Separable Potentials and Coulomb Interactions

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The scattering of two nonrelativistic charged particles interacting via a short-range separable potential and a repulsive Coulomb potential is studied. The "nuclear" phase shift is given by an explicit formula, identical to that for the neutral-particle case except that the separable potential functions are replaced by their Coulomb-modified counterparts. A simple one-term S-wave separable potential of the Yamaguchi type is used to illustrate this result. A perturbation theory is developed and used to relate the proton-proton scattering length and effective range and the corresponding neutron-neutron parameters. The result is in reasonable agreement with experiment and with previous calculations using local potentials.

I. INTRODUCTION

THERE has recently been a flurry of interest in the behavior of particles interacting through nonlocal separable potentials, especially in three-particle systems.[1–9] These studies are definitely of interest from a purely theoretical viewpoint, but there is also hope that separable potentials will be useful in explaining three-particle scattering data in terms of the results of two-particle experiments. It should be possible, for example, to calculate reasonably accurately the nucleon-deuteron scattering amplitude using information gained in nucleon-nucleon scattering experiments.[2,6,7]

There are, of course, loopholes in this program. The three-particle problem requires two-particle T-matrix elements far off the energy shell, and these we cannot hope to know with any definiteness. (The nonrelativistic potential theory itself is not meaningful for high-momentum states.) The hope is that the contribution from high-momentum states is small, but at present this hypothesis can be tested only by comparing predictions based upon it to experiment. It thus seems that the question of the usefulness of separable potentials in three-particle calculations can be settled only by a series of increasingly accurate calculations and experiments.

One incidental problem in this program will be the handling of Coulomb interactions. Charged particles are much easier to study experimentally than neutral particles, but their Coulomb interactions complicate the theory considerably. The Coulomb potential is, of course, not separable and, at low energies, with particles held together in bound or resonant states, cannot be treated as a small perturbation. It seems likely that a reasonably exact method for dealing with the Coulomb interactions will be required.

To this end, in this paper we study the behavior of two particles interacting via a short-range separable potential together with a Coulomb potential. (The corresponding problem for short-range local potentials was solved many years ago.[10,11]) We shall not consider here the analogous, much more difficult, three-body problem except to note that the two-body solution will certainly serve as a starting point. The results obtained here may also be of pedagogic interest since they can be expressed relatively simply and yet illustrate the important features of the Coulomb plus short-range potential problem. It further appears that, in certain cases, they may be useful in checking charge independence and charge symmetry in low-energy interactions.

The general problem is considered in Sec. II, where it is shown that the most important property of the separable potential is preserved when a Coulomb potential is added: in both cases the solution to the Lippmann-Schwinger equation can be given in closed form. Aside from this simplification the solution has all the properties expected from previous studies of the Coulomb plus short-range potential problem.[10,11] To illustrate the nature of the Coulomb modifications a simple S-wave problem with a single-term separable potential of the Yamaguchi type is solved in Sec. III. A perturbation expansion of this result is used to calculate the neutron-neutron scattering length and effective range from the corresponding proton-proton parameters, assuming charge independence; the results are in fair agreement with experiment and previous calculations. The results of the paper are summarized and possible further applications are discussed in Sec. IV.

II. GENERAL FORMULATION

We wish to study the properties of a system of two particles of mass m_1 and m_2 and charge e_1 and e_2,

[1] The list of references here is meant to be illustrative, not complete. References 2–4 give the foundations, while Refs. 5–9 contain applications. These papers contain many references to other work using separable potentials.
[2] C. Lovelace, Phys. Rev. **135**, B1225 (1964).
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[4] R. D. Amado, Phys. Rev. **132**, 485 (1963).
[5] Y. Yamaguchi, Phys. Rev. **95**, 1628, 1635 (1954).
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[7] A. N. Mitra and V. S. Bashin, Phys. Rev. **131**, 1265 (1963).
[8] J. H. Hetherington and L. H. Schick, Phys. Rev. **137**, B935 (1965).
[9] J. H. Naqvi, Nucl. Phys. **58**, 289 (1964).
[10] R. Yost, J. Wheeler, and G. Breit, Phys. Rev. **49**, 174 (1936); G. Breit, E. Condon, and R. Present, *ibid.* **50**, 825 (1936); G. Breit, B. Thaxton, and L. Eisenbud, *ibid.* **55**, 1018 (1939).
[11] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964), p. 263.

