THE SIZE AND SHAPE OF THE ELECTRON.[1]

BY ARTHUR H. COMPTON.

SYNOPSIS.—Attention is called to two outstanding differences between experiment and the theory of scattering of high frequency radiation based upon the hypothesis of a sensibly point charge electron. In the first place, according to this theory the mass scattering coefficient should never fall below about .2, whereas the observed scattering coefficient for very hard X-rays and γ-rays falls as low as one fourth of this value. In the second place, if the electron is small compared with the wave-length of the incident rays, when a beam of γ-rays is passed through a thin plate of matter the intensity of the scattered rays on the two sides of the plate should be the same, whereas it is well known that the scattered radiation on the emergent side of the plate is more intense than that on the incident side.

It is pointed out that the hypothesis that the electron has a diameter comparable with the wave-length of the hard γ-rays will account qualitatively for these differences, in virtue of the phase difference between rays scattered by different parts of the electron. The scattering coefficient for different wave-lengths is calculated on the basis of three types of electron: (1) A rigid spherical shell of electricity, incapable of rotation; (2) a flexible spherical shell of electricity; (3) a thin flexible ring of electricity. All three types are found to account satisfactorily for the meager available data on the magnitude of the scattering coefficient for various wave-lengths. The rigid spherical electron is incapable of accounting for the difference between the emergent and the incident scattered radiation, while the flexible ring electron accounts more accurately for this difference than does the flexible spherical shell electron.

It is concluded that the diameter of the electron is comparable in magnitude with the wave-length of the shortest γ-rays. Using the best available values for the wave-length and the scattering by matter of hard X-rays and γ-rays, the radius of the electron is estimated as about 2×10^{-10} cm. Evidence is also found that the radius of the electron is the same in the different elements. In order to explain the fact that the incident scattered radiation is less intense than the emergent radiation, the electron must be subject to rotations as well as translations.

I. THE SCATTERING OF HIGH FREQUENCY RADIATION.

THE radius of the electron is usually calculated from its kinetic energy when in motion, taking this to be identical with its magnetic energy. According to the customary assumption that the charge on an electron is uniformly distributed over the surface of a sphere, the radius of the sphere as thus calculated is about 1×10^{-13} cm. There are, however, a number of phenomena in connection with the scattering and absorption of high frequency radiation by matter, which appear to be

[1] A preliminary paper on this subject was read before the American Physical Society, December 28, 1917 (PHYS. REV., 11, 330, 1918). Cf. also J. Wash. Ac. Sci., 8, 1, 1918.

inexplicable according to classical electrodynamics if the dimensions of the electron are taken to be of this order of magnitude, whose explanation is obvious if the electron is assumed to be a flexible ring of electricity whose radius is comparable with the wave-length of short γ-rays. This paper is the first of a series of three, which will deal respectively with the scattering of high-frequency radiation, the absorption of high-frequency radiation, and the nature of the ultimate magnetic particle. The present discussion will deal with certain outstanding differences between experiment and the theory of scattering of high frequency radiation based upon the hypothesis of a sensibly point charge electron, and it will be shown that these differences may be explained on the basis of an electron of relatively large size. In order to preserve the directness of the argument, the details of the calculations will be reserved for the latter part of the paper.

Part I.

A. *The Scattering Coefficient of High Frequency Radiation.*

On the hypothesis that the electron is sensibly a point charge of electricity, Sir J. J. Thomson has shown[1] that the ratio of the energy of the electromagnetic radiation scattered by an isolated electron to the energy incident upon it is given by the expression

$$\frac{8\pi}{3}\frac{e^4}{m^2C^4}.$$

Here e and m are respectively the charge and mass of the electron, and C is the velocity of light. If the electrons in any substance act independently of each other, the scattering coefficient per unit mass of the substance will therefore be

$$\frac{\sigma}{\rho}=\frac{8\pi}{3}\frac{Ne^4}{m^2C^4}, \qquad (1)$$

where σ is the ratio of the scattered to the incident energy per unit volume of the material, ρ is its density, and N is the number of electrons in unit mass of the substance.

Since this scattered energy is lost from the primary beam, the quantity σ/ρ represents also the part of the mass absorption coefficient which is due to scattering. As Barkla has pointed out,[2] there may be absorption due to other causes, such as the production of secondary beta or cathode rays, but this absorption due to scattering must always be present. Moreover, if the electrons in the absorbing material are grouped together in regions which are small compared with the wave-length of the incident

[1] J. J. Thomson, Conduction of Electricity through Gases, 2d ed., p. 325.

[2] C. G. Barkla and M. P. White, Phil. Mag., 34, 275, 1917.

beam, the electrons do not scatter independently. The rays scattered by the electrons in this case overlap in such a manner that a certain amount of "excess scattering" occurs. There is, however, no arrangement of the electrons which will result in less scattering than when they act independently. On the hypothesis of a point charge electron, it is possible for the scattering, and hence for the mass absorption coefficient, to be smaller than that predicted by Thomson's expression (1) only in case the electrons are held in position so firmly that their natural period of vibration is shorter than the period of the incident radiation.

In making the calculations from Thomson's theory, it may be assumed that the number of electrons in an atom which are effective in scattering the incident radiation is equal to the atomic number. This assumption is supported in the case of the lighter elements by the experiments of Barkla and Dunlop[1] when X-rays of ordinary hardness are used. It would seem possible that with the higher frequency γ-rays certain electrons might be effective in scattering which are too rigidly bound to scatter X-rays. Such an effect, however, would mean an increase rather than a decrease in the scattering for the shorter wave-lengths. That the atomic number is the number of effective electrons when γ-rays are used, is confirmed by the observations of Soddy and Russell[2] and of Ishino[3] to the effect that for the shortest rays the amount of energy scattered by atoms of the different elements is accurately proportional to their atomic numbers. This means that all the electrons outside of the nucleus are effective in producing absorption when hard γ-rays are used. If the electron is sensibly a point charge of electricity, the scattered energy should therefore be at least as great as the value assigned by equation (1).

Barkla and Dunlop[4] have shown that for a considerable range of wave-lengths of X-rays the mass scattering coefficients of the lighter elements are given accurately by equation (1) if the number of electrons in the atom is taken to be approximately half the atomic weight. For elements of high atomic weight, the scattering becomes greater than this value except for very short wave-lengths, indicating that the electrons are so closely packed that "excess scattering" occurs. For wave-lengths less than 2×10^{-9} cm., however, Barkla and White[5] have shown that the total mass absorption coefficient of the light elements is less than the value theoretically calculated for the mass scattering coefficient alone; and Soddy and Russell[6] have found that for the hard γ-rays from Radium

[1] C. G. Barkla and J. G. Dunlop, Phil. Mag., 31, 222, 1916.

[2] Soddy and Russell, Phil. Mag. 18, 620, 1910; 19, 725, 1910.

[3] M. Ishino, Phil. Mag., 33, 140, 1917.

[4] C. G. Barkla and J. G. Dunlop, *loc. cit.*

[5] C. G. Barkla and M. P. White, *loc. cit.*, p. 277.

[6] Soddy and Russell, *loc. cit.*

C the absorption of substances of lower atomic weight than mercury is only a small fraction of that required by Thomson's expression. Direct measurements of the scattering of hard γ-rays confirm the conclusions based on absorption measurements. Thus it has been found[1] in the case of radiation of very high frequency that the mass scattering coefficient falls as low as one fourth of the value predicted by Thomson's theory.

As has just been pointed out, it is impossible, according to classical electrodynamics, to account for this low scattering and absorption of radiation of very high frequency by matter if the electron is taken to be sensibly a point charge of electricity. If, on the other hand, the electron is considered to have a radius comparable with the wave-length of the incident radiation, a qualitative explanation of the phenomenon of low scattering for short wave-lengths is obvious. The effect of this hypothesis is to make an appreciable phase difference between the rays scattered by different parts of the electron. Thus the radiation scattered from A, Fig. 1, traverses a longer path between S and P than does the ray scattered from the part of the electron at B. If the wave-length is many times the diameter of the electron, the phase difference between these two rays will be negligible, and the reduction in the intensity of the scattered beam will be inappreciable; if, however, the difference in the two paths is comparable with the wave-length of the incident radiation, the phase difference will be such that the intensity of the ray scattered to P will be much reduced. The assumption of a relatively large electron is therefore capable of explaining qualitatively the observed decrease in the scattering of electromagnetic radiation when the wave-length becomes very short.

