0.0000
.0500	1.0000	1.0000	1.0000	.9938	.9079	.7046	.2869	.1590	.0385	0.0000
.2500	1.0000	1.0000	.9997	.9790	.8577	.6466	.2596	.1439	.0347	0.0000
.3333	1.0000	1.0000	.9994	.9611	.8133	.6005	.2386	.1321	.0319	0.0000
.5000	1.0000	1.0000	.9896	.8752	.6755	.4745	.1840	.1018	.0246	0.0000
.6667	1.0000	.9997	.9063	.6788	.4727	.3162	.1197	.0661	.0160	0.0000
.7500	1.0000	.9931	.7921	.5312	.3537	.2319	.0869	.0480	.0116	0.0000
.9500	1.0000	.3935	.1791	.1030	.0644	.0411	.0152	.0084	.0020	0.0000
1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

For a sphere of glass where $\kappa=0.004$ and $a=10$, the columns give the temperature distribution at 0, 100, 400, 900, 1,600, 2,500, 4,900, 6,400, 10,000 seconds respectively. On the other hand, for a similar sphere where $a^2=1{,}000$, the successive times would be 0, 1,000, 4,000,

[1] The equations for cases (1b), (2a) and (3) closely resemble the final term in the corresponding equations for linear cooling. This is brought out more clearly by writing them in the shortened notation used above. They become:

(1b)
$$F=2\sum_{m=1}^{m=\infty}\frac{1}{Q_m(-1)^{m+1}}e^{-\frac{\kappa Q_m^2t}{a^2}}\cos\left(\frac{Q_mx}{a}\right).\tag{28}$$

(2a)
$$F=2\sum_{m=1}^{m=\infty}\frac{1}{R_mJ_1(R_m)}e^{-\frac{\kappa R_m^2t}{a^2}}J_0\left(\frac{R_mx}{a}\right).\tag{29}$$

(3)
$$F=2\sum_{m=1}^{m=\infty}\frac{1}{S_m(-1)^{m+1}}e^{-\frac{\kappa S_m^2t}{a^2}}\frac{a}{x}\sin\left(\frac{S_mx}{a}\right).\tag{30}$$

etc. For the earth, if the diffusivity constant be about 0.008, the times would be 0, 6.4×10^9 years, 25.6×10^9 years, etc.

The curves of Fig. 3 were obtained by plotting the fractional temperature difference $(\theta - \theta_1)/(\theta_0 - \theta_1)$ in a sphere, against the distance from the center x/a expressed in fractional parts of the radius for different values of $\kappa t/a^2$. In Fig.[1] 4, on the other hand, $(\theta - \theta_1)/(\theta_0 - \theta_1)$ is plotted against $\kappa t/a^2$ for different values of x/a.

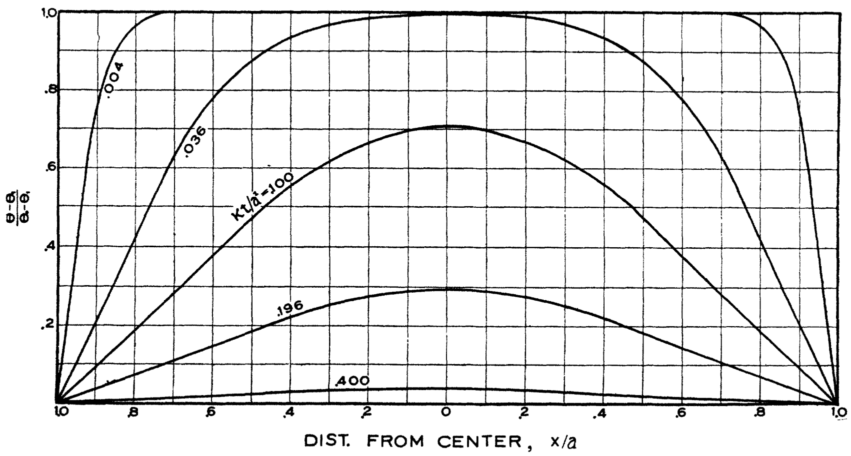

Fig. 3.

Diagram to show variation of temperature θ in a solid sphere after the temperature of the surface has been suddenly changed from θ_0 to θ_1 (temperature of sphere originally uniform throughout). The ordinate is the fractional temperature difference, $(\theta - \theta_1)/(\theta_0 - \theta_1)$, and the abscissa is the distance from the center expressed as fractional parts of the radius. The successive curves are for various values of $\kappa t/a^2$, the diffusivity multiplied by the time and divided by the square of the radius. Example: find the temperature in the center of a sphere of glass (24 cm. in diam.) originally at 100°, after being placed for 1 hour (= 3,600 sec.) in a well-stirred bath at 0°; take $\kappa = 0.004$, hence $\kappa t/a^2 = 0.1$, then from the appropriate curve it is seen that $(\theta - \theta_1)/(\theta_0 - \theta_1) = 0.71$. Hence $\theta = 71°$.

It will be noticed that equations (22)–(27) are considerably less involved than those used for the uniform heating case. In particular, the more involved are products of the less involved, *e. g.*, for Case (2) the equation is simply the product of those for (1*b*) and (2*a*) which are the simplest ones. It is therefore only necessary to calculate for (1*b*) and (2*a*), and the other follows.

Very frequently it is only the temperature at the center that is required, so we have calculated this for a few cases. When $x = 0$ all the cosine and J_0 terms are equal to unity, so the equations reduce to comparatively simple forms containing only the exponentials as variables.

