Quantum Electrodynamics without Dead Wood*

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In quantum electrodynamics one can obtain a Hamiltonian which gives reasonable field equations in the Heisenberg picture, but which does not allow of solutions of the wave equation to represent physical states in the Schrödinger picture. The inference is that the Heisenberg picture is a good picture, the Schrödinger picture is a bad picture, and the two pictures are not equivalent. The usual proof of the equivalence of the two pictures fails because the state vector of the Schrödinger picture does not remain in Hilbert space. One can set up quantum electrodynamics entirely in the Heisenberg picture and thereby avoid the worst difficulties encountered in the Schrödinger picture. The theory takes a logical form and is not merely an assembly of working rules. It can be applied to the calculation of the anomalous magnetic moment and the Lamb shift, and is then similar to the usual calculations of these effects, with a good deal of dead wood cut away. There is a problem concerned with the general interpretation of quantum mechanics when one cannot use the Schrödinger picture and some postulates are proposed for dealing with it.

CONNECTION BETWEEN THE HEISENBERG AND SCHRÖDINGER PICTURES

Quantum mechanics was discovered independently by Heisenberg and Schrödinger. They gave us two theories, which at first looked very different. Heisenberg's theory involved supposing the dynamical variables to be matrices. They thus did not satisfy the commutative axiom of multiplication. They varied with time according to Heisenberg's equation of motion

$$i\hbar(du/dt)=uH-Hu\,, \tag{1}$$

where H is the matrix representing the Hamiltonian or total energy.

Schrödinger's theory involved working with wave functions ψ to represent atomic states. They varied with time according to Schrödinger's wave equation

$$i\hbar(d\psi/dt)=H\psi\,, \tag{2}$$

where H is an operator representing the Hamiltonian.

For a while physicists had two quantum theories to work with, but it was soon found that there was a simple connection between them, resulting, as people then believed, in the two theories being equivalent. The theories may be written in an abstract form which enables one to express the connection between them more concisely.

The matrices of Heisenberg's theory can be replaced by linear operators operating on the vectors of a Hilbert space. (I use the term "Hilbert space" to mean a "separable" Hilbert space, which can be spanned by a denumerably infinite set of vectors.) It is convenient to use the notation $|A\rangle$ for a Hilbert vector labeled by A, and $u|A\rangle$ for the result of applying the dynamical variable u to $|A\rangle$. Heisenberg's equation of motion (1) reads the same whether the dynamical variables are considered as matrices or linear operators.

A wave function in Schrödinger's theory may be considered as the coordinates of a vector in Hilbert space, often called the state vector. The dynamical variables in Schrödinger's theory then become operators in Hilbert space as in the Heisenberg theory. Schrödinger's wave equation (2) becomes an equation for the time variation of the state vector, say

$$i\hbar d|A\rangle/dt=H|A\rangle\,. \tag{3}$$

Both theories now deal with Hilbert vectors and operators on them. H is the same operator in both theories. The difference between the two theories is that in Heisenberg's the dynamical variables vary with the time, according to (1), while the Hilbert vectors are fixed. In Schrödinger's the dynamical variables are fixed operators while the Hilbert vectors vary with the time, according to (3).

The two theories may be connected by the equations

$$u_{\rm H}=e^{iHt/\hbar}u_{\rm S}e^{-iHt/\hbar}\,, \tag{4}$$

$$|A_{\rm S}\rangle=e^{-iHt/\hbar}|A_{\rm H}\rangle\,, \tag{5}$$

where the subscripts S and H refer to Schrödinger and Heisenberg, respectively. One can easily check the connection by noting that, with $u_{\rm S}$ constant, (4) makes $u_{\rm H}$ vary with time according to (1), while with $|A_{\rm H}\rangle$ constant, (5) makes $|A_{\rm S}\rangle$ vary with time according to (3). Equations (4) and (5) give us solutions of the Heisenberg and Schrödinger equations of motion, connecting the variable $u_{\rm H}$ and $|A_{\rm S}\rangle$ at time t with their initial values, according to

$$u_{\rm H}(t)=e^{iHt/\hbar}u_{\rm H}(0)e^{-iHt/\hbar}\,, \tag{6}$$

$$|A_{\rm S},t\rangle=e^{-iHt/\hbar}|A_{\rm S},0\rangle\,. \tag{7}$$

With the two theories connected in this way, it became clear that there was just one quantum mechanics with two ways of looking at it, Schrödinger's picture and Heisenberg's picture. The mathematical transformation (4), (5) connects the two pictures.