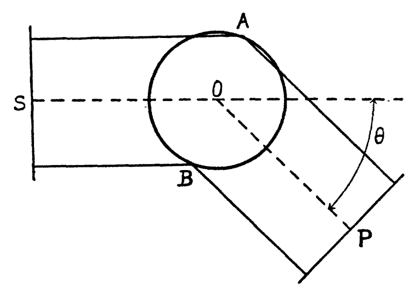

Fig. 1.

Calculation of the Scattering. I. *Rigid Spherical Electron.*—The exact manner in which the scattering will decrease with shorter wave-lengths will of course depend upon the form of electron considered. For example, taking the simplest case of the scattering due to a rigid, uniform, spherical shell of electricity, incapable of rotation, we find

$$\frac{\sigma}{\rho} = \frac{8\pi}{3}\frac{e^4N}{m^2C^4}\sin^4\left(\frac{2\pi a}{\lambda}\right)\bigg/\left(\frac{2\pi a}{\lambda}\right)^4, \tag{2}$$

[1] M. Ishino, *loc. cit.*

where a is the radius of the spherical shell and λ is the wave-length of the incident beam. The details of the derivation of this expression will be found in Part 2, Section 1. If $a = 10^{-13}$ cm., as usually assumed, this is practically identical with equation (1), even for radiation of the shortest known wave-length. The relative values of the scattering according to this expression are shown in curve *I*, Fig. 2, for different values of λ/a. In this diagram the value $\sigma/\sigma_0 = 1$ indicates the magnitude of the mass scattering if the electron were sensibly a point charge of electricity, and the calculated values are given in terms of this quantity.

2. *Flexible Spherical Electron.*—It would appear more reasonable to suppose that the spherical shell electron is subject to rotational as well as to translational displacements when traversed by a γ-ray. The scattering due to such an electron is difficult to calculate, but an approximate expression can be obtained if the electron is considered to be perfectly flexible, so that each part of it can be moved independently of the other parts. On this hypothesis it can be shown (cf. Part 2, Section 2) that the intensity of the beam scattered by an electron at an angle θ with an unpolarized beam of γ-rays is given by the expression,

$$(1) \qquad I_\theta = I\frac{e^4(1 + \cos^2\theta)}{2L^2m^2C^4}\left\{\sin^2\left(\frac{4\pi a}{\lambda}\sin\frac{\theta}{2}\right)\Big/\left(\frac{4\pi a}{\lambda}\sin\frac{\theta}{2}\right)^2\right\}.$$

Here I is the intensity of the incident beam, L is the distance at which the intensity of the scattered beam is measured, and the other quantities have the same meaning as before. The mass absorption coefficient due to the scattering by such an electron is therefore,

$$(4) \qquad \frac{\sigma}{\rho} = 2\pi NL^2\int_0^\pi \frac{I_0}{I}\sin\theta d\theta.$$

This integral may be evaluated graphically or by expansion into a series (*cf. infra*, equation 17). The values of σ/σ_0 according to equation (4) are plotted in curve *II*, Fig. 2. The values for a rigid spherical electron which is subject to rotation should lie between curves *I* and *II* for the range of wave-lengths for which curve *II* is plotted.

3. *Ring Electron.*—According to electromagnetic theory it is obvious that the mass of an electron cannot be accounted for on the basis of a uniform distribution of electricity over the surface of a sphere of radius comparable with the wave-length of γ-rays. Much the same effect, so far as the scattering of high frequency radiation is concerned, results from the conception of the electron as a ring of electricity of relatively large diameter, similar in form to the "magneton" suggested by A. L. Parson.[1] It has been shown by Webster[2] and Davisson[3] that the assump-

[1] A. L. Parson, Smithsonian Misc. Collections, Nov., 1915.

[2] D. L. Webster, PHYS. REV., 9, 484, 1917.

[3] Davisson, PHYS. REV., 9, 570, 1917.

tion of such an electron is compatible with the electromagnetic theory of mass.

The exact calculation of the scattering produced by a thin ring of electricity is difficult. A chief factor in the complexity of the problem is the fact that the effective electromagnetic mass of a short arc of the ring differs according as it is accelerated parallel to the tangent to the arc, parallel to the axis of the ring, or parallel to a radius of the ring. The ratios of the effective masses along these three axes depends moreover upon the speed with which the electricity in the ring is rotating. In order to make the problem manageable, the assumptions have been made that the mass of an arc element is the same in all directions, and that the velocity of the electricity in the ring is small compared with the velocity of light. On the basis of these assumptions the mass scattering coefficient for a flexible electronic ring is found to be

$$(5)\qquad \frac{\sigma}{\rho}=\frac{8\pi}{3}\frac{Ne^4}{m^2C^4}\left\{1-a\left(\frac{a}{\lambda}\right)^2+b\left(\frac{a}{\lambda}\right)^4-c\left(\frac{a}{\lambda}\right)^6+\cdots\right\},$$

where the coefficients a, b, c, $\cdots$ are constants which are evaluated below (cf. equation 21). The relative values of the scattering according to this expression are shown in curve *III*, Fig. 2. The scattering of γ-rays

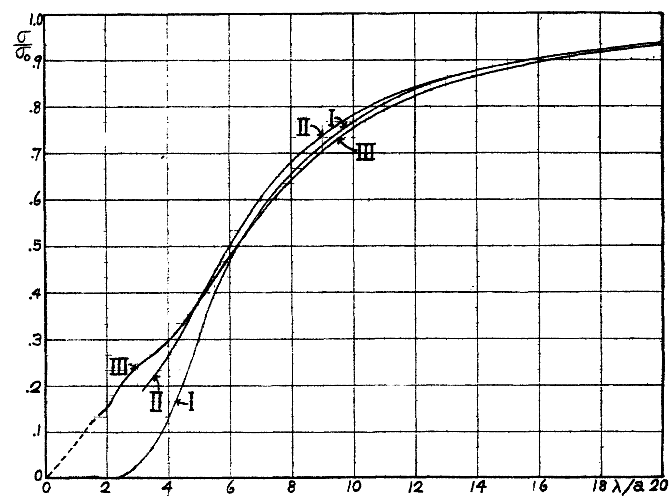

Fig. 2.

by a ring electron as thus calculated is an approximation which will doubtless correspond closely with the true value for relatively long waves, but which may differ appreciably for the shortest known radiation.

Unfortunately the experimental data are too meager to submit these formulæ to accurate quantitative test. There are, however, three points on the curve which have been established with some care. Barkla and Dunlop[1] have found that for relatively soft X-rays the light elements scatter according to equation (1), so that the part of the curves where λ/a is large is verified. Hull and Rice[2] have estimated from their absorption measurements that for X-rays and γ-rays whose wave-lengths are in the neighborhood of 0.15×10^{-8} cm. the value of σ/ρ for aluminium is 0.12. Taking the number of electrons in an aluminium atom to be 13, this gives for the relative scattering, 0.64. According to curve *I* this corresponds to an electronic radius of 1.9×10^{-10} cm. Curve *II* gives 2.0×10^{-10} cm., and curve *III*, 1.9×10^{-10} cm.[3] Ishino[4] finds that the value of σ/ρ for aluminium, using the hard γ-rays from radium C, is about 0.045, which means a value for the relative scattering of 0.24. The work of Rutherford and Andrade[5] shows that the principal part of the "homogeneous" radiations from radium C consists of a strong line $\lambda = 0.099$, and a weaker line $\lambda = .071$, $\times 10^{-8}$ cm. Both of these lines were prominent in Ishino's experiment, in which he filtered the γ-rays through a centimeter of lead. Rutherford has pointed out,[6] however, from a consideration of the velocities of the β-particles, that there must be a certain amount of radiation of much shorter wave-length. The existence of such extremely hard rays is confirmed by the fact that the absorption coefficient of the penetrating radiation of the atmosphere as determined at high altitudes is much smaller than that of the hard γ-rays from radium C, such as used by Ishino. The fact that it is impossible to detect these very short waves by crystal analysis, however, indicates that their effectiveness in the scattered beam is small compared with that of the two lines observed by Rutherford and Andrade. It seems reasonable, therefore, to take for the effective wave-length of the γ-rays used in Ishino's scattering experiments about $.08 \times 10^{-8}$ cm.[7] This gives for the value of the electronic radius, from curve *I*, 1.7, from curve *II*, 2.1 and from curve *III*, 2.7, $\times 10^{-10}$ cm.