interacting via a potential $V=V_s+V_c$ which is the sum of a separable potential V_s and the Coulomb potential V_c. To avoid the complication of the extra bound states we shall assume the Coulomb potential to be repulsive, but, as will be clear, the attractive case can be handled by essentially identical techniques. Furthermore, to escape the difficulties associated with the long-range nature of the Coulomb potential we shall make the realistic assumption that it is cut off at a shielding radius R which is finite but extremely large compared to the range of the separable potential.

It is convenient to have at our disposal several complete sets of states: the free-particle states $|\mathbf{k}\rangle$; the outgoing and ingoing Coulomb states

$$|\mathbf{k}(\pm)\rangle_c=|\mathbf{k}\rangle+[E(k)\pm i\epsilon-H_0-V_c]^{-1}V_c|\mathbf{k}\rangle; \quad (1)$$

and the outgoing and ingoing exact states

$$\begin{aligned}|\mathbf{k}(\pm)\rangle&=|\mathbf{k}\rangle+[E(k)\pm i\epsilon\\&\qquad-H_0-V_c-V_s]^{-1}(V_c+V_s)|\mathbf{k}\rangle,\\&=|\mathbf{k}(\pm)\rangle_c+[E(k)\pm i\epsilon\\&\qquad-H_0-V_c-V_s]^{-1}V_s|\mathbf{k}(\pm)\rangle_c. \quad (2)\end{aligned}$$

Here H_0 is the kinetic-energy operator, with eigenvalues $E(k)=(2\mu)^{-1}k^2$, where $\mu=(m_1^{-1}+m_2^{-1})^{-1}$ is the reduced mass. We shall take for all states the normalization

$$\langle\mathbf{k}'|\mathbf{k}\rangle=(2\pi)^3\delta^3(\mathbf{k}'-\mathbf{k}). \quad (3)$$

With these definitions one obtains the well-known[12] result for the S-matrix element

$$\begin{aligned}S(\mathbf{k}',\mathbf{k})&\equiv\langle\mathbf{k}'(-)|\mathbf{k}(+)\rangle,\\&=(2\pi)^3\delta^3(\mathbf{k}'-\mathbf{k})-2\pi i\delta[E(k')-E(k)]\\&\qquad\times[T_{sc}(E(k);\mathbf{k}',\mathbf{k})+T_c(\mathbf{k}',\mathbf{k})], \quad (4)\end{aligned}$$

where

$$T_c(\mathbf{k}',\mathbf{k})=\langle\mathbf{k}'|V_c|\mathbf{k}(+)\rangle_c, \quad (5)$$

and

$$T_{sc}(E(k);\mathbf{k}',\mathbf{k})={}_c\langle\mathbf{k}'(-)|V_s|\mathbf{k}(+)\rangle. \quad (6)$$

The function $T_c(\mathbf{k}',\mathbf{k})$ is just the usual Coulomb scattering amplitude. Our main object of interest is T_{sc}, which satisfies the integral equation[13]

$$\begin{aligned}T_{sc}(E;\mathbf{k}',\mathbf{k})&\\&={}_c\langle\mathbf{k}'(-)|V_s|\mathbf{k}(+)\rangle_c\\&\quad+\textstyle\sum_{k''}[E+i\epsilon-E(k'')]^{-1}\\&\quad\times{}_c\langle\mathbf{k}'(-)|V_s|\mathbf{k}''(-)\rangle_cT_{sc}(E;\mathbf{k}'',\mathbf{k}). \quad (7)\end{aligned}$$

At this point we make partial-wave decompositions of our matrix elements:

$$T_{sc}(E;\mathbf{k}',\mathbf{k})=\textstyle\sum_l(2l+1)P_l(\hat{k}'\cdot\hat{k})T_{sc,l}(E;k',k), \quad (8)$$

[12] Reference 11, Sec. 5.4.