DIFFICULTIES

This quantum mechanics had enormous success when applied to the simpler problems of the atomic world,

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but ran into serious trouble when one tried to set up an accurate relativistic theory involving fields. The trouble was most acute for electrodynamics, because the other atomic fields were not very well known and one could ascribe the difficulties for them to our imperfect knowledge of the forces involved. But electrodynamics was so well understood classically that it was most puzzling to find that quantum theory fails for it.

In electrodynamics we confine our attention to electrons and positrons interacting with the electromagnetic field. It is then not difficult to obtain the Hamiltonian H. One can be confident that this H is essentially correct because, when one uses it in the Heisenberg equation of motion (1), one gets sensible field equations in agreement with what one would expect from classical electrodynamics. However, when one inserts this H into Schrödinger's wave equation (2), one cannot find any solutions, not even approximate ones, to represent the physical states one is interested in. One would expect there to be a trivial solution corresponding to the vacuum state, but there is not even this trivial solution. We have here the dilemma which has hindered the development of theoretical physics for three decades.

The fact that physicists have been unable to solve the Schrödinger equation has not prevented them from wrestling with it. During the last eighteen years heroic efforts have been made to extract some useful information from it, with considerable success.

The method that people have followed consists in treating the interaction between the electrons and the electromagnetic field as a small perturbation and seeking for a solution of the Schrödinger equation as a power series in the coupling constant $e^2/\hbar c$. The solution then appears as the sum of an infinite number of terms, each of which corresponds to a Feynman diagram. The trouble is that all the terms except the first few involve divergent integrals.

Among the Feynman diagrams there are some that involve v-v (vacuum to vacuum) transitions with no external lines. These diagrams may occur either by themselves or as parts of complete diagrams. The terms corresponding to such diagrams all involve infinities. They are the worst terms, as they prevent one from getting a solution to represent even the vacuum state.

The usual procedure of physicists in quantum field theory is to neglect such terms altogether, and to excuse themselves by saying that such terms could not correspond to anything observable. This neglect involves a drastic departure from logic. It changes the whole character of the theory, from logical deduction to a mere setting up of working rules.

The other terms in the expansion, those corresponding to Feynman diagrams without v-v parts, may also contain divergent intergrals. But they can be dealt with by the technique of renormalization, which one can to some extent justify in the case of electrodynamics, where the coupling constant is small.

The Schrödinger equation, handled on these lines, enables one to deduce practical results; namely, the Lamb shift and the anomalous magnetic moment of the electron, which are found to be in very good agreement with experiment. The theory is thus undeniably a brilliant success. But the price one must pay for this success is to abandon logical deduction and replace it by working rules. This is a very heavy price and no physicist should be content to pay it.

NONEQUIVALENCE OF THE HEISENBERG AND SCHRÖDINGER PICTURES

Let us re-examine the foundations of quantum mechanics and try to find out where things go wrong in the application to electrodynamics. We have a Hamiltonian H which gives sensible results in the Heisenberg picture, but which we cannot use in the Schrödinger picture. It would appear from this that *the Heisenberg picture is a good picture, the Schrödinger picture is a bad picture, and the two pictures are not equivalent*, as physicists usually suppose.

There is, of course, the connection (4), (5) between the two pictures. If we accept the Heisenberg picture as a good picture, we can obtain from it, by the transformation (5), the vectors $|A_S\rangle$ of the Schrödinger picture. These vectors vary with the time according to (7) and satisfy Schrödinger's equation (3).

Now Schrödinger's wave function ψ consists of the coordinates of a vector $|A\rangle$ satisfying (3). It seems that we can obtain $|A\rangle$ to satisfy (3) but we cannot find ψ. This must mean that the vector $|A\rangle$ satisfying (3) cannot be represented by coordinates. Any vector in Hilbert space can be represented by coordinates. The vector $|A\rangle$ must therefore be in some more general space than a Hilbert space—a space having many more dimensions than Hilbert space.