[1] C. G. Barkla and Dunlop, *loc. cit.*

[2] A. W. Hull and M. Rice, PHYS. REV., 8, 326, 1916.

[3] In the second part of this paper, by using a more accurate formula for the mass absorption coefficient, the data of Hull and Rice will be shown to lead to a value of $(1.85 \pm .04) \times 10^{-12}$ cm., if the electron is taken to be a ring.

[4] M. Ishino, *loc. cit.*, p. 141.

[5] Sir E. Rutherford and Andrade, Phil. Mag., 28, 263, 1916.

[6] Sir E. Rutherford, Phil. Mag., 34, 153, 1917.

[7] In the paper last referred to, Rutherford estimated the effective wave-length of Ishino's γ-rays to be much shorter than the value here used. His estimate is based upon measurements of the absorption of high frequency X-rays filtered by means of a lead filter. He calculated the frequency of the X-rays according to the relation $h\nu = eV$, taking V to be the maximum voltage applied to the tube. As is apparent from the work of Rutherford, Barnes and Richardson (Phil. Mag., 30, 339, 1915), this relation does not express the *effective* fre-

The value of σ/ρ given by Hull and Rice is a mean over a relatively large range of wave-lengths, and Barkla is of the opinion[1] that Ishino's value of the scattering of the γ-rays from radium C is appreciably in in error because of a too high estimate of the true absorption. Thus, though the experimental values of the electronic radius agree best on the basis of the flexible spherical shell electron, as represented in curve II, the accuracy of the experiments is by no means sufficient to distinguish between the three hypotheses.

The important thing to notice is that if the electrons had dimensions comparable with 10^{-13} cm., as usually assumed, the scattering should be represented by the upper line of Fig. 2 where $\sigma/\sigma_0 = 1.0$. The fact that experiment gives consistently lower values when short wave-lengths are used is sufficient proof that the electron is not sensibly a point charge of electricity. On the other hand, it is possible to account for this reduced scattering within the probable experimental error if the electron has a radius of 2×10^{-10} cm.

B. The Dissymmetry of the Scattering of Hard γ-rays on the Incident and Emergent Sides of a Plate.

A second difficulty which is found with Sir J. J. Thomson's simple theory is that it predicts that if a beam of X-rays is passed through a thin plate of matter, the intensity of the scattered rays on the two sides of the plate should be the same. Barkla and Ayers[2] have shown that, for rather hard X-rays and for those substances of low atomic weight whose mass scattering coefficients can be calculated accurately by equation (1), this second prediction of Thomson's theory is also valid. On the other hand, it is well known that both in the case of relatively soft X-rays and in the case of hard γ-rays the scattered radiation on the emergent side of the plate is more intense than that on the incident side.

When heavy atoms and long waves are used, the dissymmetry between the emergent and the incident scattered radiation is accompanied by an increase in the total scattered energy. For this reason the phenomenon is described by the term "excess scattering." It is satisfactorily accounted for[3] by the fact that the electrons in the heavy atoms do not act independently in scattering the longer wave-length X-rays, since

quency of the filtered rays, especially when a lead filter is used. This doubtless accounts for the fact, which will be brought out in the following paper, that Rutherford's determinations of the absorption do not agree with those of Hull and Rice, who measured the absorption coefficient of homogeneous X-rays of known wave-length.

[1] C. G. Barkla and M. P. White, *loc. cit.*, p. 278.

[2] Barkla and Ayers, Phil. Mag., 21, 271, 1911.

[3] C. G. Darwin, Phil. Mag., 27, 329, 1914; D. L. Webster, Phil. Mag., 25, 234, 1913.

they are grouped so closely together that the rays scattered by the different electrons are in nearly the same phase. This has the effect of increasing the total scattering. But also, since the phase difference between the rays from the different electrons in an atom is less for the scattered rays which make small angles with the primary beam, there is greater reinforcement and hence greater intensity on the emergent than on the incident side of the scattering atom.

This explanation cannot be applied, however, to the case of the unsymmetrical scattering of very hard rays. This is clear for two reasons. In the first place, if an atom of medium weight is traversed by rays of increasing hardness, at first excess scattering occurs as described above; but as the wave-length becomes shorter the scattered radiation becomes nearly symmetrical until the scattered energy can be calculated according to Thomson's formula (1). The electrons now, therefore, are scattering independently, and must continue to do so for all shorter wave-lengths. Thus we see that the dissymmetry in the scattering which reappears as the wave-length becomes very short cannot be accounted for by the mutual action of the separate electrons. In the second place, the phenomenon of unsymmetrical scattering for very short waves is distinguished from the excess scattering which occurs with longer waves by the fact that in the former case the dissymmetry is accompanied not by an increase but by a decrease in the total scattering. If the phenomenon were due to the mutual action of the electrons, it would be accompanied by an increased total scattering, as before. It is thus evident that the unsymmetrical scattering of very short electromagnetic waves is due not to groups of electrons in the atoms, but to some property of the individual electrons.

The qualitative explanation of this phenomenon on the basis of our large electron hypothesis is at once apparent. Referring again to Fig. 1, it is obvious that if the diameter of the electron is comparable with the wave-length of the radiation, there will be an appreciable difference in phase between the rays scattered from different parts of the electron. Since this phase difference is greater for rays scattered at large than for those at small angles, the intensity of the incident radiation will be in the former case the more strongly reduced. In order to explain this phenomenon it is not sufficient, however, merely to assume that the electron is relatively large. For example, the hypothesis of the electron as a rigid spherical shell, incapable of rotation, though resulting in a reduced total scattering, would give rise to symmetrical scattering on the incident and emergent sides of a plate. To account for the observed dissymmetry, the further assumption must be made that the incident electromagnetic wave is capable of moving the different parts of the electron relatively to each other.

If the electron is sensibly a point charge of electricity, the intensity of the beam scattered by an electron at an angle θ with the incident beam is[1]

$$(6) \qquad I_\theta = I\frac{e^4(1 + \cos^2\theta)}{2L^2m^2C^4}.$$

The corresponding expression for an electron in the form of a flexible spherical shell of electricity has already been given:

$$(3) \qquad I_\theta = I\frac{e^4(1 + \cos^2\theta)}{2L^2m^2C^4}\left\{\sin^2\left(\frac{4\pi a}{\lambda}\sin\frac{\theta}{2}\right)^2 \Big/ \left(\frac{4\pi a}{\lambda}\sin\frac{\theta}{2}\right)^2\right\}.$$

For the ring type electron we obtain

$$(7) \qquad I_\theta = I\frac{e^4(1 + \cos^2\theta)}{2L^2m^2C^4}\left\{1 - \alpha\left(\frac{4\pi a}{\lambda}\sin\frac{\theta}{2}\right)^2 + \beta\left(\frac{4\pi a}{\lambda}\sin\frac{\theta}{2}\right)^4 - \cdots\right\},$$

where the values of the constants α, β, γ, etc., are those determined below (equation 20). When a/λ remains small, the scattering according to both expressions (3) and (7) approaches the value for a point charge electron (6).

