Absorptive Processes and Single-Particle Exchange Models at High Energies. I. General Theory*

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It has been shown by several authors that the effect of competition from other open channels can materially affect the predictions of single-particle exchange models for high-energy particle reactions. The general theory of these absorptive effects in the distorted-wave Born approximation is discussed in detail for several situations in nonrelativistic potential scattering using the WKB approximation. The case of many coupled two-body channels is considered in detail. If the transition potential in question is of short range compared with most of the other potentials, the distorted-wave Born approximation for the S matrix in the presence of this potential is given by $S=S_0+A^TS^BA$, where S_0 is the S matrix in the absence of the perturbing potential, S^B is the Born approximation for the transition S matrix, and A is a known unitary matrix such that $S_0=A^TA$. Under appropriate circumstances, A may be approximated by the square root of the S matrix; one then obtains the generalization to the many-channel problem of a result derived previously for two channels, $S=S_0+S_0^{1/2}S^BS_0^{1/2}$. For long-range transition potentials, S is given by $S=S_0+\frac{1}{2}[S_0S^B+S^BS_0]$. These results do not depend on the details of the potentials. It is therefore plausible that they may be used in relativistic calculations. The generalizations necessary for the application to relativisitic particle reactions, including the effects of the particle spins, are discussed in detail. The properties of the rotation coefficients of the second kind encountered in this analysis are discussed in an Appendix. A second Appendix deals with the general Fourier–Bessel integral representation of the scattering amplitudes for particles with spin. Finally, the possibility of adapting our results to a K-matrix formalism, appropriate when the distorted-wave Born approximation fails, is discussed briefly.

I. INTRODUCTION

SINGLE–PARTICLE exchange models have been applied with apparent success to the theoretical analysis of a number of high-energy reactions involving elementary particles or resonant systems, for example, single[1] and double[2] pion production in nucleon-nucleon scattering, pion production in pion-nucleon collisions,[3] the production of nucleon isobars[4] and hyperon-antihyperon pairs[5,6] in antiproton-proton annihilations, and the production of K^* mesons and nucleon isobars[7] in kaon-nucleon scattering. It is characteristic of these reactions that the differential-reaction cross sections are strongly peaked at small scattering angles, or more properly, in the case of resonance production, at low-momentum transfers. Although this feature of the cross sections suggests strongly that the reactions proceed primarily through a long-range interaction such as that induced by the exchange of a particle of low mass (Fig. 1), the experimental cross sections are generally too sharply peaked at forward angles, and too small in magnitude, to be consistent with the predictions of the simple single-particle exchange models. Moreover, the simple models are generally inconsistent from a theoretical point of view, with the low partial-wave amplitudes exceeding in magnitude the limits imposed by unitarity. Although those amplitudes are certainly not given correctly by a theory which includes only the longest range part of the interaction, they may affect the predicted-reaction cross section significantly even at forward angles. It has been customary to attempt to circumvent these problems by introducing form factors at the interaction vertices and on the propagator of the exchanged particle, and adjusting their dependence on the momentum transfer to bring the theoretical cross sections into agreement with experiment.[8] This procedure may be reasonable in some cases. On the other hand, it may be criticized on several counts, and it is not clear that the phenomenological form factors so derived have any deep physical significance. It should be noted first that the form factors modify primarily the medium-to-short-

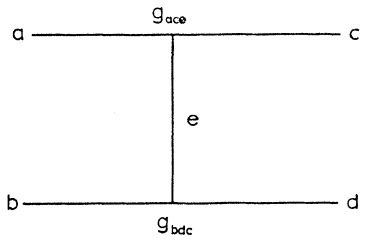

FIG. 1. The Feynman diagram for the single-particle exchange process $ab(e)cd$.