We are thus led to look upon electrodynamics in the Schrödinger picture in the following way. The interaction is so violent that if we start with a particular state vector in Hilbert space to represent the initial state, it gets knocked right out of Hilbert space in any time interval, however short. It moves about in some more general space, in which it cannot be represented by coordinates, and thus one cannot construct a Schrödinger wave function. We may call these more general vectors "ket vectors."

The dynamical variables of the Heisenberg picture cannot now be operators in Hilbert space, but must operate in the more general ket vector space.

A MODEL HAMILTONIAN

To illustrate the relationship between the Heisenberg and Schrödinger pictures one may consider a model Hamiltonian which is sufficiently simple for one to be able to solve the equations accurately, but yet contains features that correspond to the essential difficulties of quantum electrodynamics.

Let us take as our dynamical variables at any time an infinite set of fermion emission operators $\eta_r (r=1,2,\cdots,\infty)$ and their conjugate absorption operators η_r^*. They satisfy

$$\eta_r\eta_s+\eta_s\eta_r=0\,,$$
$$\eta_r\eta_s^*+\eta_s^*\eta_r=\delta_{rs}\,. \tag{8}$$

Take as the Hamiltonian

$$H=\tfrac{1}{2}a_{rs}(\eta_r\eta_s-\eta_r^*\eta_s^*)\,, \tag{9}$$

where the a_{rs} are real numbers and $a_{rs}=-a_{sr}$. This H is physically real.

Introduce a ket vector $|S\rangle$ satisfying

$$\eta_r^*|S\rangle=0 \quad \text{for all } r. \tag{10}$$

Let us try to get a solution of the Schrödinger equation (3) with the state vector $|A\rangle$ initially equal to $|S\rangle$. We have

$$H|S\rangle=\tfrac{1}{2}a_{rs}\eta_r\eta_s|S\rangle\,,$$
$$H^2|S\rangle=\tfrac{1}{4}a_{rs}a_{pq}\eta_r\eta_s\eta_p\eta_q|S\rangle-\tfrac{1}{4}a_{rs}a_{pq}\eta_r^*\eta_s^*\eta_p\eta_q|S\rangle.$$

The second term reduces to

$$-\tfrac{1}{2}a_{rs}a_{sr}|S\rangle.$$

The coefficient of $|S\rangle$ here may very well be infinite, even for a bounded matrix a. As a simple example we may take a to be such that its square is minus the unit matrix. (The minus is needed because the matrix a is skew). Under these conditions $H^2|S\rangle$ does not exist.

There is then no solution of the Schrödinger equation (3) with the vector $|A\rangle$ initially equal to $|S\rangle$, if $|A\rangle$ is restricted to lie in the Hilbert space provided by power series in the η's applied to $|S\rangle$. Is there a solution if $|A\rangle$ is not so restricted? According to (7) the solution should be

$$e^{-iHt/\hbar}|S\rangle\,, \tag{11}$$

so the question is, "Can we give a meaning to this quantity, in spite of our failure to do so when we expand the exponential?"

Let us work in the Heisenberg picture. The Heisenberg equations of motion give

$$i\hbar\dot{\eta}_r=-\tfrac{1}{2}a_{pq}(\eta_r\eta_p^*\eta_q^*-\eta_p^*\eta_q^*\eta_r)\,,$$
$$=-a_{rp}\eta_p^*\,,$$
$$i\hbar\dot{\eta}_r^*=a_{rp}\eta_p\,,$$

so

$$\hbar^2\ddot{\eta}_r=a_{rp}a_{pq}\eta_q\,.$$

Taking the special example with $a^2=-1$, this gives

$$\hbar^2\ddot{\eta}_r=-\eta_r\,,$$

whose solution with any given initial condition is

$$\eta_r(t)=\cos(t/\hbar)\eta_r(0)+i\sin(t/\hbar)a_{rs}\eta_s^*(0)\,.$$

Thus the solution of the Heisenberg equations of motion leads to no infinities.