[13] For an attractive Coulomb potential we would have to include a sum over the bound states.

and

$$\begin{aligned}{}_c\langle\mathbf{k}'(-)|V_s|\mathbf{k}(\pm)\rangle_c&\\&=\textstyle\sum_l(2l+1)P_l(\hat{k}'\cdot\hat{k})V_{sc,l}^{(\pm)}(k',k), \quad (9)\end{aligned}$$

with similar expansions for $T_c(\mathbf{k}',\mathbf{k})$ and $\langle\mathbf{k}'|V_s|\mathbf{k}\rangle$. The integral equation (7) then gives

$$\begin{aligned}T_{sc,l}(E;k',k)&=V_{sc,l}^{(+)}(k',k)+(2\pi^2)^{-1}\\&\quad\times\int_0^\infty\frac{k''^2dk''V_{sc,l}^{(-)}(k',k'')T_{sc,l}(E;k'',k)}{E+i\epsilon-E(k'')}. \quad (10)\end{aligned}$$

With our state normalization the requirements of unitarity are satisfied if we write the on-shell Coulomb and total partial-wave amplitudes, respectively, as

$$T_{c,l}(k,k)=-2\pi(\mu k)^{-1}\sin\delta_{c,l}(k)\exp[i\delta_{c,l}(k)], \quad (11)$$

and

$$\begin{aligned}T_{c,l}(k,k)+T_{sc,l}(E(k);k,k)&\\&=-2\pi(\mu k)^{-1}\sin\delta_l(k)\exp[i\delta_l(k)], \quad (12)\end{aligned}$$

where $\delta_{c,l}$ and δ_l are the real phase shifts.

The above considerations apply to any two-potential problem[12] and have been included here only for completeness. We now, however, take advantage of the separable nature of V_s:

$$V_{s,l}(k',k)=\textstyle\sum_i\lambda_{l,i}g_{l,i}(k')g_{l,i}(k). \quad (13)$$

The distinctive feature of a potential of this form is that, at least in the absence of the Coulomb potential, it leads to an integral equation for the T-matrix element with a degenerate kernel and therefore with a simple algebraic solution. When the Coulomb potential is present the same integral equation holds for $T_{sc,l}$, except that $V_{s,l}$ is replaced by $V_{sc,l}^{(\pm)}$, related to it by

$$\begin{aligned}V_{sc,l}^{(\pm)}(k',k)&=(2\pi^2)^{-2}\int_0^\infty q'^2dq'\int_0^\infty q^2dq\\&\quad\times{}_{c,l}\langle k'(-)|q'\rangle V_{s,l}(q',q)\langle q|k(\pm)\rangle_{c,l}, \quad (14)\end{aligned}$$

where the partial-wave momentum-space Coulomb wave function $\langle q|k(\pm)\rangle_{c,l}$ is defined by the expansion

$$\langle q|\mathbf{k}(\pm)\rangle_c=\textstyle\sum_l(2l+1)P_l(\hat{q}\cdot\hat{k})\langle q|k(\pm)\rangle_{c,l}. \quad (15)$$

Thus, when $V_{s,l}$ has the separable form (13), $V_{sc,l}^{(\pm)}$ is also separable:

$$\begin{aligned}V_{sc,l}^{(\pm)}(k',k)&=\exp[i\delta_{c,l}(k')]\\&\quad\times\textstyle\sum_i\lambda_{l,i}g_{c,l,i}(k')g_{c,l,i}(k)\\&\quad\times\exp[\pm i\delta_{c,l}(k)], \quad (16)\end{aligned}$$

where

$$\begin{aligned}g_{c,l,i}(k)\exp[\pm i\delta_{c,l}(k)]&\\&=(2\pi^2)^{-1}\int_0^\infty q^2dqg_{l,i}(q)\langle q|k(\pm)\rangle_{c,l}. \quad (17)\end{aligned}$$