D. C. H. Florance[2] has determined experimentally the values of the relative intensity of the radiation scattered at different angles when the hard γ-rays from radium bromide traverse a plate of iron. His values are indicated in Fig. 3 by circles. Taking the effective wave-length of these rays to be $.09 \times 10^{-8}$ cm., and using in equations (3) and (7) $a = 2 \times 10^{-10}$ cm. as above estimated, the relative scattering at different angles may be calculated. The intensity of the radiation scattered at different angles by a point charge electron is indicated in Fig. 3 by the outer solid curve, that due to the spherical shell electron by the inner solid curve, and that from the ring form electron by the broken curve. Inasmuch as the γ-rays used by Florance were heterogenous, and as the softer rays are scattered relatively more strongly at larger angles, the agreement of the experimental values with either of the inner curves is as good as can be expected. The important point to be noticed is, however, that the experimental values are entirely out of harmony with

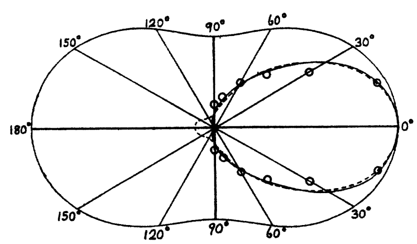

Fig. 3.

[1] J. J. Thomson, Conduction of Electricity through Gases, 2d ed., p. 326.

[2] D. C. H. Florance, Phil. Mag., 20, 921, 1910.

what is to be expected if the electron is sensibly a point charge of electricity.

A better quantitative test of this explanation of the dissymmetry of scattered γ-radiation is afforded by determinations of the ratio of the total radiation scattered on the incident side of a plate struck by hard γ-rays to that scattered on the emergent side. The theoretical value of this ratio is

$$\frac{I_i}{I_e} = \int_{\pi/2}^{\pi} I_\theta \sin\theta d\theta \Bigg/ \int_0^{\pi/2} I_\theta \sin\theta d\theta. \tag{8}$$

The curves of Fig. 4 give the values of this ratio for different values of λ/a, the broken curve being calculated on the basis of the spherical

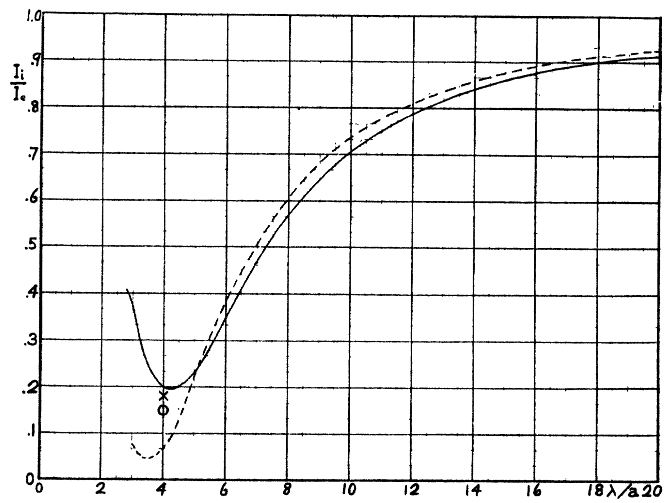

Fig. 4.

electron. These curves doubtless explain at least in part the observation of Florance[1] that the "incident" scattered rays are softer than the "emergent" and the primary rays, since they show that the relative amount of rays scattered backward is much greater for soft or long wavelength γ-rays than for the harder radiation.

The ratio of the incident to the emergent radiation has been determined for the hard γ-rays from radium C by Madsen,[2] who found the value 18 per cent., and Ishino,[3] who found 15 per cent. Assuming for the effective wave-length in this case $\lambda = 0.08 \times 10^{-8}$ cm., and for the radius of the electron 2×10^{-10} cm., as estimated above, Ishino's datum

[1] D. C. H. Florance, Phil. Mag., 27, 225, 1914.
[2] Madsen, Phil. Mag., 17, 423, 1909.
[3] M. Ishino, *loc. cit.*, p. 138.

for this ratio is represented by the open circle in Fig. 4 and Madsen's datum by the cross. It is possible that neither of the theoretical values agree within the probable experimental error with these determinations, but in view of the approximate method of calculating the scattering by a ring electron the difference is not serious. The fact that the predicted values are of the proper order of magnitude is strong evidence that the dissymmetry in the scattering of γ-rays by matter is due to the interference of the rays scattered by different parts of the electron. Thus not only must the electron have a size comparable with the wave-length of γ-rays, but it must also be subject to rotations or be sufficiently flexible for γ-rays to move its different parts relatively to each other.

C. *Conclusions.*

As has been pointed out, therefore, according to classical electrodynamics the mass scattering coefficient for X-rays and γ-rays passing through matter should never fall below the value 0.18, as calculated on the basis of Thomson's theory, if an electron of the usual dimensions is postulated. Experiment shows, however, that for very high frequency radiation the scattering is much less than this. It is possible that certain assumptions regarding the conditions for scattering radiant energy, contrary to classical theory, might be made which would account for the observed low value of the scattering for very high frequencies. As long as the idea of the point charge electron is retained, however, no such assumptions can account for the observed dissymmetry between the incident and the emergent scattered radiations. Unless the theory that X-rays and γ-rays consist of waves or pulses is abandoned, the only possible explanation of this dissymmetry would seem to be that the scattering particles have dimensions comparable with the wave-length of the rays which they scatter. Since the scattering particles have been shown to be the electrons, the statement may therefore be made with confidence that *the diameter of the electron is comparable in magnitude with the wave-length of the shortest γ-rays.*

According to the best available values for the wave-length and the scattering by matter of hard X-rays and γ-rays, *the radius of the electron is about 2×10^{-10} cm.*

The fact that the scattering of hard γ-rays by atoms of the different elements is proportional to the atomic number shows that if the number of the electrons in an atom which are effective in scattering is equal to the atomic number, *the radius of the electron is the same in the different atoms.* This is clear, since if the electron had smaller dimensions in the atoms of one element, the scattering coefficient of this element would not decrease as rapidly for shorter wave-lengths, and the scattering by the different atoms would not be proportional to the number of electrons in the atoms.

In order to explain the fact that the emergent scattered radiation is more intense than the incident radiation, it is necessary to assume further that the different parts of the charge of the electron can possess certain motions independently of each other. That is, *the electron is subject to rotations as well as translations.*

PART 2.

1. *To calculate the energy scattered by a rigid spherical shell electron, incapable of rotation, whose diameter is comparable in magnitude with the wave-length of the incident radiation.*

Let us first derive an expression for the acceleration to which such an electron is subject when traversed by an electromagnetic wave, and then determine the energy scattered by integrating the intensity of the beam due to this acceleration over the surface of a sphere drawn with the electron at the center.