Associated with the Heisenberg variables $\eta_r(t),\eta_r^*(t)$ at time t, we may introduce a ket vector $|S_t\rangle$ satisfying

$$\eta_r^*(t)|S_t\rangle=0 \text{ for all } r. \tag{12}$$

We may do this for each value of t. Let us determine the connection between the $|S_t\rangle$ for different values of t.

From (6),

$$\eta_r^*(t)=e^{iHt/\hbar}\eta_r^*(0)e^{-iHt/\hbar}\,,$$

so (12) gives

$$\eta_r^*(0)e^{-iHt/\hbar}|S_t\rangle=0\,.$$

Now

$$\eta_r^*(0)|S_0\rangle=0\,, \tag{13}$$

so we get

$$|S_t\rangle=e^{iHt/\hbar}|S_0\rangle\,,$$

apart from an arbitrary numerical coefficient.

The $|S_0\rangle$ of (13) is just the same as the $|S\rangle$ of (10) in the Schrödinger picture, so $|S_{-t}\rangle$ is the quantity (11), which is a solution of the Schrödinger equation. Thus *we can give a meaning to the quantity* (11). In the Heisenberg picture it is on the same footing as the quantity $|S_0\rangle$ and merely refers to a different time. The Heisenberg picture thus leads us to introduce ket vectors more general than Hilbert vectors, defined by (12).

Although we cannot get a solution in Hilbert space for the Schrödinger equation with the initial vector $|S\rangle$, we can get a solution with a different initial vector. Consider a ket $|Y\rangle$ satisfying

$$(\eta_r^*+a_{rs}\eta_s)|Y\rangle=0\,. \tag{14}$$

The various conditions here for different r values are consistent since, as one easily checks, the quantities $\eta_r^*+a_{rs}\eta_s$ and $\eta_p^*+a_{pq}\eta_q$ anticommute.

We now find

$$H=\tfrac{1}{2}(\eta_r+a_{rp}\eta_p^*)(\eta_r^*+a_{rs}\eta_s)-\tfrac{1}{2}(\eta_r\eta_r^*+\eta_r^*\eta_r)$$

The last term here is an infinite constant and may be discarded. Then $H|Y\rangle=0$ and the constant vector $|Y\rangle$ is a solution of the Schrödinger equation.

THE NEED FOR THE HEISENBERG PICTURE

The Hamiltonian of quantum electrodynamics is essentially similar to that of this simple mode. The physical states that we are interested in are close to the physical vacuum, with all negative-energy electron states occupied. These correspond to kets close to $|S\rangle$ in the model. Such kets in the Schrödinger picture do not remain in a Hilbert space. The kets close to $|Y\rangle$ of the model correspond to states in electrodynamics for which nearly all the negative-energy electron states are occupied. Such states differ too much from physical reality to be used in calculations of physical problems.

Thus we see that we cannot use Schrödinger wave functions in electrodynamics. The Heisenberg equations of motion remain available to us, although they are, of course, considerably more complicated than in the model.

The dynamical variables in the Heisenberg picture cannot be represented as matrices or as operators in Hilbert space. They are something more general, whose precise mathematical nature is unknown. We may call them q numbers. All that is known about them is that they are noncommuting quantities satisfying definite commutation relations. We may carry out calculations in the Heisenberg picture by making algebraic deductions from the commutation relations. We may do this without knowing the mathematical nature of the quantities with which we are working.

Our lack of knowledge of what q numbers are prevents us from setting up a theory of them with mathematical rigor. We cannot define limits of q numbers, so we cannot give a rigorous meaning to processes of integration or differentiation applied to q numbers. Even though we cannot aspire to complete rigor, we may set up a theory with a reasonable practical standard of logic, rather like the way engineers work. Engineers do not aim at complete rigor. In their calculations they continually neglect quantities which they believe can be neglected without invalidating their results, basing this belief on previous experience, or maybe just feeling. The physicist working with q numbers will have to develop a similar feeling for what can be neglected.

