Gravitational Radiation From a Spinning Ellipsoid of Uniform Density

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(Received 30 March 1965)

The gravitational radiation from a spinning ellipsoid and a spinning ellipse of uniform density is calculated. Upon comparison, the numerical coefficients are found to be smaller than that for a spinning rod. Radiation power vanishes when the ellipsoid and the ellipse are reduced to a spheroid and a circle, respectively. A classical rotating mass of uniform density bound by its gravitational field may be shown to be unstable against bifurcation into an ellipsoid if the period of rotation is short enough. Gravitational radiation can dissipate the angular momentum when bifurcation takes place. The calculation is used to estimate the energy-loss rate of a collapsing neutron star. It is shown that the relaxation time for dissipating angular momentum is around one second.

I. INTRODUCTION

BY studying the weak-field solutions of the general-relativistic field equation

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=(8\pi/c^4)GT_{\mu\nu}, \tag{1}$$

where $g_{\mu\nu}$ is the metric tensor, $T_{\mu\nu}$ the stress-energy tensor, $R_{\mu\nu}$ the Ricci curvature tensor and the curvature scalar $R=g^{\mu\nu}R_{\mu\nu}$. Einstein[1,2] proposed the existence of gravitational waves and calculated the radiated power from a uniform rod of length $2a$ spinning at an angular velocity ω to be

$$P=(32/45)a^4\omega^6M^2(G/c^5). \tag{2}$$

In this paper we shall present a calculation of the gravitational radiation of a spinning ellipsoid of uniform density. It is known that a spinning, nonrelativistic self-gravitating body of uniform density may bifurcate into an ellipsoid (Jacobi ellipsoid).[3] It has been suggested that such a bifurcation process may cause the angular momentum of a collapsing star to be dissipated by emitting gravitational radiation.[4]

II. CALCULATION

Assuming a weak field, we can write the metric as Lorentz metric[5] plus a small quantity[2,6]

$$g_{\mu\nu}=g^{(L)}{}_{\mu\nu}+h_{\mu\nu}. \tag{3}$$

Define $\varphi_\mu{}^\nu$ by

$$\varphi_\mu{}^\nu=h_\mu{}^\nu-\tfrac{1}{2}g^{(L)}{}_\mu{}^\nu h. \tag{4}$$

Then the field equation is reduced to

$$\Box^2\varphi_\mu{}^\nu=-16\pi T_\mu{}^\nu, \tag{5}$$

with the supplementary condition

$$\varphi_\mu{}^\nu{}_{,\nu}=0. \tag{6}$$

By analogy with electrodynamics, the solution of Eq. (5) can be shown to be

$$\varphi_\mu{}^\nu=4\int\frac{(T_\mu{}^\nu)_{\text{retarded}}d^3r'}{|r-r'|}=\frac{4}{r}\int(T_\mu{}^\nu)_{\text{retarded}}d^3r' \tag{7}$$

at large distance r, where the prime refers to the coordinates of the spinning ellipsoid.

The stress-energy tensor satisfies, to a first approximation, the conservation law

$$T_{0k,k}-T_{00,0}=0, \tag{8}$$

$$T_{jk,k}-T_{j0,0}=0. \tag{9}$$

Multiplying the last equation by x^i and integrating by parts, neglecting the surface term which vanishes at infinity, we obtain

$$\int T_{ij}d^3r=-\frac{1}{2}\left[\int T_{i0}x^j+T_{j0}x^id^3r\right]_{,0}. \tag{10}$$

Multiplying Eq. (8) by x^ix^j and integrating by parts, neglecting the surface term which vanishes at infinity, we obtain

$$\left[\int T_{00}x^ix^jd^3r\right]_{,0}=-\int T_{i0}x^j+T_{j0}x^id^3r. \tag{11}$$

Combining Eqs. (7), (10), and (11), we find an expression for φ_{ij}

$$\varphi_{ij}=\frac{2}{r}\frac{\partial^2}{\partial t^2}\int(T_{00})_{\text{retarded}}x^ix^jd^3r'. \tag{12}$$

The uniform ellipsoid $x'^2/a^2+y'^2/b^2+z'^2/c^2=1$ which lies at $t=0$ with the x' axis along the space x axis, spins with an angular velocity ω about the z axis. Applying Eq. (12) to the ellipsoid under consideration, we obtain an