In Fig. 5, let us suppose that the γ-ray traverses the electron along the axis Z. We shall let A represent the amplitude of the incident wave, X its electric intensity at the plane z, and λ its wave-length. The radius of the electron we shall call a, and η will represent the charge on the surface of the electron between two planes z_1 and z_2 placed unit distance apart. As the electricity is by hypothesis distributed uniformly over the surface of the sphere, η is constant between $z = -a$ and $z = +a$, and the total charge on the electron is $e = 2a\eta$. The electric intensity at the plane z at any instant may be expressed by the relation,

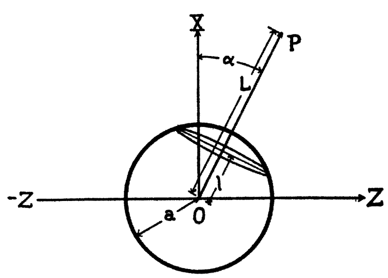

Fig. 5.

$$X = A \cos 2\pi \left(\frac{\delta - z}{\lambda} \right),$$

where $2\pi\delta/\lambda$ is the phase angle at $z = 0$ at that instant. The total force acting on the electron at this instant is therefore,

$$F = \int_{-a}^{a} A \cos 2\pi \left(\frac{\delta - z}{\lambda} \right) \eta dz$$

$$= \frac{A\eta\lambda}{\pi} \cos 2\pi \frac{\delta}{\lambda} \sin 2\pi \frac{a}{\lambda},$$

and the acceleration of the electron is

$$\ddot{x} = \frac{A\eta\lambda}{\pi m}\cos\frac{2\pi\delta}{\lambda}\sin\frac{2\pi a}{\lambda}. \tag{9}$$

Let us now calculate the intensity of the scattered ray at a distance L along the line OP which makes an angle α with the direction OX of the acceleration. According to classical theory the electric intensity at P due to a point charge electron at O subject to this acceleration would be

$$\frac{e\ddot{x}\sin\alpha}{LC^2} \tag{10}$$

where C is the velocity of light. Replacing e by ηdl and $\ddot{x}$ by its value as defined above, and integrating over the electron along the axis L from $l = -a$ to $l = +a$, the electric intensity of the scattered beam at L, α becomes:

$$\begin{aligned} R_{L,\alpha} &= \frac{\sin\alpha}{LC^2}\int_{-a}^{a}\frac{A\eta\lambda}{\pi m}\sin 2\pi\frac{a}{\lambda}\cos\frac{2\pi}{\lambda}(\delta - L + l)\eta dl \\ &= \frac{\eta^2\lambda A}{\pi m LC^2}\sin\alpha\sin 2\pi\frac{a}{\lambda}\int_{-a}^{a}\cos\frac{2\pi}{\lambda}(\delta - L + l)dl \\ &= \frac{\eta^2\lambda^2 A}{\pi^2 LmC^2}\sin^2 2\pi\frac{a}{\lambda}\sin\alpha\cos 2\pi\frac{\delta - L}{\lambda}. \end{aligned}$$

The amplitude of the electric vector of the scattered wave at the point L, α is therefore

$$A_{L,\alpha} = \frac{\eta^2\lambda^2 A}{\pi^2 LmC^2}\sin^2 2\pi\frac{a}{\lambda}\sin\alpha,$$

or substituting for η its value $e/2a$,

$$A_{L,\alpha} = \frac{Ae^2}{LmC^2}\frac{\sin^2(2\pi a/\lambda)}{(2\pi a/\lambda)^2}\cdot\sin\alpha.$$

The intensity of the radiation at this point is

$$I_{L,\alpha} = cA_{L,\alpha}^2 = \frac{cA^2e^4}{L^2m^2C^4}\frac{\sin^4(2\pi a/\lambda)}{(2\pi a/\lambda)^4}\sin^2\alpha,$$

so that the total energy scattered by the electron is

$$\begin{aligned} E_s &= \int_0^{\pi} I_{L,\alpha}\cdot 2\pi L\sin\alpha\cdot L d\alpha \\ &= \frac{2\pi cA^2e^4}{m^2C^4}\cdot\frac{\sin^4(2\pi a/\lambda)}{(2\pi a/\lambda)^4}\int_0^{\pi}\sin^3\alpha d\alpha \\ &= \frac{8\pi cA^2e^4}{3m^2C^4}\cdot\frac{\sin^4(2\pi a/\lambda)}{(2\pi a/\lambda)^4}. \end{aligned}$$

The energy incident on unit area at the electron is, however, $I = cA^2$, so that the fraction of the incident energy scattered by the electron is

$$E_s/I = \frac{8\pi e^4}{3m^2C^4}\frac{\sin^4(2\pi a/\lambda)}{(2\pi a/\lambda)^4}. \tag{11}$$

When a is small compared with λ this becomes

$$\frac{e^4}{m^2C^4}, \tag{12}$$

which is identical with the value given by Thomson[1] for a sensibly point charge electron.

If there are N electrons per unit mass of any substance, the mass scattering coefficient of the substance is therefore

$$\frac{\sigma}{\rho} = \frac{8\pi}{3}\frac{e^4N}{m^2C^4}\frac{\sin^4(2\pi a/\lambda)}{(2\pi a/\lambda)^4}, \tag{13}$$

where σ is the scattering coefficient per unit volume and ρ is the density of the scattering material.

2. *To calculate the energy scattered by an electron in the form of a flexible spherical shell of electricity.*

We shall treat this problem as if the mass of an element of the spherical shell were independent of the rest of the electron, being equal to $dm = m\cdot ds/s$, where s is the area of the surface of the electron, and m is its mass. As has been pointed out in part I. of this paper, the electro-magnetic mass of a spherical electron of the size here considered would be negligible. This form of electron is therefore only a convenient hypothesis to use in calculating the general effect on the scattering to be expected with any form of electron when the wave-length of the incident radiation approaches its largest dimensions.

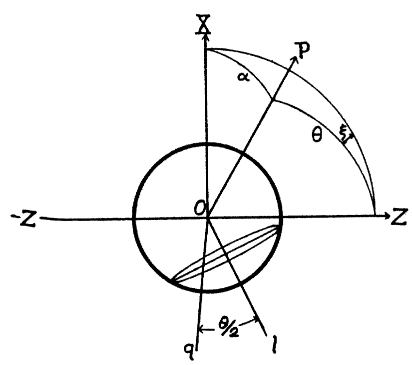

Fig. 6.

Let us suppose that the γ-ray strikes the electron when moving in the direction $-ZOZ$, and determine the intensity of the beam scattered in the direction OP. As shown in Fig. 6, OP makes the angle θ with the incident ray, and the angle α with the direction OX of the electric vector of the incident ray. The plane POZ is inclined at the dihedral angle ξ

[1] J. J. Thomson, *loc. cit.*

with the plane XOZ. We shall draw also the lines Ol and Oq in the plane POZ, line Oq being perpendicular to OZ, and Ol at an angle $\theta/2$ with Oq. The line Ol therefore bisects the external angle $-ZOP$. If now we consider the beam scattered in the direction OP, all the rays scattered from the element of the electron included between the planes l and $l + dl$ are in the same phase at P.

The amplitude of the electric vector at P due to the rays scattered by this element is, in accordance with expression (10),

$$\eta dl \cdot A \frac{\eta}{\mu} \cdot \sin \alpha / LC^2,$$

where η is the charge and μ the mass of the electron per unit distance along Ol, A is the amplitude of the electric vector of the incident beam, and as before L is the distance OP. If $2\pi\delta/\lambda$ is the phase of the ray at P scattered from the element of the electron at $l = 0$ at a given instant, the electric intensity at that instant of the ray scattered from any element to P is

$$\frac{A\eta^2}{\mu} \frac{dl}{LC^2} \sin \alpha \cdot \cos \frac{2\pi}{\lambda}\left(\delta - 2l \sin \frac{\theta}{2}\right),$$

and the electric intensity due to the whole electron is

$$R_{L,\, \alpha,\, \theta,} = \frac{A\eta^2}{\mu} \frac{\sin \alpha}{LC^2} \int_{-a}^{a} \cos \frac{2\pi}{\lambda}\left(\delta - 2l \sin \frac{\theta}{2}\right) dl$$

$$= \frac{A\eta^2\lambda}{2\pi\mu LC^2} \cdot \frac{\sin \alpha}{\sin \theta/2} \cdot \sin\left(\frac{4\pi a}{\lambda} \sin \frac{\theta}{2}\right) \cos \frac{2\pi\delta}{\lambda}.$$

Here, as before, a represents the radius of the electron. The amplitude is obviously the value of this quantity when $\cos(2\pi\delta/\lambda) = 1$, and the intensity of the ray scattered ot P by the electron is therefore

$$I_{L,\, \alpha,\, \theta} = cA_{L,\, \alpha,\, \theta}^2$$

$$= \frac{cA^2\eta^4\lambda^2}{4\pi^2\mu^2L^2C^4} \cdot \frac{\sin^2 \alpha}{\sin^2 \theta/2} \sin^2\left(\frac{4\pi a}{\lambda} \sin \frac{\theta}{2}\right).$$