Using the Heisenberg picture for quantum electrodynamics, we may try to solve the equations by a similar perturbation method to that used for the Schrödinger equation, expanding everything in powers of $e^2/\hbar c$. We get the solution as the sum of an infinite number of terms, which again correspond to Feynman diagrams, with similar divergent integrals. There is great similarity to the calculations in the Schrödinger picture, but there is the important difference that terms corresponding to v-v diagrams no longer arise.

The reason for the difference is that, if one starts with zero in the Heisenberg picture, it remains zero, while if one starts with the vacuum in the Schrödinger picture, it does not remain the vacuum. In that way the Heisenberg picture avoids the worst difficulties of the Schrödinger picture. Those that are left, the divergent integrals in the non-v-v terms, can be tackled by limiting processes involving renormalization.

We now see that, if we want a logical quantum electrodynamics, we must work entirely with q numbers in the Heisenberg picture. All references to Schrödinger wave functions must be cut out as dead wood. The Schrödinger wave functions involve infinities, associated with v-v Feynman diagrams, which destroy all hope of logic.

SOLUTION OF THE HEISENBERG EQUATIONS OF MOTION

There are two kinds of solutions one might consider:

I. One might solve the equation (1) directly and get u as a function of t and of constants of the motion, say α. The α's, of course, are q numbers, and $d\alpha/dt=0$. The α's might be the q's and p's at some standard time t_0, or they might be other constants.

If one considers this method relativistically, it reads as follows: One takes a field quantity $F(x)$ at a general point in space-time x_0,x_1,x_2,x_3 and expresses it in terms of constants of the motion and the four parameters x. The constants of the motion are q numbers. The question arises, what should one take them to be? There are two interesting possibilities:

I-a *One might take them to be the field quantities at some standard time* t_0.

I-b. *One might take them to be the ingoing fields (or the outgoing ones).*

I-a has the advantage that there are known simple commutation relations between the constants of the motion, while I-b has the advantage that it gives manifestly relativistic solutions. With I-b one would have to calculate the commutation relations between the ingoing field quantities by successive approximations.

II. One might find a q number K involving the time explicitly as well as involving the Heisenberg dynamical variables, such that it is a constant of the motion. Then K satisfies

$$i\hbar dK/dt=i\hbar\partial K/\partial t+KH-HK=0. \qquad (15)$$

In a relativistic theory, K would involve the field quantities at time x_0 for all $x_1\, x_2\, x_3$, as well as involving x_0 explicitly.

With II, as with I-a, the result will not be manifestly covariant. II has the advantage over I-a that only one time variable appears in the solution. I-a has the two times, the time of the field point and the standard time t_0. II has the further advantage in that, in looking for a simple integral of the Heisenberg equations, one might be led to a quantity of physical importance, e.g., the operator of simultaneous creation of a particle and its entourage of associated particles at a certain time.

I have found II the most convenient method to work with, in spite of its not being manifestly covariant. It can be set up in terms of an interaction representation analogous to the usual interaction representation of the Schrödinger picture.

Let the Hamiltonian be

$$H=H_0+V,$$

where H_0 is the energy of the electrons alone plus the energy of the electromagnetic field alone and V is the interaction energy. Suppose each q number ξ is transformed to a new q number $\xi\dagger$ by

$$\xi\dagger=e^{iH_0t/\hbar}\xi e^{-H_0t/\hbar}.$$

There is no doubt about the existence of the unitary operator $e^{iH_0t/\hbar}$ because H_0 is quite a simple operator.

The explicit dependence of $\xi\dagger$ on t is, of course, different from that of ξ. The connection between them is

$$i\hbar\frac{\partial\xi\dagger}{\partial t}=e^{iH_0t/\hbar}\left(i\hbar\frac{\partial\xi}{\partial t}-H_0\xi+\xi H_0\right)e^{-iH_0t/\hbar}.$$

If we take ξ to be the K of Eq. (15), the transformed $K\dagger$ satisfies

$$i\hbar\frac{\partial K\dagger}{\partial t}=V\dagger K\dagger-K\dagger V\dagger. \tag{16}$$

The only explicit dependence of $K\dagger$ on t is caused by the interaction.

