THE MOST PROBABLE VALUE OF THE PLANCK CONSTANT *h*.

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Synopsis.

The most probable value of Planck's Constant *h* has been computed, together with the probable error. The latest value of each of the other constants appearing in the equations has been used; and the probable error in each has been taken into consideration. The mean value from seven distinct methods is (6.5543 ± 0.0025) x 10^{-27} erg. sec., and each of the seven residuals lies within the limits of error. The indicated error in the final mean is practically exclusive of any error in *e* (assumed 4.774). However, by assuming Lewis and Adams' theory of ultimate rational units, it is possible to compute, in a number of different ways, an independent value of *e*, or of *h*. One such computation, using Blake and Duane's X-ray data, yields $e = 4.7705 \pm 0.0046$, $h = 6.550 \pm 0.013$.

In March, 1916, Millikan[1] published the results of a determination of the value of *h* from photo-electric data. The probable error was estimated at 0.5 per cent., and this constituted by far the most accurate determination of *h*, up to that time. Since then, however, all methods of attack have been improved in accuracy until, at the present time, we have seven distinct methods, each accurate to 0.5 per cent. or less. It is therefore of considerable importance to determine the most probable value of *h*, resulting from all known methods.

In carrying out the computations, the author has found it necessary in some cases to recompute previous results. For there are many other constants which appear in the formula for *h*, and the latest values of these constants should be used. Thus Millikan, in his photo-electric work, used 3.000 for the velocity of light (*c*). Most authors at the present time use 2.9986, as given by Kaye and Laby.[2] Since *c* enters as the second power, in the determination of *h*, this change of value produces a change of 0.1 per cent., in *h*,—an amount negligible at the time of Millikan's work, but not at all negligible at the present time.

In the following work I have assumed the author's own estimate of his probable experimental error, when such error is explicitly stated. In other cases the error has been judged as well as possible from the results themselves, together with the author's discussion of them. This final

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[1] Phys. Rev. (2), 7, 355, 1916.

[2] Physical and Chemical Constants, p. 69, 1918 edition.

error has then been combined, by the ordinary least square formulæ, with the probable errors in each of the other quantities appearing in the equation for h. The seven values of h, each with its resulting probable error, have then been combined to give the final most probable value, together with the probable error in this value.

The results are as follows:

METHOD I. FROM THE STEFAN-BOLTZMANN TOTAL RADIATION CONSTANT a, OR σ.

$$E = \frac{ac}{4} T^4 = \sigma T^4, \tag{1}$$

where E = the total radiation from unit surface of a black body (*i. e.*, through solid angle 2π) per unit time.

a is connected with h through the Planck radiation formula, the resulting equation being[1]

$$a = \frac{48\pi\alpha k^4}{c^3h^3}. \tag{2}$$

But

$$\alpha = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \text{etc.} = \frac{\pi^4}{90}. \tag{3}$$

Therefore

$$a = \frac{8\pi^5 k^4}{15c^3h^3}, \tag{4}$$

and

$$h = \sqrt[3]{\frac{8\pi^5 k^4}{15c^3 a}}, \tag{5}$$

where

$$k = \frac{p_0 V_0 e E}{TSc} = (1.372 \pm 0.0014) \times 10^{-16}, \tag{6}^2$$

k being the Boltzmann entropy constant (or, in other words, the gas constant for one molecule) and having practically the same probable error[3] as e (= 4.774 ± 0.005).

The best experimental determination of σ (= $ac/4$) is that of Coblentz,[4] the value being

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[1] See page 171, Masius, "Planck's Heat Radiation."

[2] Millikan, PHYS. REV. (2), 2, 142, 1913.

[3] Millikan, in his evaluation of k used for E (the electro-chemical equivalent of silver) 0.001118, defining the International ampere. This combined with 107.88 for S (the atomic weight of silver), gives 96,494 coulombs for the value of the Faraday. But the best value for E (Kaye and Laby, page 8, 1918 edition) is 0.00111827, giving 96,470 for the Faraday. Millikan likewise uses 1,013,700 for p_0, instead of 1,013,200 (K. and L., page 5). The two errors cancel to within 0.02 per cent. Finally, he uses 273.11 for T, which is still about the best value (K. and L., page 44).