Since $2a\eta = e$, $2a\mu = m$ and $cA^2 =$ the intensity I of the incident beam, the first factor of this expression becomes

$$\frac{Ie^4}{m^2L^2C^4}\left(\frac{\lambda}{4\pi a}\right)^2.$$

When α is expressed in terms of θ and ξ we obtain

$$\cos \alpha = \sin \theta \cos \xi,$$

i. e.,

$$\sin \alpha = \sqrt{1 - \sin^2 \theta \cos^2 \xi}.$$

We may therefore write:

$$I_{L,\xi,\theta} = \frac{Ie^4}{m^2L^2C^4}\left(\frac{\lambda}{4\pi a}\right)^2 \sin^2\left(\frac{4\pi a}{\lambda}\sin\frac{\theta}{2}\right)\frac{1-\sin^2\theta\cos^2\xi}{\sin^2(\theta/2)}.$$

The intensity of the beam scattered by an electron at an angle θ with an unpolarized incident beam is the average of this quantity for all values of ξ, or

$$\begin{aligned} I_{L,\theta} &= \frac{1}{2\pi}\int_0^{2\pi} I_{L,\xi,\theta}\,d\xi \\ &= \frac{Ie^4}{2m^2L^2C^4}\left(\frac{\lambda}{4\pi a}\right)^2 \sin^2\left(\frac{4\pi a}{\lambda}\sin\frac{\theta}{2}\right)\frac{(\cos^2\theta+1)}{\sin^2(\theta/2)}. \end{aligned} \tag{14}$$

The total energy scattered in unit time by the electron is given by the quantity

$$\begin{aligned} E_s &= \int_0^{\pi} I_{L,\theta}\cdot 2\pi L^2 \sin\theta d\theta \\ &= \frac{\pi Ie^4}{m^2C^4}\left(\frac{\lambda}{4\pi a}\right)^2\cdot\int_0^{\pi}\sin^2\left(\frac{4\pi a}{\lambda}\sin\frac{\theta}{2}\right)\frac{(\cos^2\theta+1)}{\sin^2(\theta/2)}\sin\theta d\theta. \end{aligned} \tag{15}$$

This integral may be evaluated either graphically or by means of a series. To integrate by series, substitute $\theta/2 = x$ and $4\pi a/\lambda = b$. The integral factor then becomes:

$$\int_0^{\pi/2}\sin^2(b\sin x)\frac{4\sin^4 x - 4\sin^2 x + 2}{\sin^2 x}\cdot 2\sin x\cos x\cdot 2\,dx.$$

Writing $b\cdot\sin x = z$, this reduces to

$$8\int_0^b \sin^2 z\left(\frac{2z^3}{b^4} - \frac{2z}{b^2} + \frac{1}{z}\right)dz.$$

If $\sin^2 z$ is expanded into the series

$$\sin^2 z = z^2 - \left(\frac{1}{3!}+\frac{1}{3!}\right)z^4 + \left(\frac{1}{5!}+\frac{1}{3!3!}+\frac{1}{5!}\right)z^6 - \cdots,$$

each term may be integrated separately, the result being,

$$8b^2\{\alpha - \beta b^2 + \gamma b^4 - \cdots\},$$

where

$$\alpha = \frac{1}{3} \qquad = .33333,$$

$$\beta = \left(\frac{1}{4}-\frac{1}{3}+\frac{1}{4}\right)\left(\frac{1}{3!}+\frac{1}{3!}\right) \qquad = .05556,$$

$$\gamma = \left(\frac{1}{6}-\frac{1}{4}+\frac{1}{5}\right)\left(\frac{1}{5!}+\frac{1}{3!3!}+\frac{1}{5!}\right) \qquad = .00519,5,$$

$$\delta = \left(\frac{1}{8}-\frac{1}{5}+\frac{1}{6}\right)\left(\frac{1}{7!}+\frac{1}{5!3!}+\frac{1}{3!5!}+\frac{1}{7!}\right) = .00029,10,$$

$\epsilon = .00001, 076$ $\eta = .00000, 00053, 5,$

$\zeta = .00000, 0280$ $\theta = .00000, 00000, 77.$

The total energy scattered by the electron in unit time is therefore

$$E_s = \frac{8\pi I e^4}{m^2C^4}\left\{\alpha - \beta\left(\frac{4\pi a}{\lambda}\right)^2 + \gamma\left(\frac{4\pi a}{\lambda}\right)^4 - \cdots\right\}. \tag{16}$$

When the wave-length is large compared with the radius of the electron, all terms after the first are negligible, in which case

$$(E_s/I)_{a/\lambda=0} = \frac{8\pi}{3}\frac{e^4}{m^2C^4},$$

as it should. Writing as before N as the number of electrons per unit mass, σ as the scattering coefficient per unit volume and ρ as the density, the mass scattering coefficient of a substance composed of flexible spherical electrons is

$$\frac{\sigma}{\rho} = \frac{8\pi e^4 N}{m^2C^4}\left\{\alpha - \beta\left(\frac{4\pi a}{\lambda}\right)^2 + \gamma\left(\frac{4\pi a}{\lambda}\right)^4 - \cdots\right\}. \tag{17}$$

3. *To calculate the energy scattered by an electron in the form of a thin, flexible, circular ring of electricity.*

In order to account for the electromagnetic mass of a ring electron, Webster[1] and Davisson[2] have shown that the ring must be very thin compared with its diameter. As a result of this fact, the inertia of any element of the ring is practically dependent only upon those parts of the ring immediately adjacent to it. Unless the wave-length is much smaller than the diameter of the electron, therefore, it is permissible to treat the mass of an element of the electronic ring as having a definite value.

Difficulties arise in this calculation, however, from the fact that the electromagnetic mass of an element differs according as the element is accelerated perpendicular or parallel to the tangent to the electronic ring at that point. The effective perpendicular mass depends also upon the speed with which the ring of electricity is rotating. For purposes of calculation I have assumed that the speed with which the electricity in the ring is moving is small compared with the velocity of light, and also that the mass of an element of the electron is independent of the direction of the acceleration. Admittedly the latter assumption makes the calculated value of the scattering only approximate, but it is probable that except possibly for the hardest γ-rays the approximation is close. I have further assumed, as in the case of the sphere, that the ring electron

[1] D. L. Webster, *loc. cit.*

[2] Davisson, *loc. cit.*

is flexible, *i. e.*, that the different parts of the charge are free to move relatively to each other. As was pointed out when the flexible sphere was considered, for comparatively long waves such an electron will scatter in practically the same manner as will a rigid electron which is free to rotate about any axis; for very short waves, however, the scattering by the two types of electron will not be exactly the same. The expression derived below for the scattering by a ring electron may therefore be relied upon for any except very short γ-rays.

In Fig. 7, imagine a beam of γ-rays going in the direction $-ZOZ$, and being scattered by an electron of radius a, represented by the heavy ring. Let us first determine the energy scattered in the direction OP, at

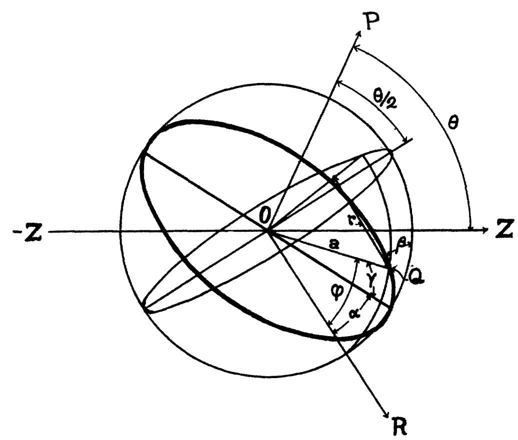

Fig. 7.

an angle θ with OZ. As is evident from the figure, the plane of the electronic ring makes a dihedral angle β with the plane ZOP, and the line of intersection of these two planes makes an angle α with the line OR bisecting the external angle $-ZOP$.