If we count V as small we can solve (16) by a perturbation method, putting

$$K\dagger=K\dagger^{(0)}+K\dagger^{(1)}+K\dagger^{(2)}+\cdots.$$

We find

$$\partial K\dagger^{(0)}/\partial t=0, \tag{17}$$

$$i\hbar\partial K\dagger^{(n+1)}/\partial t=V\dagger K\dagger^{(n)}-K\dagger^{(n)}V\dagger \quad n=0,1,2,\cdots. \tag{18}$$

from which the various $K\dagger^{(n)}$ may be obtained by successive integrations. We can then transform back to get the various $K^{(n)}$. Equation (17) tells us that $K\dagger^{(0)}$ is a dynamical variable at time t, not containing t explicitly. We may start off with any such $K\dagger^{(0)}$ and proceed to calculate $K\dagger$ and K.

APPLICATION TO THE ANOMALOUS MAGNETIC MOMENT AND THE LAMB SHIFT

The interaction energy between the electrons and the electromagnetic field is $\int A_\mu j^\mu d^3x$. To calculate the anomalous magnetic moment, we must suppose there is a static magnetic field present. To calculate the Lamb shift we must suppose there is a static electric field present. Static fields may be described by potentials, $\mathcal{A}_\mu$ say, which are functions of x_1, x_2, x_3 but do not vary with the time. They are thus not dynamical variables, but c numbers. They give rise to a further term in the Hamiltonian, namely, $\int \mathcal{A}_\mu j^\mu d^3x$, which should be included in H_0, not in the perturbation V.

As our working dynamical variables we take the Fourier components $A_{\mu k}$, $A_{\mu k}^*$ of the electromagnetic potentials, defined at each instant of time by

$$A_{\mu x}=\int\left\{A_{\mu k}e^{-i(\mathbf{k}\cdot\mathbf{x})}+A_{\mu k}^*e^{i(\mathbf{k}\cdot\mathbf{x})}\right\}d^3k,$$

$$dA_{\mu x}/dx_0=i\int|k|\left\{A_{\mu k}e^{i(\mathbf{k}\cdot\mathbf{x})}-A_{\mu k}^*e^{i(\mathbf{k}\cdot\mathbf{x})}\right\}d^3x,$$

the suffix x on the left here meaning $x_1\,x_2\,x_3$. We also take the ψ_n, ψ_n^* variables referring to emission and absorption of an electron into various stationary states for the static field $\mathcal{A}_\mu$.

As we are interested in a one-electron problem, we take for $K\dagger^{(0)}$ the operator of emission of an electron into some positive-energy state, say ψ_i. Such a choice for $K\dagger^{(0)}$ satisfies (17). We proceed to calculate $K\dagger^{(1)}$ and $K\dagger^{(2)}$. This gives us $K\dagger$, and hence K, to the accuracy $e^2/\hbar c$.

We get K as a function of our working dynamical variables $A_{\mu k}, A_{\mu k}^*, \psi_n, \psi_n^*$. These are noncommuting quantities, q numbers, so the form in which K appears depends on the order in which we arrange them. To get a standard form for K, we must arrange the q numbers in some normal order in every term in K. We assume this normal order to be that in which all emission operators are to the left of all absorption operators, in conformity with the usual practice. The emission operators are $A_{\mu k}$ for all μ, ψ_n^* for positive-energy states, and ψ_n for negative-energy states.

With K arranged in the standard form, we pick out those terms that refer to the emission of one electron and do not contain any other emission or absorption operators. We find that such terms are a multiple of the initial emission operator ψ_i^*. The time variation of the coefficient then fixes the energy of the emitted electron. We find that $K^{(1)}$ provides no correction to the energy and $K^{(2)}$ provides a correction of order $e^2/\hbar c$.

In this way we can calculate in the Heisenberg picture the change in the energy of an electron produced by the perturbation $\int A_\mu j^\mu d^3x$. The results are similar to those obtained from the usual calculation in the Schrödinger picture, with discard of the v-v terms. For the free electron there is a logarithmic infinity, which can be eliminated by a renormalization of the mass-parameter in the Hamiltonian. In the calculation of the Lamb shift a further logarithmic infinity arises, which can be eliminated by a renormalization of the charge-parameter in the Hamiltonian. The remaining terms give just the usual anomalous magnetic moment and Lamb shift, to the order $e^2/\hbar c$.