The function $g_{c,l,i}(k)$ can also be given in terms of $w_l(k,r)$, the configuration-space radial-wave function for the Coulomb potential, which we here define with the asymptotic normalization

$$w_l(k,r)\sim\sin[kr-\tfrac{1}{2}l\pi+\delta_{c,l}(k)]. \quad (18)$$

If we define

$$G_{l,i}(r)=(2\pi^2)^{-1}\int_0^\infty q^2dqg_{l,i}(q)j_l(qr), \quad (19)$$

where j_l is a spherical Bessel function, then

$$g_{c,l,i}(k)=4\pi k^{-1}\int_0^\infty rdrG_{l,i}(r)w_l(k,r). \quad (20)$$

The wave-function $w_l(k,r)$ is a solution of the Schrödinger equation with a cut-off, rather than exact, Coulomb potential, but[14] for $kR\gg l(l+1)+\eta^2(k)$ and $r<R$ we have

$$\begin{aligned} w_l(k,r)&=F_l(kr)\\ &=(2i)^{-l-1}C_l(\eta)M_{i\eta,l+(1/2)}(2ikr), \end{aligned} \quad (21)$$

where $M_{k,\mu}(z)$ is the Whittaker function,[15]

$$C_l(\eta)=2^l[(2l+1)!]^{-1}|\Gamma(l+1-i\eta)|\exp[-\tfrac{1}{2}\pi\eta] \quad (22)$$

is the barrier-penetration factor, and

$$\eta=\eta(k)=\mu e_1e_2k^{-1}. \quad (23)$$

The radial Coulomb wave function $F_l(kr)$, originally given by Yost, Wheeler, and Breit,[10] has the asymptotic form

$$F_l(kr)\sim\sin[kr-\eta\ln 2kr-\tfrac{1}{2}l\pi+\sigma_l], \quad (24)$$

where σ_l, usually known as the Coulomb phase shift, is given by

$$\lambda_l=\arg\Gamma(l+1+i\eta). \quad (25)$$

Although the exact value of $\delta_{c,l}$ depends upon the details of the cutoff, we expect, for k not too small,

$$\begin{aligned} \delta_{c,l}&=\sigma_l-\eta\ln 2kR, \quad l\ll kR,\\ \delta_{c,l}&=0, \quad l\gg kR. \end{aligned} \quad (26)$$

Since $V_{sc,l}^{(\pm)}(k',k)$ is separable the integral equation (10) can be solved algebraically. Setting

$$\begin{aligned} T_{sc,l}(E;k',k)=\exp[i\delta_{c,l}(k')]&\\ \times\textstyle\sum_{i,j}g_{c,l,i}(k')\tau_{ij}^{(l)}(E)g_{c,l,j}&\\ \times\exp[i\delta_{c,l}(k)],& \end{aligned} \quad (27)$$

we obtain for the matrix τ (dropping the index l) the equation

$$\tau(E)=\Lambda-\Lambda I(E)\tau(E), \quad (28)$$

where Λ is the diagonal matrix

$$\Lambda_{ij}=\delta_{ij}\lambda_i, \quad (29)$$

and the elements of the matrix $I(E)$ are given by

$$I_{ij}(E)=(2\pi^2)^{-1}\int_0^\infty\frac{k^2dkg_{c,i}(k)g_{c,j}(k)}{E(k)-E-i\epsilon}. \quad (30)$$

Solving (30) for $\tau(E)$ we find

$$\tau(E)=[1+\Lambda I(E)]^{-1}\Lambda. \quad (31)$$

From (27) and (31) we see that, except for the Coulomb phase factors, $T_{sc,l}(E;k',k)$ can be obtained from the corresponding T-matrix element in the absence of the Coulomb potential merely by replacing $g_{l,i}$ by $g_{c,l,i}$.

If we define a Coulomb-corrected "nuclear" phase shift $\delta_{sc,l}$ from the on-the-energy-shell matrix element by

$$\begin{aligned} \exp[-2i\delta_{c,l}(k)]T_{sc,l}(E(k);k,k)&\\ =\textstyle\sum_{i,j}g_{c,l,i}(k)\tau_{ij}^{(l)}(E(k))g_{c,l,j}(k),&\\ =-2\pi(\mu k)^{-1}\sin\delta_{sc,l}(k)\exp[i\delta_{sc,l}(k)],& \end{aligned} \quad (32)$$

then, as can easily be seen from (11) and (12), the total phase shift is just the sum of the Coulomb and "nuclear" phase shifts, $\delta_l(k)=\delta_{c,l}(k)+\delta_{sc,l}(k)$, and