Consider first the energy scattered toward P due to the component of the electric vector which is perpendicular to the plane of the paper. If L is the distance, in the direction OP, at which the scattered radiation is evaluated, the electric displacement due to an element of the electron at $Q(a, \alpha, \beta, \gamma)$ is, in accordance with expression (10),

$$\frac{\eta a d\gamma \cdot A_1 \frac{\eta}{\mu}}{LC^2} \cdot \cos\frac{2\pi}{\lambda}\left(\delta - 2r \sin\frac{\theta}{2}\right).$$

In this statement, η is the charge and μ the mass per unit length of the circumference of the ring, A_1 is the amplitude of the electric vector perpendicular to the plane of the paper, r is the distance from Q to a plane drawn through O perpendicular to OR, and $2\pi\delta/\lambda$ is the phase angle of a ray scattered to P from this plane.

The displacement at P due to the whole electron is therefore

$$\frac{A_1\eta^2 a}{\mu L C^2}\int_0^{2\pi}\cos\frac{2\pi}{\lambda}\left(\delta - 2r\sin\frac{\theta}{2}\right)d\gamma.$$

This quantity is obviously a maximum when $\delta = 0$, so the amplitude at P due to the rays scattered by the whole electron is

$$\frac{A_1\eta^2 a}{\mu L C^2}\int_0^{2\pi}\cos\frac{4\pi}{\lambda}\left(r\sin\frac{\theta}{2}\right)d\gamma,$$

and the corresponding intensity is

$$(18)\qquad \frac{cA_1^2\eta^4 a^2}{\mu^2 L^2 C^4}\left\{\int_0^{2\pi}\cos\frac{4\pi}{\lambda}\left(r\sin\frac{\theta}{2}\right)d\gamma\right\}^2.$$

This expression represents the intensity at P due to an electron with the particular orientation defined by the values of α and β. The probable intensity at P due to polarized rays scattered by an electron at O is the average of this quantity for all values of α and β; *i. e.*,

$$\frac{cA_1^2\eta^4 a^2}{\mu^2 L^2 C^4}\int_0^{\pi}\frac{d\alpha}{\pi}\int_0^{\pi}\frac{d\beta}{\pi}\left\{\int_0^{2\pi}\cos\frac{4\pi}{\lambda}\left(r\sin\frac{\theta}{2}\right)d\gamma\right\}^2.$$

In a similar manner, the probable intensity at P due to the component of the incident electric vector which is parallel with the plane of the paper is

$$\frac{cA_2^2\eta^4 a^2\cos^2\theta}{\mu^2 L^2 C^4}\int_0^{\pi}\frac{d\alpha}{\pi}\int_0^{\pi}\frac{d\beta}{\pi}\left\{\int_0^{2\pi}\cos\frac{4\pi}{\lambda}\left(r\sin\frac{\theta}{2}\right)d\gamma\right\}^2,$$

where A_2 is the component of the incident amplitude parallel to the plane of the paper. Since on the average $A_1 = A_2$, and since the intensity of the unpolarized incident beam is $I = c(A_1^2 + A_2^2)$, we may write as the intensity of the beam scattered to P by an electron at O traversed by an unpolarized γ-ray,

$$I_{L,\theta} = \frac{I\eta^4 a^2}{\pi^2\mu^2 L^2 C^4}\,\frac{(1+\cos^2\theta)}{2}\int_0^{\pi}d\alpha\int_0^{\pi}d\beta\left\{\int_0^{2\pi}\cos\frac{4\pi}{\lambda}(r\sin\theta/2)d\lambda\right\}^2.$$

Remembering that $2\pi a\eta = e$ and $\eta/\mu = e/m$, the intensity of the beam scattered at the angle θ by an electron is

$$(19)\quad I_{L,\theta} = \frac{Ie^4(1+\cos^2\theta)}{8\pi^4 m^2 L^2 C^4}\int_0^{\pi}d\alpha\int_0^{\pi}d\beta\left\{\int_0^{2\pi}\cos\frac{4\pi}{\lambda}(r\sin\theta/2)d\gamma\right\}^2.$$

To evaluate this expression it is necessary to write r in terms of α, β and γ. In Fig. 7,

$$r = \alpha\cos\varphi$$

and

$$\cos\varphi = \cos\alpha\cos\gamma - \sin\alpha\cos\beta\sin\gamma.$$

Thus

$$r = a\,(\cos\alpha\cos\gamma - \sin\alpha\cos\beta\sin\gamma).$$

The first integral of equation (19) then becomes

$$F_1 = \int_0^{2\pi} \cos\frac{4\pi a}{\lambda}\left\{\sin\frac{\theta}{2}\,(\cos\alpha\cos\gamma - \sin\alpha\cos\beta\sin\gamma)\right\}d\gamma.$$

Substitute

$$k = \frac{4\pi a}{\lambda}\sin\frac{\theta}{2}\cos\alpha,$$

and

$$l = \frac{4\pi a}{\lambda}\sin\theta/2\sin\alpha\cos\beta.$$

Then

$$F_1 = \int_0^{2\pi}\cos(k\cos\gamma - l\sin\gamma)d\gamma.$$

We may write

$$k\cos\gamma - l\sin\gamma = m\sin(\gamma + \Delta),$$

where m is the maximum value of $k\cos\gamma - l\sin\gamma$, *i. e.*,

$$\begin{aligned} m &= \sqrt{k^2 + l^2} \\ &= \frac{4\pi a}{\lambda}\sin\frac{\theta}{2}\sqrt{1 - \sin^2\alpha\sin^2\beta}, \end{aligned}$$

and Δ is the appropriate phase angle. With this substitution,

$$F_1 = \int_0^{2\pi}\cos\{m\sin(\gamma + \Delta)\}d\gamma.$$

Since the integration extends from 0 to 2π, the value of Δ is immaterial, and may therefore be put to equal zero. The integral then becomes,

$$\begin{aligned} F_1 &= \int_0^{2\pi}\cos(m\sin\gamma)\,d\gamma \\ &= 2\pi\cdot\frac{1}{\pi}\int_0^{\pi}\cos(m\sin\gamma)\,d\gamma \\ &= 2\pi\cdot J_0(m), \end{aligned}$$

where

$$J_0(m) = 1 - \frac{m^2}{2^2} + \frac{m^4}{2^2\cdot 4^2} - \frac{m^6}{2^2\cdot 4^2\cdot 6^2} + \cdots,$$

i. e., Bessel's J function of the zero order. Thus

$$F_1 = 2\pi J_0\left(\frac{4\pi a}{\lambda}\sin\frac{\theta}{2}\sqrt{1 - \sin^2\alpha\sin^2\beta}\right).$$

The second integral of expression (19) may be written

$$F_2 = \int_0^\pi F_1^2 d\beta = \int_0^\pi \left\{ 2\pi J_0 \left(\frac{4\pi a}{\lambda} \sin\frac{\theta}{2} \sqrt{1 - \sin^2\alpha \sin^2\beta} \right) \right\}^2 d\beta.$$

By substituting

$$\frac{4\pi a}{\lambda} \sin \theta/2 = k$$

and $\sin \alpha = l$ this is reduced to the form

$$F_2 = 4\pi^2 \int_0^\pi J_0^2(k \sqrt{1 - l^2 \sin^2\beta}) d\beta.$$

The integral can be evaluated by expansion into a series of the form

$$J_0^2(x) = 1 - Ax^2 + Bx^4 - Cx^6 + \cdots,$$

and integrating term by term. In this series

$$A = \frac{1}{2^2} + \frac{1}{2^2},$$

$$B = \frac{1}{2^2 \cdot 4^2} + \frac{1}{2^2 \cdot 2^2} + \frac{1}{2^2 \cdot 4^2},$$

$$C = \frac{1}{2^2 \cdot 4^2 \cdot 6^2} + \frac{1}{2^2 \cdot 4^2 \cdot 2^2} + \frac{1}{2^2 \cdot 2^2 \cdot 4^2} + \frac{1}{2^2 \cdot 4^2 \cdot 6^2},$$

$$D = \frac{1}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} + \frac{1}{2^2 \cdot 4^2 \cdot 6^2 \cdot 2^2} + \frac{1}{2^2 \cdot 4^2 \cdot 2^2 \cdot 4^2} + \frac{1}{2^2 \cdot 2^2 \cdot 4^2 \cdot 6^2} + \frac{1}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2},$$

etc.