[1] A. Einstein, Sitzber. Preuss. Akad. Wiss. Phys. Math. Kl. **1916**, 688 (1916); **1918**, 154 (1918).
[2] A. S. Eddington, Proc. Roy. Soc. (London) **A102**, 268 (1923).
[3] A series of papers by S. Chandrasekhar and associates on the stability of a rotating body of uniform density have been published in Vol. 138 of the *Astrophysical Journal.*
[4] H.-Y. Chiu, Ann. Phys. (N. Y.) **26**, 364 (1964).
[5] We use the followlng convention: Greek indices range from 0 to 3, Latin from 1 to 3; Lorentz metric $(+---)$. Units $G=c=1$.
[6] J. Weber, *General Relativity and Gravitational Waves* (Interscience Publishers, Inc., New York, 1961), pp. 87–97.

expression for φ_{11}:

$$\varphi_{11}=\frac{2}{r}\frac{\partial^2}{\partial t^2}\int_{\text{ellipsoid}}\rho[x'\cos\omega(t-r)-y'\sin\omega(t-r)]^2d^3r'$$
$$=\frac{2}{r}(\partial^2/\partial t^2)\{(4\pi/15)\rho abc[a^2\cos^2\omega(t-r)+b^2\sin^2\omega(t-r)]\}$$
$$=-\tfrac{4}{5}(\omega^2/r)M(a^2-b^2)\cos 2\omega(t-r),$$

where M is the total mass of the ellipsoid. Similarly, we obtain the rest of φ_{ij}:

$$\varphi_{22}=\tfrac{4}{5}M(\omega^2/r)(a^2-b^2)\cos 2\omega(t-r),$$
$$\varphi_{12}=-\tfrac{4}{5}M(\omega^2/r)(a^2-b^2)\sin 2\omega(t-r),$$
$$\varphi_{13}=\varphi_{23}=\varphi_{33}=0.$$

The supplementary condition as given by Eq. (2) will give us the remaining elements of $\varphi_{\mu\nu}$.

$$\partial\varphi_{10}/\partial t=\partial\varphi_{11}/\partial x+\partial\varphi_{12}/\partial y+\partial\varphi_{13}/\partial z$$
$$=\partial\varphi_{11}/\partial x+\partial\varphi_{12}/\partial y$$
$$=-\tfrac{4}{5}(\omega^2/r)M(a^2-b^2)[\sin 2\omega(t-r)](2\omega)(x/r)+\tfrac{4}{5}(\omega^2/r)M(a^2-b^2)[\cos 2\omega(t-r)](2\omega)(y/r),$$

where we have dropped terms of higher inverse powers than r^{-1} since these terms do not contribute to radiation power across the surface of an infinite sphere.

Similarly, we obtain

$$\partial\varphi_{20}/\partial t=\tfrac{4}{5}(\omega^2/r)M(a^2-b^2)[\cos 2\omega(t-r)]2\omega(x/r)+\tfrac{4}{5}(\omega^2/r)M(a^2-b^2)[\sin 2\omega(t-r)]2\omega(y/r),$$
$$\partial\varphi_{30}/\partial t=0,$$
$$\partial\varphi_{00}/\partial t=\tfrac{4}{5}(\omega^2/r)M(a^2-b^2)\times[\sin 2\omega(t-r)]2\omega[(x^2/r^2)-(y^2/r^2)]-(8/5)\times(\omega^2/r)M(a^2-b^2)[\cos 2\omega(t-r)]2\omega(xy/r^2).$$