$$\delta_{sc,l}(k)=-\arg\{\det[1+\Lambda_lI_l(E(k))]\}. \quad (33)$$

In general the exact nature of the Coulomb screening is not known. As long as R is much larger than the range of V_s, however, under most experimental conditions the differential cross section is given accurately by[11]

$$d\sigma/d\Omega=|f_c(\theta)+f_{sc}(\theta)|^2, \quad (34)$$

where

$$f_c(\theta)=k^{-1}\textstyle\sum_l(2l+1)\sin\sigma_l\exp(i\sigma_l)P_l(\cos\theta) \quad (35)$$

is the usual Coulomb amplitude, and

$$\begin{aligned} f_{sc}(\theta)=k^{-1}\textstyle\sum_l(2l+1)\sin\delta_{sc,l}&\\ \times\exp(i\delta_{sc,l}+2i\sigma_l)P_l(\cos\theta),& \end{aligned} \quad (36)$$

with σ_l and $\delta_{sc,l}$ given by (25) and (33), respectively.

It should be emphasized that the results here are not really new, but merely an application of what are now standard methods to a particular class of potentials. In the next section we shall see, through an example, the considerable simplication which results from the use of a separable, rather than the more usual local, potential.

III. A SIMPLE EXAMPLE

To illustrate some of the features of the formulas developed in Sec. II we shall apply them to the simple problem of calculating the S-wave phase shift $\delta_{sc,0}$ for a single-term separable potential of the Yamaguchi

[14] This is required in order that $F_l(kr)$ has its asymptotic form when $r=R$.

[15] See, for example, W. Magnus and F. Oberhettinger, *Special Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1949), p. 88.

type[5]:

$$V_{s,0}(k',k)=\lambda g(k')g(k)\,, \tag{37}$$

where

$$g(k)=(k^2+\beta^2)^{-1}. \tag{38}$$

Then, using (21),

$$G(r)=(4\pi r)^{-1}\exp(-\beta r)\,, \tag{39}$$

and, from (22) and (23), assuming k not too small and $\beta\gg R^{-1}$,

$$g_c(k)=k^{-1}\int_0^\infty dr\,\exp(-\beta r)F_0(kr)\,. \tag{40}$$

The Laplace transform of the Whittaker function is given in standard tables[16]; the resulting expression for $g_c(k)$ is

$$g_c(k)=g(k)C_0(\eta)\exp[2\eta\tan^{-1}(k\beta^{-1})]. \tag{41}$$

Here η is given by (25) and, taking $l=0$ in (24),

$$C_0(\eta)=\{2\pi\eta[\exp(2\pi\eta)-1]^{-1}\}^{1/2}. \tag{42}$$

In this case the dimension of the matrices τ and I is one, and, from (35), the phase shift is given by

$$\cot\delta_{sc}(k)=-\mathrm{Re}(1+\lambda I)[\lambda\,\mathrm{Im}I]^{-1}. \tag{43}$$

Since

$$\mathrm{Im}I=(2\pi)^{-1}\lambda\mu kC_0^2(\eta)\exp[4\eta\tan^{-1}(k\beta^{-1})]\times(k^2+\beta^2)^{-2}, \tag{44}$$

and

$$\mathrm{Re}I\equiv\bar{I}(k)=(2\pi^2)^{-1}P\int_0^\infty\frac{q^2dqg_c^2(q)}{E(q)-E(k)}, \tag{45}$$

this can be rewritten as

$$kC_0^2(\eta)\cot\delta_{sc}=-2\pi(\lambda\mu)^{-1}(k^2+\beta^2)^2\times\exp[-4\eta\tan^{-1}(k\beta^{-1})][1+\lambda\bar{I}(k)]. \tag{46}$$

From (46) we see that as $k\rightarrow 0$ the factor $C^2(\eta)$ requires a phase shift which approaches zero as $\exp(-2\pi e_1e_2\mu k^{-1})$, rather than as k. This is of course because low-energy particles cannot penetrate the Coulomb barrier and therefore cannot feel the effects of V_s. This result is not really true all the way down to $k\rightarrow 0$ if R is finite, but it will hold for $k\ll\mu e_1e_2$, where the exponential behavior is apparent.