Performing the integration we obtain

$$F_2 = 4\pi^3 \{M + Nl + Ol^2 + Pl^3 + \cdots\},$$

where

$$M = J_0^2(k),$$

$$N = \tfrac{1}{2}(Ak^2 - 2Bk^4 + 3Ck^6 - \cdots),$$

$$O = \tfrac{1}{2} \cdot \tfrac{3}{4}(Bk^4 - 3Ck^6 + 6Dk^8 - 10Ek^{10} + \cdots),$$

$$P = \tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{5}{6}(Ck^6 - 4Dk^8 + 10Ek^{10} - 20Fk^2 + \cdots),$$

etc.

The third integral is

$$F^3 = \int_0^\pi F_2 d\alpha = 4\pi^3 \int_0^\pi (M + N \sin^2\alpha + o \sin^4\alpha + \cdots) d\alpha$$

$$= 4\pi^4 \left(M + \tfrac{1}{2} N + \tfrac{1}{2} \cdot \tfrac{3}{4} O + \tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{5}{6} P + \cdots\right).$$

Substituting the above values of M, N, O, etc., this may be written

$$F^3 = 4\pi^4 (1 - \alpha k^2 + \beta k^4 - \gamma k^6 + \cdots),$$

where

$$\alpha = (1 - \tfrac{1}{4})A = .37500,$$

$$\beta = (1 - \tfrac{2}{4} + \tfrac{1}{4}\cdot\tfrac{9}{16})B = .06006, 0,$$

$$\gamma = (1 - \tfrac{3}{4} + 3\cdot\tfrac{1}{4}\cdot\tfrac{9}{16} - \tfrac{1}{4}\cdot\tfrac{9}{16}\cdot\tfrac{25}{36})C = .00498, 48,$$

$$\delta = (1 - \tfrac{4}{4} + 6\cdot\tfrac{1}{4}\cdot\tfrac{9}{16} - 4\cdot\tfrac{1}{4}\cdot\tfrac{9}{16}\cdot\tfrac{25}{36} + \tfrac{1}{4}\cdot\tfrac{9}{16}\cdot\tfrac{25}{36}\cdot\tfrac{49}{64})D = .00025, 060,$$

$$\epsilon = (1 - \tfrac{5}{4} + 10\cdot\tfrac{1}{4}\cdot\tfrac{9}{16} - 10\cdot\tfrac{1}{4}\cdot\tfrac{9}{16}\cdot\tfrac{25}{36} + 5\cdot\tfrac{1}{4}\cdot\tfrac{9}{16}\cdot\tfrac{25}{36}\cdot\tfrac{48}{64} - \tfrac{1}{4}\cdot\tfrac{9}{16}\cdot\tfrac{25}{36}\cdot\tfrac{49}{64}\cdot\tfrac{81}{100})E = .00000, 84241,$$

$$\zeta = .00000, 02023, 6,$$

$$\eta = .00000, 00036, 51,$$

$$\theta = .00000, 00000, 5056.$$

The scattering at the angle θ is therefore

$$(20)\quad I_{L,\theta} = \frac{Ie^4(1 + \cos^2\theta)}{2m^2L^2C^4}\left\{1 - \alpha\left(\frac{4\pi a}{\lambda}\sin\frac{\theta}{2}\right)^2 + \beta\left(\frac{4\pi a}{\lambda}\sin\frac{\theta}{2}\right)^4 - \cdots\right\}.$$

The rate at which energy is scattered by an electron is obtained by integrating this expression over the surface of a sphere of radius L. That is

$$E_s = \int_0^\pi I_{L,\theta}\cdot 2\pi L^2 \sin\theta d\theta,$$

and since

$$\frac{\sigma}{\rho} = \frac{N}{I}E_s,$$

$$\frac{\sigma}{\rho} = \frac{Ne^4}{2m^2C^4}\int_0^\pi (1 + \cos^2\theta)\sin\theta\left\{1 - \alpha\left(\frac{4\pi a}{\lambda}\sin\frac{\theta}{2}\right)^2 + \beta\left(\frac{4\pi a}{\lambda}\sin\frac{\theta}{2}\right)^4 - \cdots\right\}d\theta.$$

If for $\sin^2(\theta/2)$ we substitute $1/2(1 - \cos\theta)$, this expression is immediately integrable, and we obtain for the mass scattering coefficient,

$$(21)\quad \frac{\sigma}{\rho} = \frac{8\pi}{3}\frac{Ne^4}{m^2C^4}\left(1 - a\left(\frac{a}{\lambda}\right)^2 + b\left(\frac{a}{\lambda}\right)^4 - c\left(\frac{a}{\lambda}\right)^6 + \cdots\right),$$

where

$$a = \tfrac{3}{4}(8\pi^2)(1 + \tfrac{1}{3})\alpha = 29.60881,$$

$$b = \tfrac{3}{4}(8\pi^2)^2(1 + \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{5})\beta = 524.1827,$$

$$c = \tfrac{3}{4}(8\pi^2)^3(1 + \tfrac{1}{3} + \tfrac{3}{3} + \tfrac{3}{5})\gamma = 5{,}397.801,$$

$$d = \tfrac{3}{4}(8\pi^2)^4(1 + \tfrac{1}{3} + \tfrac{6}{3} + \tfrac{6}{5} + \tfrac{1}{5} + \tfrac{1}{7})\delta = 35{,}619.04,$$

$$e = \tfrac{3}{4}(8\pi^2)^5(1 + \tfrac{1}{3} + \tfrac{10}{3} + \tfrac{10}{5} + \tfrac{5}{5} + \tfrac{5}{7})\epsilon = 162{,}501.7,$$

$$f = \frac{3}{4}(8\pi^2)^6\left(1 + \frac{1}{3} + \frac{15}{3} + \frac{15}{5} + \frac{15}{5} + \frac{15}{7} + \frac{1}{7} + \frac{1}{9}\right)\zeta = 541{,}970.2,$$

g = 1,377,792,	n[1] = 3,717,000,
h = 2,757,220,	o[1] = 2,356,000,
i = 4,455,520,	p[1] = 1,334,000,
j = 5,935,500,	q[1] = 682,000,
k[1] = 6,632,700	r[1] = 318,000,
l[1] = 6,311,200,	s[1] = 136,000,
m[1] = 5,182,000,	t[1] = 54,000.

The right hand member of this equation is convergent for all values of a/λ, but the convergence is very slow when λ approaches equality with a. The values of σ/ρ calculated according to this expression for different values of λ/a are shown in Fig. 2 by the solid part of curve *III*.

Elementary considerations suffice to determine the manner in which the scattering by a ring electron depends upon the wave-length when the frequency is very high. If we consider waves shorter than the diameter of the electron, it is apparent that the length of the arc of the ring which may be considered to vibrate as a unit due to the action of the incident beam will be proportional to the wave-length of the incident rays. The amplitude of the beam scattered by such a unit will therefore, by equation 10, be proportional to λe, and the intensity to $\lambda^2 e^2$. Since, however, the total number of such units in each electron will be inversely proportional to the wave-length, the intensity of the beam scattered by the whole electron will be proportional to $\lambda^2 e^2/\lambda$, *i. e.*, proportional to the wave-length. The solid part of curve *III* (Fig. 2) shows, as we should expect, that this relation holds approximately even for waves considerably longer than the diameter of the electron. I have therefore extrapolated curve *III* according to this law for values of λ/a too small for the practical application of formula 21, indicating these approximate values by the broken part of the curve.

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March 17, 1919.

[1] These values were determined by an approximation formula.