The radial component of the Poynting vector [2,6] in our case is given explicitly by

$$\begin{aligned}t_4{}^r=&-(1/32\pi)[(\partial\varphi_{11}/\partial t)(\partial\varphi_{11}/\partial r)+(\partial\varphi_{22}/\partial t)\\&\times(\partial\varphi_{22}/\partial r)+(\partial\varphi_{33}/\partial t)(\partial\varphi_{33}/\partial r)+\tfrac{1}{2}(\partial\varphi_{00}/\partial t)\\&\times(\partial\varphi_{00}/\partial r)+2(\partial\varphi_{12}/\partial t)(\partial\varphi_{12}/\partial r)+2(\partial\varphi_{31}/\partial t)\\&\times(\partial\varphi_{31}/\partial r)+2(\partial\varphi_{32}/\partial t)(\partial\varphi_{32}/\partial r)-2(\partial\varphi_{10}/\partial t)\\&\times(\partial\varphi_{10}/\partial r)-2(\partial\varphi_{20}/\partial t)(\partial\varphi_{20}/\partial r)\\&-2(\partial\varphi_{30}/\partial t)(\partial\varphi_{30}/\partial r)]\\=&-(1/32\pi)[(\partial\varphi_{11}/\partial t)(\partial\varphi_{11}/\partial r)+(\partial\varphi_{22}/\partial t)\\&\times(\partial\varphi_{22}/\partial r)+\tfrac{1}{2}(\partial\varphi_{00}/\partial t)(\partial\varphi_{00}/\partial r)+2(\partial\varphi_{12}/\partial t)\\&\times(\partial\varphi_{12}/\partial r)-2(\partial\varphi_{10}/\partial t)(\partial\varphi_{10}/\partial r)\\&-2(\partial\varphi_{20}/\partial t)(\partial\varphi_{20}/\partial r)]\\=&+(1/32\pi)[(\tfrac{4}{5}(\omega^2/r)M)^2(a^2-b^2)^2(2\omega)^2\\&\times(2(z^2/r^2)+\tfrac{1}{4}(x^2+y^2)^2/r^4)\\&+\text{oscillating terms}].\end{aligned}\tag{13}$$

The total flow of energy per second across an infinite sphere is simply given by

$$P=\int t_4{}^r r^2\sin\theta d\theta d\varphi$$
$$=(32/125)(a^2-b^2)^2M^2\omega^6G/c^5,\tag{14}$$

where we have restored the gravitational constant G and speed of light c.

If a similar calculation is carried out for a two-dimensional ellipse $x^2/a^2+y^2/b^2=1$, spinning about the z axis at an angular velocity ω, we find the radiation power to be

$$P=(18/45)(a^2-b^2)^2M^2\omega^6G/c^5.\tag{15}$$

III. DISCUSSION

When the spinning body is axially symmetric $(a=b)$, it will not radiate gravitational energy as shown by expressions (14) and (15). Furthermore, the numerical coefficient for a spinning rod is the largest, while that for a spinning ellipsoid is the smallest. These are consistent with the usual conception that the gravitational radiation depends on the geometry of the spinning body.

In order to estimate the time scale of a neutron star for energy loss in the form of gravitational readiation, we apply expression (14) to a rotating ellipsoid of uniform density. To a/b and ω, we assign those values corresponding to bifurcation as calculated by Darwin[7] using classical hydrodynamics:

$$a\sim 1.9R,$$
$$b\sim 0.8R,\tag{16}$$
$$\omega_b{}^2\sim 0.14\times 2\pi G\rho,\tag{17}$$

where $R^3=abc$.

The potential energy of an ellipsoid is of the same order as that of a sphere

$$U\sim -M^2G/a.\tag{18}$$

Substituting Eq. (16) in Eq. (14) and dividing U by P, we obtain a rough estimate of the time scale

$$\tau\sim U/p\sim\tfrac{1}{4}(c/R)^5\omega^{-6}.\tag{19}$$

Substituting Eq. (17) into Eq. (19), we obtain

$$\tau\sim\tfrac{1}{4}(c/R)^5\omega_b{}^{-6}\sim 0.4(c/R)^5(1/G^3\rho^3).\tag{20}$$

For a neutron star of uniform density $\rho=10^{14}$ g/cc spinning at a frequency corresponding to bifurcation, the time scale is estimated to be of the order of one second. As τ is related to ω by

$$\tau\propto 1/\omega^6,$$

the time scale will be increased by a factor of 10^6 if the spinning rate is reduced to one-tenth of the frequency corresponding to bifurcation.

ACKNOWLEDGMENT

I would like to thank Dr. Hong-Yee Chiu for suggesting the problem.

[7] J. Jeans, *Astronomy and Cosmology* (Dover Publications, Inc., New York, 1961), p. 219.