[4] Proc. Nat. Acad. Sci. 3, 504, 1917.

$$\sigma = (5.72 \pm 0.012) \times 10^{-5} \text{ erg. cm}^{-2} \text{ deg}^{-4} \text{ sec}^{-1}. \qquad (7)$$

Therefore

$$h = \sqrt[3]{\frac{8\pi^5\{(1.372 \pm 0.0014) \times 10^{-16}\}^4}{15(2.9986 \times 10^{10})^2 \times 4 \times (5.72 \pm 0.012) \times 10^{-5}}} \qquad (8)$$

$$= (6.551 \pm 0.009) \times 10^{-27} \text{ erg sec.}$$

Method 2. From the Wien Constant c_2 (= C, of Coblentz).

$$\lambda_{\max} T = c_2/\beta = b \; (= A, \text{ of Coblentz}), \qquad (9)$$

where $\beta = 4.9651$. (The root of the equation $e^{-\beta} + \beta/5 - 1 = 0$.) c_2 is connected with h by the Planck equation,[1] giving

$$c_2 = b\beta = ch/k. \qquad (10)$$

The best value of c_2 is that of Coblentz.[2] The author states that the value of c_2 lies almost certainly between 1.43 and 1.44 cm. degrees and recommends[3] the tentative adoption of 1.433. I therefore use this value and judge from the above maximum variation, a probable error of 0.0025, giving

$$c_2 = (1.433 \pm 0.0025) \text{ cm. degrees.} \qquad (11)$$

Therefore

$$h = \frac{(1.433 \pm 0.0025) \times (1.372 \pm 0.0014) \times 10^{-16}}{2.9986 \times 10^{10}} = 6.557 \pm 0.013. \qquad (12)$$

Method 3. From the Rydberg Constant N_0, by Bohr's Theory of Atomic Structure.

$$N_0 = \frac{2\pi^2 e^4}{h^3(e/m)}, \qquad (13)^1$$

where the term $(1 + m/M)$ has been omitted, since it differs from unity by only one part in 1852.6. The best value of N_0 is[5]

$$109{,}678.6 \times 2.9986 \times 10^{10} = 3.28882 \times 10^{15} \text{ sec}^{-1}.$$

The mean value of e/m, for negative electrons[6] is 1.772×10^7 E.M.U. From the Zeeman effect[3] the mean value is 1.775. The former group of experiments is probably the more accurate, and hence I have assumed

$$e/m = (1.773 \pm 0.002) \times 10^7 \text{ E.M.U.} \qquad (14)$$

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[1] Masius, *loc. cit.*

[2] Bulletin Bur. of Standards, 13, 459, 1916.

[3] Private communication to the author. Mendenhall (Phys. Rev. (2), 10, 515, 1917) obtains 1.440 and concludes that there is a real discrepancy between the radiation and electrical values of *h*. The results presented in this paper show no such discrepancy.

[4] Bohr, Phil. Mag., 26, 1, 1913.

[5] Birge, Science 48, 47, 1918.

[6] Kaye and Laby, pp. 98–99, 1918 edition.

Therefore

$$h = \sqrt[3]{\frac{2\pi^2\{(4.774 \pm 0.005) \times 10^{-10}\}^5}{(1.773 \pm 0.002) \times 10^7 \times 2.9986 \times 10^{10} \times 3.2888 \times 10^{15}}} \qquad (15)$$

$$= 6.542 \pm 0.011.$$

METHOD 4. FROM EINSTEIN'S PHOTO-ELECTRIC EQUATION.

$$V \cdot e = h \cdot \nu - p. \qquad (16)$$

Millikan[1] gives for the final value of h, from his photoelectric work, 6.57 ± 0.5 per cent. However, this is the result of the work with sodium only. If the value obtained with lithium (6.584 ± 1.0 per cent.) is included also, we have for the final mean 6.572 ± 0.4 per cent. But this corresponds to a value of $c = 3.000$. Using, instead, 2.9986, we obtain

$$h = 6.578 \pm 0.026. \qquad (17)$$

METHOD 5. FROM THE QUANTUM RELATION, AS APPLIED TO GENERAL X-RADIATION.

$$V \cdot e = h \cdot \nu, \qquad (18)$$

where

$$\nu = \frac{1}{2d \sin \theta}, \qquad (19)$$

and

d = calcite grating space.

If V is expressed in volts, and ν in frequency numbers (number of wave-lengths in one cm.) we have

$$h = \frac{2 \cdot d \cdot \sin \theta \cdot e \cdot V \cdot 10^8}{c^2}. \qquad (20)$$

The procedure of Blake and Duane[2] is so accurate that the probable error of the experimental work itself ($V \cdot \sin \theta$) is about 0.02 per cent. (from the results for h on page 634 loc cit.). The authors therefore rightly conclude that the final error in h is due almost entirely to the uncertainty in the values of e and d. Blake and Duane use for d Compton's value[3] of 3.028. Compton, in a later article[4] concludes that the most probable value is

$$d = (3.0281 \pm 0.0010) \times 10^{-8} \text{ cm}. \qquad (21)$$

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[1] PHYS. REV. (2), 7, 355, 1916.