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[3] F. Salzman and G. Salzman, Phys. Rev. **120**, 599 (1961).
[4] The Ferrari theory (Ref. 2) as modified for the $\bar{p}p$ problem has been applied to this reaction in Refs. 8 and 9.
[5] N. J. Sopkovich, dissertation, Carnegie Institute of Technology, 1962 (unpublished), and Nuovo Cimento **26**, 186 (1962). The results given in this paper are incorrect in detail because of the use of an inconsistent phase convention in the treatment of the antiparticle spins.
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range parts of the interaction, or, equivalently, affect most strongly the low partial-wave transition amplitudes. However, these amplitudes are also affected by other short-range modifications of the interaction, for example, by multiple exchanges or the exchange of more massive particles, and it is perhaps unrealistic at the present time to attempt to describe any but those effects having the longest range. These do not always arise from the form factors. For example, the reaction $\bar{p}p \to \bar{N}^*N^*$ can proceed by the exchange of a single pion, and has been treated quite successfully at[8,9] 3.7 and[10] 7 BeV/c on the assumption that this is the dominant interaction. However, the pion-nucleon form factors required to fit the data are only marginally consistent with present knowledge of the properties of multipion systems: The effective mass in a single-pole approximation for the form factors is roughly 30 $m_\pi{}^2$, while there is no evidence for strong interactions of three or five pions in a $T=1$, $J^{PG}=0^{--}$ state for masses below 50 $m_\pi{}^2$. In contrast, the strong annihilation of the $\bar{p}p$ system into pions takes place with characteristic range of about 1.2 F.[9] It therefore seems probable that the most important medium-to-short-range modifications of the single-pion exchange model will arise in this case from the strong absorption of the low partial waves in the entrance and exit channels into the numerous competing channels. In the present paper, we will be concerned primarily with such absorptive effects, but will also consider briefly the case of the self-damping of exchange reactions through multiple exchanges. The latter appears to be the most probable mechanism for the suppression of the low partial-wave amplitudes in several cases, and is particularly relevant to situations in which only a few inelastic channels are important. The theory in this case will be based on the K-matrix approach to the multichannel scattering problem; the results provide a natural framework within which different reactions can be correlated.

Several attempts have been made to incorporate the requirements of unitarity and the effects of absorptive processes into modified single-particle exchange models. An early attack on the problem by Baker and Blankenbecler[11] involved the approximate solution of the coupled many-channel unitarity equations for the transition and scattering amplitudes on the physical cuts. Although the model did not encompass the case of complete or nearly complete absorption in any given channel, this work indicated clearly that previous results derived using unmodified single-particle exchange models were likely to require extensive revision. A more satisfactory solution to the problem was given by Sopkovich[5] for the specific reaction $\bar{p}p \to \bar{\Lambda}\Lambda$. The argument in this case was based on potential-scattering theory using the Glauber high-energy approximation for the scattering amplitude,[12] and an optical potential to describe the scattering in the initial and final states. The importance of absorptive effects was again demonstrated. Interest in such effects has developed rapidly in recent months. It was observed by Dar *et al.*[13] that the differential cross sections for particle reactions in the presence of strong absorption seem to display diffraction structure similar to that observed in nuclear reactions. In particular, some features of antiproton interactions at high energies could be reproduced qualitatively using a very simple model, essentially Fraunhofer diffraction from an illuminated annulus. Preliminary results of the present more detailed work have been published elsewhere.[14–16] Essentially equivalent results for a number of reactions have been obtained independently by Gottfried *et al.*[17,18] Ross and Shaw[19] have considered the effects of strong absorption on the reaction $\pi N \to \rho N$ using a model which is different in detail, but equivalent in its predictions, to that of Baker and Blankenbecler.[11] The model of Dar and Tobocman[20,21] for reactions involving spinless particles appears as a special case of the present theory and that of Gottfried and Jackson for a black-sphere model of the absorption (sharp cutoff in the absorption as a function of the angular momentum), but in general does not appear to be very realistic. A somewhat different approach has been used by Arnold,[22] who has applied the Blankenbecler-Goldberger[23] impact-parameter representation of the scattering amplitude to the np charge-exchange reaction; the results suggest that self-damping of the reaction amplitude may be important in this case. Finally, in interesting nonquantitative discussions, Squires[24] and Omnes[25] have attempted to derive entirely

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[25] R. Omnes, Phys. Rev. **137**, B649 (1965).

within the context of relativistic S-matrix theory the basic results on the absorptive modification of partial-wave transition amplitudes derived for potential scattering by Sopkovich,[5] Gottfried and Jackson,[17] and the present authors.[14,15] The arguments, while suggestive, are not entirely convincing, and the application to relativistic particle reactions of results proved only for nonrelativistic potential scattering is perhaps a weak point of the theory.