The calculations in the Heisenberg picture are quite logical, except for the occurrence of infinities which can be handled by renormalization. To make the theory completely logical one would have to cut off the high-energy part of the interaction, so as to make δm and δe finite. As the infinities are only logarithmic one can choose the cutoff at a fairly high energy value, say of the order 10^9 eV, and have $\delta m/m$ and $\delta e/e$ small, of the order 3%, so that they can legitimately be looked upon as small corrections.

The cutoff, of course, results in the theory not being relativistic. However, the departure from Lorentz invariance becomes serious only for processes involving energies comparable with the cutoff energy and one cannot in any case expect quantum electrodynamics to be accurate for energies much beyond 10^8 eV, because it neglects mesons and neutrinos. The Hamiltonian of quantum electrodynamics is thus necessarily an incomplete one and the cutoff does not reduce its domain of applicability to a serious extent.

PHYSICAL INTERPRETATION

We have the equations of quantum electrodynamics in the Heisenberg picture. To get a complete physical theory we need to set up some general method for their physical interpretation. The usual interpretation of quantum mechanics involves the Schrödinger wave function and so cannot be used. We must find a new one. The preceding calculations of the anomlaous magnetic moment and Lamb shift use some special assumptions for physical interpretation, which we need to generalize.

An important feature of these calculations was the normal ordering of the various factors in each term. This normal ordering of a q number becomes significant when it is multiplied into a ket $|0_t\rangle$ such that every absorption operator at time t gives zero when multiplied into $|0_t\rangle$. Then when the q number is multiplied into $|0_t\rangle$, only those terms of the normally ordered expression survive that contain no absorption operators.

The ket $|0_t\rangle$ may be considered as representing the vacuum state at time t. This state is special to the time t with respect to one particular Lorentz frame of reference. There is no universal vacuum state in the present theory. The ket $|0_t\rangle$ of course does not satisfy the Schrödinger equation.

For the general physical interpretation of the theory I propose two assumptions:

I. Each physical state corresponds to a q number, say K, that is a constant of the motion in the Heisenberg picture. K may be expressed as a function of the dynamical variables at time t and of t explicitly, and then satisfies (15).

II. For the state corresponding to K, what one can observe at time t is determined by the product $K|0_t\rangle$. Thus if K is expressed in terms of emission and absorption operators at time t and is arranged in the normal order, only those terms with no absorption operators will contribute anything observable. The other terms are latent at time t. Each of the terms with no absorption operators is associated with certain particles in certain states. We may assume that the square of the modulus of its coefficient is an *intensity* for these particles being observed at time t.

I use the word intensity and not the usual work probability because these quantities cannot be normalized. If one normalized them at one time, they would not remain normalized. They are related to probabilities in the sense that, if the intensity is zero, the corresponding probability is zero; if the intensity is large, the probability is large. I cannot at present make the concept of intensity more definite. The interpretation for quantum mechanics in the Heisenberg picture is thus rather vague.

You may be dissatisfied with a theory for which the equations are definite but the physical interpretation is vague. But if you refer to the historical development of the ordinary quantum mechanics, you will see that the equations came before their physical interpretation. For both the Heisenberg and Schrödinger theories, first came the equations of motion. Then applications were made to various simple examples for which it was not hard to guess a physical interpretation. It needed a few years and quite a number of successfully worked-out examples before people were led to a complete understanding of the uncertainty relations and the general physical interpretation. With the present reformulation of the basic equations, one may expect that it will again need a few years of development and a number of successful examples before their physical interpretation becomes completely clear.

The above interpretation in the Heisenberg picture should be compared and contrasted with the usual one in the Schrödinger picture. The present ket $|0_t\rangle$ plays the role of the usual vacuum ket $|0\rangle$, with the difference that the present one depends on the time t while the usual vacuum ket does not. The quantity $K|0_t\rangle$ of the present theory corresponds to the usual wave function at time t. They both determine what is observable at time t, according to similar rules. But there is no determinism for $K|0_t\rangle$ as there is for the wave function. The value of $K|0_t\rangle$ at time t does not determine the value of $K|0_{t'}\rangle$ at time t'. This may be illustrated by a simple example. We may take K to be just an absorption operator at a certain time t_0. Then $K|0_{t0}\rangle=0$. But we may very well have $K|0_t\rangle\neq 0$ for $t\neq t_0$. The present theory thus involves a greater amount of indeterminacy than the usual quantum mechanics.