[2] PHYS. REV. (2), 10, 624, 1917. In addition, A. Müller (Phys. Zeits., 19, 489, 1918; Science Abstracts, No. 303, March, 1919) obtains $h = 6.58 \pm 0.07$, while E. Wagner (Ann. d. Physik, 57, 401, 1918; Science Abstracts, No. 440, April, 1919) obtains $h = 6.49$, the accuracy not being stated in the abstract.

[3] PHYS. REV. (2), 7, 655, 1916.

[4] PHYS. REV. (2), 11, 430, 1918.

Uhler[1] considers this estimate of error too small, as it includes only the error in e, which appears here to the one third power, *i.e.*,

$$d = \left(\frac{\frac{1}{2} M_1 e E}{\rho_1 \phi(\beta_1) S c} \right)^{1/3}, \tag{22}$$

where M_1 = molecular weight of $CaCO_3$ = 100.075.
ρ_1 = density of the calcite crystal = 2.7116 gr./c.c.
$\phi(\beta_1)$ = volume of the calcite rhombohedron = 1.0963.
E = electro-chemical equivalent of silver = 0.00111827 × 10 gr. per E.M.U.
S = atomic weight of silver = 107.88.

Compton uses Millikan's value of $N\ (= Sc/eE) = 6.062 \times 10^{23}$, based on $E = 0.001118$. The corrected value of $N = 6.0594$.

Uhler's discussion of the above data indicates that, outside of the error in e, there may be a probable error in d of about 0.06 per cent. In obtaining the final probable error in h, it is not permissible to combine the total probable error in d, with that in e, and in the experimental results. Instead, the error in d, aside from that due to e, must be combined with the total per cent. error in $e^{4/3}$ (of which $e^{1/3}$ appears implicitly in d) and with the experimental error (in $V \cdot \sin\theta$). The error in c^2 is negligible.

Therefore the probable error in h equals

$$\sqrt{0.06^2 + 0.13^2 + 0.02^2} = 0.144 \text{ per cent.} \tag{23}$$

Therefore,

$$h = 6.555 \pm 0.009. \tag{24}$$

Method 6. From Lewis and Adams' Theory of Ultimate Rational Units.

This theory[2] gives directly the value of the Wien constant σ. To obtain h, it is necessary to assume the truth of Planck's formula. Making such an assumption, there results (page 101, *loc. cit.*)

$$h = \frac{16\pi^2 e^2}{c} \sqrt[3]{\frac{8\pi^5}{15}}, \tag{25}$$

where h is expressed as a definite function of the charge on the electron, so that the only error in h is that due to the error in e. This gives

$$h = 6.560 \pm 0.014. \tag{26}$$

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[1] Phys. Rev. (2), 12, 39, 1918.

[2] Phys. Rev. (2), 3, 92, 1914.

METHOD 7. FROM THE QUANTUM RELATION, APPLIED TO IONIZATION AND RESONANCE POTENTIALS.

Foote and Mohler[1] give a list of six ionization potentials, and seven resonance potentials. The arithmetic mean of the thirteen resulting values of h is 6.55. In order however to obtain the most probable value of h, it is best to plot voltage against frequency, and to obtain the slope of this straight line. For

$$h \cdot \nu = \frac{e \cdot V \cdot 10^8}{c^2}, \tag{27}$$

where V = potential in volts,
ν = frequency in wave-numbers.

Therefore

$$h = \frac{e}{c^2} \times \frac{dV}{d\nu}. \tag{28}$$

The thirteen values just mentioned have since been supplemented by data on As, Rb, and Cs[2] and on Ca,[3] but these latter results are rather inaccurate and so can well be omitted. The author accordingly used only the thirteen points and obtained, by least square formulæ, the most probable constants of the voltage-frequency curve. Assuming first the more general relation

$$\text{Voltage} = a \times \text{frequency} + b, \tag{29}$$

there results

$$a = (1.2471 \pm 0.0078) \times 10^{-4},$$

giving

$$h = 6.615 \pm 0.041, \tag{30}$$

while

$$b = -0.041 \pm 0.037 \text{ volts}. \tag{31}$$

The probable error in b is practically equal to the actual intercept, and hence it seems justifiable to assume that the intercept is zero, *i.e.*, to assume the usual quantum relation (27) or (28). Performing, on this assumption, the least square solution, I obtain

$$h = 6.579 \pm 0.021, \tag{32}$$

a result considerably higher than the arithmetic mean. The error in this latter procedure is due to the equal weighting of all points on the curve, regardless of their distance from the origin. The least square solution gives the proper weighting to each point. Equation (32) includes the error in e (0.1 per cent.) as well as that due to the experimental results (0.3 per cent.).