The modifications of the usual single-particle exchange models which are necessary in the presence of strong absorption in the initial and final states, and the effects of these modifications on the predictions of the models, seem now to be fairly well understood. Two effects should be emphasized. First, the angular dependence and magnitude of the reaction cross sections predicted by the modified models may differ drastically from those predicted by the unmodified models. The differential cross sections are more strongly peaked at forward angles and may not display the kinematic features characteristic of the simple models, for example, the predicted vanishing of the cross section at 0° for the process $np(\pi)pn$.[16,26] In addition, the reaction cross sections are generally reduced in magnitude. As a consequence of these differences, results derived using unmodified models may be quite misleading with respect to the validity of a supposed reaction mechanism, the magnitude of any unknown coupling constants, and the variation of form factors as functions of the momentum transfer. Second, the relative weights of various polarization states of the final particles, hence, the decay angular distributions of unstable reaction products, or spin correlations between stable particles, can be changed markedly by absorptive effects. Since the corrected theories involve multiple exchanges, such simple tests for the various single-particle exchange mechanisms as the Treiman-Yang[27] and Goldhaber[28] tests for scalar or pseudoscalar exchange, the Sakurai-Stodolsky tests for vector exchange,[29] and the simple predictions for the angular distributions in the decay of unstable reaction products,[30] are no longer relevant. However, it should be emphasized that one has replaced a simple model and very general kinematical considerations with a detailed dynamical model. Equivalent tests of the model using spin correlations or decay angular distributions are still available, but the predictions of the theory must be calculated separately for each case.

In the present paper, we will develop in some detail the theory of single-particle exchange reactions in the presence of initial- and final-state interactions. The application of the general theory to specific reactions will be considered in detail in a future paper; some preliminary results have already been reported.[15,16] In Sec. II, we will consider the case of potential scattering, first for a two-channel problem with absorptive potentials, then for the more realistic case of many coupled two-body channels. It will be shown that, under suitable conditions, the modified partial-wave transition amplitudes at high energies can be expressed in terms of the Born approximation amplitudes and the S-matrix elements for elastic scattering in the initial and final states. This result is applied in Sec. III to the discussion of reactions involving spinless particles, and then to the more general case of particles with spin. The mathematical properties of the rotation coefficients of the second kind which appear in this analysis are treated in Appendix A, and the general Fourier-Bessel integral representation for the scattering amplitude, in Appendix B.

Although much-improved compared to the simple single-particle exchange models, the model to be discussed in this paper is incomplete in at least three respects: (1) The effects of medium-to-short-range contributions to the basic reaction mechanisms have not been considered completely. These include in some cases the effects of the form factors which are certainly present. The theory thus remains a small-angle (or small-momentum-transfer) approximation. Although more generally valid than the unmodified models, the present theory cannot be expected to account in detail for large-angle reaction cross sections. To the extent to which it does, the fit must be regarded as accidental, signifying only that the correct transition amplitudes for the low partial waves do not differ drastically from the present estimates. On the other hand, a significant test of a more detailed dynamical theory of the short-range effects would require much better experimental data than are currently available. (2) The effects of indirect processes, in which, for example, the $\bar{p}p$ channel in the reaction $\bar{p}p \to \bar{\Lambda}\Lambda$ is connected to the $\bar{\Lambda}\Lambda$ channel through, say, $\bar{Y}^*Y^*$ or multipion intermediate states, have not been considered. The total contribution from such processes is expected to be small, but this has not been proved. These effects should be most prominent in the large-angle reaction cross section. (3) It is probable that the basic reaction mechanism should involve the exchange of a Regge pole rather than an elementary particle: The absorptive effects are not sufficiently strong to eliminate the violations of unitarity which result at high energies from the exchange of an elementary particle of spin greater than $\frac{1}{2}$. This problem will be treated in a future paper.

II. ABSORPTIVE EFFECTS IN POTENTIAL SCATTERING

In the present section, we will derive our basic results on the modification of partial-wave transition ampli-

[26] We denote by $ab(e,f,\cdots)cd$ the reaction $a+b \to c+d$, assumed to proceed through the exchange of particles e, f, $\cdots$ between the pairs a, c, and b, d.