GAUGE INVARIANCE

There are some constraints on the Hamiltonian in electrodynamics, associated with the gauge invariance of the theory. With Fermi's form of the theory, as was used here in the section on anomalous magnetic moment and Lamb shift, the constraints are

$$\partial A_\mu/\partial x_\mu \approx 0$$

holding at all points in space-time. They are written as weak equations, with the sign $\approx$, because they cannot be used freely, like identities. In the classical theory one may not use them inside Poisson brackets. In the quantum theory one may multiply them by factors on the left, but not by factors on the right, in general.

Developing the Heisenberg picture with these constraints, one introduces the subset of kets $|P\rangle$ satisfying

$$\partial A_\mu/\partial x_\mu|P\rangle=0$$

at all points of space-time, and the subset of q numbers that commute with $\partial A_\mu/\partial x_\mu$ at all points of space-time. These kets and q numbers are gauge-invariant. They are the only physically important quantities in the theory.

We may eliminate quantities that are not gauge-variant from the Hamiltonian. This is the so-called transformation to the Coulomb gauge. The variables describing the longitudinal components of the electromagnetic field then disappear from the Hamiltonian and in their place we get the Coulomb energy between the electrons. The transformation may be carried out without reference to the Schrödinger picture, so it is applicable to the present theory.

In our calculation of the anomalous magnetic moment and Lamb shift, we started from a $K\dagger^{(0)}=\psi_i^*$ which is not gauge-invariant, and refers to the emission of a bare electron. It would be better to start with a modified ψ_i^* which is gauge-invariant and refers to the emission of an electron together with its Coulomb field. This modified ψ_i^* may be defined as in my paper.[1] Likewise we worked with a ket $|0_t\rangle$ which is not gauge-invariant. It would be better to work with a modified ket which is gauge-invariant and which is more appropriate to describe the physical vacuum at time t.

The calculation in terms of these gauge invariant quantities is more complicated as many new terms appear in the equations. These terms, however, are not of the right quality to influence the energy, to the order $e^2/\hbar c$, so the results of the calculation stand. It is rather unsatisfactory, however, to have these new terms appearing in the equations because their physical significance is not clear.

The source of these complications is that, although our total Hamiltonian is gauge-invariant, we split it into two parts, H_0 and V, which are not separately gauge-invariant, and then assume that V is small. It is not very satisfactory to have a whole perturbation technique based on the assumption of the smallness of a quantity $\int A_\mu j^\mu d^3x$ which is not gauge invariant. However, one sees from the results that the assumption works for one-electron problems. It would not work for two-electron problems in which the Coulomb force is important.

[1] P. A. M. Dirac, Can. J. Phys. 33, 650 (1955).

Further development of the theory will require a better understanding of which quantities can be assumed to be small.

CONCLUSION

The present theory based on the Heisenberg picture is closely connected with the usual one based on the Schrödinger picture. There is much similarity in the details of calculations in the two theories. But there is the underlying difference that the present calculations all follow logically from certain general assumptions applied to a suitable Hamiltonian, while the previous calculations made use of working rules without a logical connection between them.

The treatment of quantum electrodynamics described here does not break new ground. It clears away dead wood on the old ground, showing up the essential good features of the theory and enabling one to avoid the difficulties and bad logic that arise merely from the use of the wrong picture. All the brilliant successes of the older theory are retained.

Of course the development of quantum theory proposed here should not be considered as detracting from the value of Schrödinger's work. The Schrödinger picture is a very good one for all those problems in which only a finite number of degrees of freedom are effective, such as problems in which the electromagnetic interaction can be represented by Coulomb forces, and it will continue to be used extensively as it is then simpler and more convenient than Heisenberg's. Only when one goes to an infinite number of degrees of freedom does one find that the Schrödinger picture is inadequate and that the Heisenberg picture has more fundamental validity.