[1] In plotting $(\theta - \theta_1)/(\theta_0 - \theta_1)$ against t or $\kappa t/a^2$, it is convenient to use "semi-log," paper.

(1b) Slab, at center.

$$\frac{\theta - \theta_1}{\theta_0 - \theta_1} = \frac{4}{\pi} \sum_{m=1}^{m=\infty} \frac{(-1)^{m+1}}{2m - 1} e^{-\kappa \frac{(2m-1)^2 \pi^2}{4a^2} t}. \tag{31}$$

(2a) Cylindrical rod, at center.

$$\frac{\theta - \theta_1}{\theta_0 - \theta_1} = 2 \sum_{m=1}^{m=\infty} \frac{1}{R_m J_1(R_m)} e^{-\kappa \frac{R_m^2}{a^2} t}. \tag{32}$$

(3) Sphere, at center.

$$\frac{\theta - \theta_1}{\theta_0 - \theta_1} = 2 \sum_{m=1}^{m=\infty} (-1)^{m+1} e^{-\kappa \frac{m^2 \pi^2}{a^2} t}. \tag{33}$$

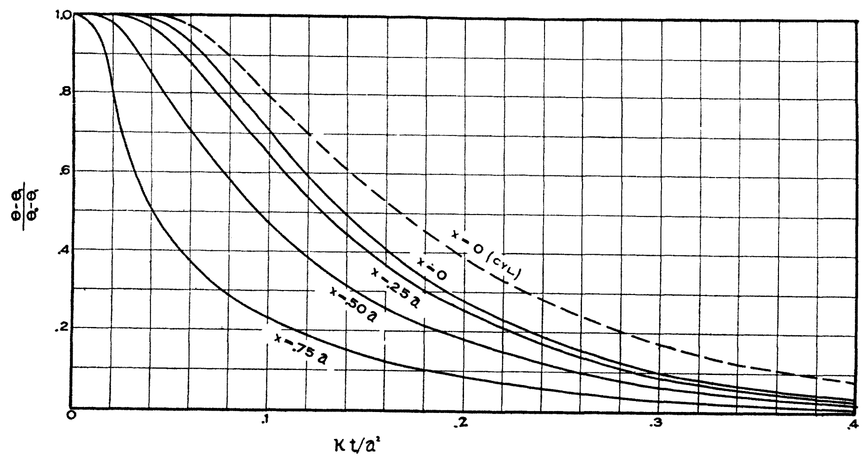

Fig. 4.

For same conditions as Fig. 3 except that $(\theta - \theta_1)/(\theta_0 - \theta_1)$ is plotted against $\kappa t/a^2$ for various values of x/a. These graphs are therefore cooling curves for various points in the sphere. In addition the figure contains a broken-line cooling curve which is a cooling curve for the center of a cylinder whose length is very great compared to its diameter.

Tables of exponentials are easily available, and in Table VI. the first ten values of R_n and $J_1(R_n)$ are given since these may not always be conveniently obtained.

TABLE VI.[1]

Values of R_n (roots of $J_0(x) = 0$) and $J_1(R_n)$.

n.	R_n.	$J_1(R_n)$.	n.	R_n.	$J_1(R_n)$.
1	2.4048	0.51915	6	18.071	−.1877
2	5.5201	− .34026	7	21.212	.1733
3	8.654	.2714	8	24.352	−.1617
4	11.792	− .2325	9	27.493	.1522
5	14.931	.2065	10	30.635	−.1422

[1] The values in this table are taken from Gray and Matthews, Treatise on Bessel Functions (Macmillan Co.).

In Table VII. are tabulated the values of $(\theta - \theta_1)/(\theta_0 - \theta_1)$ at the center under various conditions and with solids of various shapes.

TABLE VII.

Values of $(\theta - \theta_1)/(\theta_0 - \theta_1)$ at the center of solids of various shapes for case of sudden change in temperature of surface. In the heading of column 1, a is the radius, t is the time and κ is the diffusivity constant.

$\kappa t/a^2$.	Slab.	Square Bar.	Cube.	Cylinder of Infinite Length.	Cylinder of Length = Diam.	Sphere.
0	1	1	1	1	1	1
.032	.99983	.9997	.9995	.9990	.9988	.9975
.080	.9752	.9510	.9274	.9175	.8947	.8276
.100	.9493	.9012	.8555	.8484	.8054	.7071
.160	.8458	.7154	.6051	.6268	.5301	.4087
.240	.7022	.4931	.3462	.3991	.2802	.1871
.320	.5779	.3340	.1930	.2515	.1453	.0850
.800	.1768	.0313	.00553	.0157	.00277	.000745
1.600	.0246	.00060	....	.00015	....	....
3.200	.00047	....	....	....	....	....

Approximate Formulæ for Short Times.—When the time interval from the beginning is so small that the heating effect is negligible at the center—at least to the order of magnitude considered—the problem may be considered as that of a heat flow into an infinite solid. For still shorter times when only the surface layers need be considered, the curvature may be disregarded in the case of the cylinder and sphere and the case of the slab alone need be considered. In this case the well-known formula used by Lord Kelvin[1] is very convenient if tables of the so-called probability integral are available. In the notation used in this paper the formula would be

$$\frac{\theta - \theta_1}{\theta_0 - \theta_1} = \frac{2}{\sqrt{\pi}} \int_0^{x/2\sqrt{\kappa t}} e^{-z^2} dz, \tag{34}$$

where z is merely an integration variable.

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Washington, D. C., April, 1919.

[1] Thomson and Tait's Treatise on Natural Philosophy.