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tudes at high energies in the presence of strong interactions in the initial and final states. The derivations will be given within the context of nonrelativistic potential scattering theory. However, it will be shown that, for a suitable class of potentials, including those of physical interest, the modification of the partial-wave amplitudes requires knowledge only of the elastic-scattering amplitudes for the initial and final states. Since these are directly accessible experimentally, and the smoothness conditions on the interaction required by the approximations are quite general, it is plausible that the results can be applied even to relativistic interactions for which the potential has at best a phenomenological significance.

Some of our results have been derived or suggested by other authors. However, previous derivations have generally been confined to the case of spinless particles and are in most instances fragmentary. We hope here to make fairly precise the assumptions basic to the approximations, and to elucidate several points which have caused difficulty in the past. The derivations will therefore be given in more detail than would be otherwise warranted, and will be arranged for pedagogical purposes as follows: Sec. IIA is concerned with the formulation of the problem for the case of spinless particles interacting with complex optical potentials. The approximate evaluation of transition amplitudes in the distorted-wave Born approximation for short- and long-range transition potentials is considered for this model in Secs. IIB and IIC, respectively. The restriction to spinless particles and optical potentials is removed in Sec. IID by extending the previous considerations to the case of many coupled two-body channels. Finally, the self-damping of large partial-wave amplitudes, and the possibility of treating several reactions simultaneously in a self-consistent manner, are discussed briefly in Sec. IIE using the K-matrix approach. For the benefit of those readers prepared to accept the approximations without proof, we note that the basic results of this section are contained in Eqs. (49) and (57), which express the modified partial-wave transition amplitudes in terms of the Born amplitude and the amplitudes for elastic scattering in the initial and final states.

A. Formulation of the Two-Channel Problem

The basic problem to be solved involves a number of coupled channels which interact through a real symmetric potential matrix V. We will assume that the potential $V_{\alpha\beta}$ which couples channels α and β is sufficiently weak that it can be treated as a perturbation. The truncated problem in which $V_{\alpha\beta}$ is set equal to zero we will furthermore assume to be solvable. The element $S_{\alpha\beta}$ of the S matrix which connects the two channels will, in general, be nonvanishing even for $V_{\alpha\beta}=0$. However, there is no reason to expect the various terms in which α and β are coupled through other channels γ to add coherently, and the individual contributions to $S_{\alpha\beta}$ are expected to be small in most situations. We will consequently assume that the main contribution to $S_{\alpha\beta}$ arises from the potential $V_{\alpha\beta}$ itself, and calculate this contribution in the distorted-wave Born approximation. This requires knowledge of the wave functions in the ingoing and outgoing channels only; we will assume initially that these can be constructed in terms of complex optical potentials which describe the effects on the wave functions in the entrance and exit channels of the absorption into the numerous competing channels; it will become clear later that this assumption, although convenient, is not necessary. The problem thus reduces to the solution of the coupled two-channel partial-wave equations[31]

$$\left[\frac{d^2}{dr^2}+k_\alpha{}^2-2m_\alpha V_\alpha(r)-\frac{l(l+1)}{r^2}\right]u_l{}^\alpha(r) = 2m_\alpha V_{\alpha\beta}(r)u_l{}^\beta(r), \quad (1a)$$

$$\left[\frac{d^2}{dr^2}+k_\beta{}^2-2m_\beta V_\beta(r)-\frac{l(l+1)}{r^2}\right]u_l{}^\beta(r) = 2m_\beta V_{\alpha\beta}(r)u_l{}^\alpha(r). \quad (1b)$$

Here V_α and V_β are the optical potentials in channels α and β, m_α and m_β are the reduced masses, and the u_l are the radial wave functions for orbital angular momentum l. The equations are to be solved subject to the boundary condition that there be an incident plane wave plus outgoing waves in channel β, and no incident wave in channel α. We are interested in the outgoing wave in channel α. The formal solution of the problem is readily expressed in terms of the outgoing-wave Green's functions for the problem with $V_{\alpha\beta}=0$:

$$u_l{}^\alpha(r)=2m_\alpha\int_0^\infty G_l{}^\alpha(k_\alpha;r,r')V_{\alpha\beta}(r')u_l{}^\beta(r')dr', \quad (2a)$$