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[1] Phil. Mag., 37, 33, 1919.

[2] Foote, Rognley, and Mohler, PHYS. REV., 13, 59, 1919.

[3] Foote, personal communication to the author.

The seven values of h are given in Table I. The least square mean of these values is

$$h = (6.5543 \pm 0.0025) \times 10^{-27} \text{ erg sec.} \tag{33}$$

The residuals also are given in Table I. In all cases they are less than, or approximately equal to, the probable error, proving that all seven methods are consistent among themselves. Considering all the circumstances, this is a most surprising result.

Table I.

Value of h.	Residual.	Method.	Equation.	Dependence on e.
6.551 ± 0.009	− 0.0033	a	(8)	$e^{4/3}$
6.557 ± 0.013	+ 0.0027	c_2	(12)	e
6.542 ± 0.011	− 0.0123	N_0	(15)	$e^{5/3}$
6.578 ± 0.026	+ 0.0237	photo-elec.	(17)	e
6.555 ± 0.009	+ 0.0007	X-rays	(24)	$e^{4/3}$
6.560 ± 0.014	+ 0.0057	Lewis and Adams.	(26)	e^2
6.579 ± 0.021	+ 0.0247	Ion. Pot.	(32)	e

Mean value of h = 6.5543 ± 0.0025.

It is not to be concluded, from this result, that the probable absolute error in h is only 0.04 per cent. For all seven of the above methods include e to a positive power, and any error in e would therefore affect all of them in approximately the same manner. (The power of e is given in Table I., last column.) But if Lewis and Adams' relation between e and h is a true one, it is possible to eliminate e from each of the other six equations, thus obtaining a value of h independent of Millikan's value of e. Or, by eliminating h, it is possible to obtain an independent value of e.

The most accurate of these six methods is that of Blake and Duane. Accordingly, the elimination will be shown for this one method only. Using equations (20), (22) and (25), we obtain

$$e = \left(\frac{2V\sin\theta\cdot 10^8}{c^2}\right)^{3/2}\left(\frac{1}{2}\frac{M_1E}{\rho_1\phi(\beta_1)Sc}\right)^{1/2}\left(\frac{c}{16\pi^2}\sqrt[3]{\frac{15}{8\pi^5}}\right)^{3/2} \tag{34}$$

and

$$h = \left(\frac{2V\sin\theta\cdot 10^8}{c^2}\right)^{3}\left(\frac{1}{2}\frac{M_1E}{\rho_1\phi(\beta_1)Sc}\right)\left(\frac{c}{16\pi^2}\sqrt[3]{\frac{15}{8\pi^5}}\right)^{2}. \tag{35}$$

Computing backwards from Blake and Duane's results,

$$\left(\frac{2V\sin\theta\cdot 10^8}{c^2}\right) = \frac{h}{e\cdot d} = \frac{6.555 \times 10^{-27}}{4.774 \times 10^{-10} \times 3.028 \times 10^{-8}} \tag{36}$$
$$= 4.5346 \times 10^{-10}.$$

Using for the other quantities in (34) and (35) the values already specified, we obtain

$$e = 4.7705 \pm 0.0050, \tag{37}$$

$$h = 6.550 \pm 0.012. \tag{38}$$

The errors in e and h are obtained from the previously assumed error of 0.02 per cent. in $(V \cdot \sin \theta / c^2)$, and of 0.06 per cent. in $(\frac{1}{2} M_1 E / \rho_1 \phi(\beta_1) S c)^{1/3}$ The velocity of light appears in e as c^{-2}, and in h as c^{-5}. The error in c is probably not over 0.01 per cent.

This new value of e has therefore the same probable error as Millikan's value. Its correctness depends however on the following assumptions:

1. The truth of Lewis and Adams' theory of ultimate rational units.
2. The truth of the relation between a and h, as given by Planck's radiation formula.
3. The truth of the quantum relation, as applied by Blake and Duane to X-ray data.

Because of the more direct method used by Millikan, his value of e is likely to be more nearly correct. We may therefore conclude that

$$\mathbf{h} = (\mathbf{6.5543} \pm \mathbf{0.0025}) \times \mathbf{10^{-27}}\ \textbf{erg sec.},$$

remembering however that in addition to the probable error, this value assumes the correctness of 4.774 as the value of e, and actually possesses a per cent. error somewhat greater than the per cent. error in e, and in the same sense.

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