The above formulas can be considerably simplified when $\mu e_1e_2\beta^{-1}\ll 1$. In this case we can expand the factor $\exp[4\eta\tan^{-1}(q\beta^{-1})]$ which occurs in the integrand in (45) to obtain a perturbation series for $\bar{I}(k)$. The first term in the expansion gives

$$\bar{I}_0(k)=(2\pi^2)^{-1}P\int_0^\infty\frac{q^2dq\,C_0^2(\eta(q))}{(q^2+\beta^2)^2(E(q)-E(k))}. \tag{47}$$

[16] See, for example, Ref. 15, p. 130.

This integral can be done using an integral representation for the ψ (digamma) function,[17]

$$\psi(\eta(\beta))=\ln(\mu e_1e_2\beta^{-1})-\beta(2\mu e_1e_2)^{-1}-\beta^2(\mu e_1e_2)^{-1}\frac{1}{\pi}\int_0^\infty\frac{dq\,C_0^2(\eta(q))}{q^2+\beta^2}, \tag{48}$$

and the related formula

$$\begin{aligned}\eta(\eta(k))&\equiv\mathrm{Re}\psi(-i\eta)-\ln\eta\,,\\&=(\mu e_1e_2)^{-1}k^2\frac{P}{\pi}\int_0^\infty\frac{dq\,C_0^2(\eta(q))}{q^2-k^2}.\end{aligned} \tag{49}$$

The next term in the expansion is

$$\bar{I}_1(k)=(2\pi^2)^{-1}4\mu e_1e_2P\int_0^\infty\frac{qdq\tan^{-1}(q\beta^{-1})C_0^2(\eta(q))}{(q^2+\beta^2)^2(E(q)-E(k))}. \tag{50}$$

We shall here keep terms only to order $\mu e_1e_2\beta^{-1}$, ignoring terms of order $(\mu e_1e_2\beta^{-1})^2\ln(\mu e_1e_2\beta^{-1})$. We can then replace $C_0^2(\eta(q))$ by 1 in (50), the integral becoming elementary. Consistent with this approximation, in evaluating (47) with the help of (48), we can set

$$\psi(\mu e_1e_2\beta^{-1})\approx-\beta(\mu e_1e_2)^{-1}-\gamma\,, \tag{51}$$

where $\gamma=0.5772\cdots$ is Euler's constant. With these approximations we find

$$\bar{I}(k)=\beta\mu\pi^{-1}(k^2+\beta^2)^{-2}\{(4\beta^2)^{-1}(\beta^2-k^2)+\beta^{-1}\mu e_1e_2h(\eta)+\beta^{-1}\mu e_1e_2[\ln(4\mu e_1e_2\beta)-\ln(\beta^2+k^2)+\gamma]\}; \tag{52}$$

and, substituting into (46),[18]

$$\begin{aligned}kC_0^2(\eta)\cot\delta_{sc}(k)+2\mu e_1e_2h(\eta)\\=k\cot\delta_s(k)[1-4\mu e_1e_2k^{-1}\tan^{-1}(k\beta^{-1})]\\-2\mu e_1e_2[\ln(4\mu e_1e_2\beta)-\ln(\beta^2+k^2)+\gamma],\end{aligned} \tag{53}$$

where $\delta_s(k)$ is the corresponding phase shift in the absence of the Coulomb potential:

$$k\cot\delta_s(k)=-2\pi(\lambda\mu)^{-1}(k^2+\beta^2)^2-(2\beta)^{-1}(\beta^2-k^2). \tag{54}$$

From this formula we obtain the scattering length and effective range[19]:

$$-a_{sc}^{-1}=-a_s^{-1}[1-4\mu e_1e_2\beta^{-1}]-2\mu e_1e_2[\ln(4\mu e_1e_2\beta^{-1})+\gamma], \tag{55}$$

$$\tfrac{1}{2}r_{sc}=\tfrac{1}{2}r_s[1-4\mu e_1e_2\beta^{-1}]+2\mu e_1e_2\beta^{-2}[1-2(3\beta a_s)^{-1}], \tag{56}$$

where

$$-a_s^{-1}=-2\pi\beta^4(\lambda\mu)^{-1}-\tfrac{1}{2}\beta \tag{57}$$

[17] Reference 15, p. 3.

[18] The left-hand side of (53) is the function used for effective-range expansions in the presence of a Coulomb potential. See, for example, Ref. 11, p. 295.