$$u_l{}^\beta(r)=\bar{u}_l{}^\beta(r)+2m_\beta\int_0^\infty G_l{}^\beta(k_\beta;r,r') \times V_{\alpha\beta}(r')u_l{}^\alpha(r')dr', \quad (2b)$$

where the Green's functions satisfy the equations

$$\left[\frac{d^2}{dr^2}+k^2-2mV(r)-\frac{l(l+1)}{r^2}\right]G_l(k;r,r')=\delta(r-r'), \quad (3)$$

and $\bar{u}_l{}^\beta$ is a solution of Eq. (1b) with $V_{\alpha\beta}$ set equal to zero. If we treat $V_{\alpha\beta}$ as a perturbation, and ignore its effects on $u_l{}^\beta$, the outgoing wave in channel α is given

[31] We must apologize for repeating in the following paragraphs a derivation which is so well known for the case of real potentials. However, the presence of the phase shift factors in Eq. (8) is important for the ensuing arguments, and it is perhaps not clear *a priori* how the usual result is to be transcribed to the case of complex potentials.

by Eq. (2a) with u_l^β replaced by $\bar{u}_l^\beta$. The boundary conditions on u_l^β require that

$$\bar{u}_l^\beta(r)=e^{i\delta_l^\beta}F_l^\beta(k_\beta,r), \tag{4}$$

where δ_l^β is the complex elastic-scattering phase shift in channel β, and F_l^β is the standing wave solution of Eq. (1b) in the case $V_{\alpha\beta}=0$. Asymptotically,

$$F_l^\beta(k_\beta,r)\to\sin(k_\beta r+\delta_l^\beta-\tfrac{1}{2}l\pi), \quad k_\beta r\to\infty. \tag{5}$$

The outgoing wave Green's function in channel α may be constructed in terms of the solutions of Eq. (1a) with $V_{\alpha\beta}$ set equal to zero:

$$G_l^\alpha(k_\alpha;r,r')=-k_\alpha^{-1}H_{l,+}^\alpha(k_\alpha,r_>) \times F_l^\alpha(k_\alpha,r_<)\to -k_\alpha^{-1}e^{i(k_\alpha r+\delta_l^\alpha-\frac{1}{2}l\pi)}F_l^\alpha(k_\alpha r'), \quad k_\alpha r\to\infty, \tag{6}$$

where $H_{l,+}^\alpha(k_\alpha,r)$ is the outgoing wave solution of Eq. (1a) defined by the boundary condition

$$H_{l,\pm}^\alpha(k_\alpha,r)\to e^{\pm i(k_\alpha r+\delta_l^\alpha-\frac{1}{2}l\pi)}, \quad k_\alpha r\to\infty. \tag{7}$$

The desired result for u_l^α can now be written down, and the distorted-wave Born approximation for $S_{\alpha\beta}^l$, extracted:

$$S_{\alpha\beta}^l(k_\alpha,k_\beta)=e^{i\delta_l^\alpha}\tilde{B}_{\alpha\beta}^l(k_\alpha,k_\beta)e^{i\delta_l^\beta}, \tag{8}$$

where $\tilde{B}_{\alpha\beta}^l$ is the distorted-wave Born approximation matrix element,

$$\tilde{B}_{\alpha\beta}^l=-4i(v_\alpha v_\beta)^{-1/2} \times\int_0^\infty F_l^\alpha(k_\alpha,r)V_{\alpha\beta}(r)F_l^\beta(k_\beta,r)dr, \tag{9}$$

and v_β and v_α are the velocities of the particles in the entrance and exit channels. The complete scattering amplitude $M_{\alpha\beta}$ is given by the usual partial-wave expansion,

$$M_{\alpha\beta}=\sum_{l=0}^\infty(2l+1)M_{\alpha\beta}^l(k_\alpha,k_\beta)P_l(\cos\theta), \tag{10}$$

$$M_{\alpha\beta}=S_{\alpha\beta}/2ik_\beta.$$

The normalization is such that the differential reaction cross section is given by

$$d\sigma_{\alpha\beta}/d\Omega=|M_{\alpha\beta}|^2. \tag{11}$$

Thus far, the results are familiar. In particular, the distorted-wave Born approximation matrix element in Eq. (9) would be expected to depend critically on the details of the potentials V_α and V_β. However, we will now show that under appropriate circumstances, the function $\tilde{B}$ can be replaced by the usual Born approximation matrix element calculated with free wave functions. The only surviving dependence of the partial-wave transition amplitudes on the initial- and final-state interactions is then contained in the phase-shift factors in Eq. (8).