[19] Equation (55) is similar in form to an approximate expression for the proton-proton scattering length developed by J. D. Jackson and J. M. Blatt, Rev. Mod. Phys. 22, 77 (1950).

and

$$\tfrac{1}{2}r_s = -4\pi\beta^2(\lambda\mu)^{-1} + \tfrac{1}{2}\beta^{-1} \tag{58}$$

are the corresponding parameters for neutral particles in the same separable potential.

We can check that these results are reasonable by applying them to the nucleon-nucleon system.[20,21] The proton-proton scattering length and effective range are reasonably well known from experiment; translating to our notation we have[20]

$$a_{pp} = a_{sc} = -7.81 \text{ F}, \quad r_{pp} = r_{sc} = 2.80 \text{ F}. \tag{59}$$

These are consistent with Eqs. (55)–(58) if

$$a_s = -18.0 \text{ F}, \quad r_s = 2.93 \text{ F}, \quad (4\pi)^{-1}\lambda\mu = 1.20 \text{ F}^{-3}, \tag{60}$$

and

$$\beta = 1.10 \text{ F}^{-1}. \tag{61}$$

(Then $4\pi e_1e_2\beta^{-1} \approx 0.06$, so that the perturbation expansion should be valid.) These values for a_s and r_s, although perhaps a bit large, are in reasonable agreement with previous calculations, based upon charge symmetry, and experimental values for the neutron-neutron parameters (but not with the experimental neutron-proton parameters).[20]

It should be pointed out that the simple separable potential used above does not provide a good fit to the high-energy proton-proton data. For a more accurate calculation, a two-term separable potential, such as that suggested by Naqvi,[9] could be used. A comparison of the different calculated results for the neutron-neutron scattering length a_{nn},[20] however, seems to indicate that this parameter is relatively independent of the shape of the potential.

IV. CONCLUSION

We have shown that the problem of determining the "nuclear" phase shift, δ_{sc}, due to a short-range separable potential acting between two charged particles, is essentially the same as the corresponding problem with neutral particles. The effect of the Coulomb potential is completely accounted for by the use of the Coulomb-modified functions g_c, rather than the original separable-potential functions g. It seems clear that this would also be true in a more ambitious calculation including vacuum polarization, nuclear-size effects, etc.; these would merely modify the Coulomb wave function used in obtaining g_c from g.

Our results may be useful in finding the correct separable-potential parameters for the scattering of two charged particles, especially in those cases where there are no neutral counterparts. With this information the off-shell two-particle T matrix, $T = T_c + T_{sc}$, needed as input in the Faddeev equations[22] of the three-body problem, can be calculated. One has still the difficult problem of handling the nonseparable T_c, but it may prove true that the main Coulomb effects will be included when T_{sc} is used instead of T_s. This is because T_{sc} accounts exactly for the Coulomb effects for pairs of particles in bound or pseudo-bound states, while T_c acts upon less strongly correlated pairs.

Another possible application of the results obtained above is to the problem of charge symmetry and charge independence. As illustrated in Sec. III, checking charge symmetry amounts to determining whether or not the same separable-potential parameters can be used to fit the scattering of a family of particles, with the Coulomb modifications included when needed.

It is hoped that the results of this paper will be useful in answering the question of whether separable potentials are meaningful in situations other than on-shell two-particle scattering.

[20] Recent discussions of low-energy nucleon-nucleon singlet scattering can be found in L. Heller, P. Signell, and N. R. Yoder, Phys. Rev. Letters **13**, 577 (1964); R. E. Schneider and R. M. Thaler, Phys. Rev. **137**, B874 (1965); R. P. Haddock, R. M. Salter, Jr., M. Zeller, J. B. Czirr, and D. R. Nygren, Phys. Rev. Letters **14**, 318 (1965); and H. P. Noyes, Nucl. Phys. (to be published). These papers contain references to earlier work. The author would like to thank Dr. Noyes for a pre-publication copy of his paper.

[21] From arguments given in Refs. 2 and 3 we expect the nucleon-nucleon S-wave scattering to be a good subject for treatment with separable potentials because of the virtual bound state.

[22] See Refs. 2 and 3.