B. Derivation of the High-Energy Approximation for $S_{\alpha\beta}$: Short-Range Interactions

We are primarily interested in particle reactions at high energies. The particle wavelengths are short compared to the typical ranges of interaction in the elastic channels, and the optical potentials necessary to reproduce the elastic-scattering data are rather weak and vary slowly with r. The conditions necessary for the validity of the WKB approximation are therefore satisfied, and the wave functions may be approximated in terms of Bessel functions as

$$F_l(k,r)\approx\left[\frac{\pi kW_l}{6q_l}\right]^{1/2}[J_{1/3}(W_l)+J_{-1/3}(W_l)]. \tag{12}$$

Here

$$W_l(r)=\int_{r_l}^r q_l(r)dr, \tag{13a}$$

$$q_l^2(r)=k^2-2mV(r)-(l+\tfrac{1}{2})^2/r^2, \tag{13b}$$

and r_l is the classical turning point, $q_l(r_l)=0$. We will assume initially that the transition potential $V_{\alpha\beta}$ is of short range compared to V_α and V_β, but is of long range relative to the wavelengths in the initial and final states, that is[32]

$$k_\alpha\gg\mu\gg\nu_\alpha, \quad k_\beta\gg\mu\gg\nu_\beta, \tag{A}$$

where μ is the range parameter for the transition potential (the mass of the exchanged particle), and ν_α and ν_β are range parameters for the optical potentials. These relations are appropriate, for example, to the reaction $\bar{p}p(K,K^*)\bar{\Lambda}\Lambda$ at energies of a few BeV. Because of the short range of $V_{\alpha\beta}$, accurate values of the wave functions are needed in Eq. (9) only for r in the region $r_l\lesssim r\lesssim r_l+\mu^{-1}$: for $r<r_l$, the wave functions are suppressed in magnitude by the angular-momentum barrier, while for $r\gg r_l+\mu^{-1}$, the transition potential $V_{\alpha\beta}$ becomes quite small. If the optical potentials vary slowly over distances on the order of μ^{-1}, we can neglect this variation, and replace the potentials by their values at the classical turning points, or better, by appropriate

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[32] A more familiar limit is $\mu\gg k\gg\nu$. In this case, the corrections for initial- and final-state interactions involve only the changes in the values of the wave functions at the origin, and can be expressed in terms of the Jost functions $f_l(-k)$,

$$S_{\alpha\beta}^l\approx[f_l^\alpha(-k_\alpha)]^{-1}B_{\alpha\beta}^l(k_\alpha,k_\beta)[f_l^\beta(-k_\beta)]^{-1},$$

$$f_l(-k)=\exp\left[\frac{1}{\pi}\int_{k_0^2}^\infty\frac{\delta_l(k')dk'^2}{k'^2-k^2-i\epsilon}\right].$$

If it is argued that $\delta_l(k)$ varys slowly with k, hence, that the principal value integral in $f_l(-k)$ is small, this result reduces to that given in Eq. (18) (essentially this argument has been used by Omnes, Ref. 25, but in a slightly different context); however, it must be emphasized that the two approximations are in fact different.

average values $\bar{V}_l$ for the above range of r. Thus, for $r_l \lesssim r \lesssim r_l + \mu^{-1}$,

$$q_l^2(r) \approx k^2 - 2m\bar{V}_l - (l+\tfrac{1}{2})^2/r^2 = \tilde{k}^2 - (l+\tfrac{1}{2})^2/r^2, \quad (14)$$

and the function q_l is essentially identical to that for a free wave function for a shifted (complex) momentum $\tilde{k}$. Comparing the normalizations of the two wave functions, we find in fact that

$$F_l(k,r) \approx [k\tilde{k}]^{1/2} r j_l(\tilde{k}r), \quad r_l \leq r \leq r_l + \mu^{-1}. \quad (15)$$

The distorted-wave Born approximation matrix element in Eq. (9) can be approximated accordingly by the Born amplitude evaluated for modified momenta,

$$\tilde{B}_{\alpha\beta}{}^l(k_\alpha, k_\beta) \approx B_{\alpha\beta}{}^l(\tilde{k}_\alpha, \tilde{k}_\beta), \quad (16)$$

where

$$B_{\alpha\beta}{}^l(k_\alpha, k_\beta) = -4i[v_\alpha v_\beta]^{-1/2} \times \int_0^\infty j_l(k_\alpha r) V_{\alpha\beta}(r) j_l(k_\beta r) k_\alpha k_\beta r^2 dr. \quad (17)$$

The foregoing result depends on the optical potentials only through their average values in the neighborhood of turning points. If the optical potentials are sufficiently weak, $\tilde{k} \approx k$, and even this dependence on the potentials disappears. This final approximation requires essentially that $2m\bar{V}_l \ll k^2$, and is generally valid except for the lowest partial waves. Since the low partial waves are in any case strongly absorbed into the competing channels in the situations of interest, we shall henceforth assume that the partial-wave S-matrix elements are adequately approximated for all l as

$$S_{\alpha\beta}{}^l(k_\alpha k_\beta) \approx e^{i\delta_l{}^\alpha} B_{\alpha\beta}{}^l(k_\alpha, k_\beta) e^{i\delta_l{}^\beta}, \quad (18)$$

with $B_{\alpha\beta}{}^l$ as defined in Eq. (17). It should perhaps be emphasized that we have not used the fact that the potentials V_α and V_β are primarily absorptive. The results in Eqs. (16)–(18) are therefore applicable to potential scattering problems without strong absorption, for example, to the estimation of spin-coupling parameters in nucleon-nucleon scattering. This will *not* be true of the results derived in Sec. IIC.

The present result for $S_{\alpha\beta}{}^l$ has been derived independently by Sopkovich[5] and by Gottfried and Jackson[17] using a different, somewhat less precise method based on the Glauber high-energy approximation for potential scattering.[12] The disadvantage of the latter procedure lies in the apparent connection between the approximations which are made, and the presence of the phase shift factors in $S_{\alpha\beta}{}^l$. In fact, the appearance of the phase shift factors in the distorted-wave Born approximation matrix element is a consequence of the boundary conditions alone: The essence of the approximation is the replacement of $\tilde{B}_{\alpha\beta}{}^l$ by $B_{\alpha\beta}{}^l$, and the neglect of the "indirect" contributions to $S_{\alpha\beta}{}^l$.

A somewhat different result for the modified partial-wave matrix element $S_{\alpha\beta}{}^l$ has been suggested by Ross and Shaw.[19] Those authors approximate the wave functions $F_l(k,r)$ in Eq. (9) by $\cos\delta_l kr j_l(kr)$, and obtain

$$S_{\alpha\beta}{}^l = e^{i\delta_l{}^\alpha} \cos\delta_l{}^\alpha B_{\alpha\beta}{}^l \cos\delta_l{}^\beta e^{i\delta_l{}^\beta} = \tfrac{1}{4}(1+S_l{}^\alpha) B_{\alpha\beta}{}^l (1+S_l{}^\beta), \quad \text{(Ross-Shaw)}, \quad (19)$$

where S_l is the partial-wave amplitude for elastic scattering in the initial or final state. (The second form of this result can also be obtained by a straightforward generalization of the Baker-Blankenbecler solution of the coupled many-channel unitarity equations.[11] However, this generalization is not unique, and we will present an alternative result in the next section.) Since the product $e^{i\delta}\cos\delta$ has the limit $\frac{1}{2}$ for phase shifts with large imaginary parts, the transition amplitude in Eq. (19) cannot be completely suppressed, no matter how strong the competition from other inelastic channels. The result is clearly unsatisfactory in this limit. However, for small phase shifts, $\cos\delta_l \approx 1$, and the foregoing result approaches that given in Eq. (18).

The result for $S_{\alpha\beta}{}^l$ given in Eq. (18) is remarkable in that all reference to the details of the potentials has disappeared: The initial- and final-state interactions enter only through the phase shifts δ_l, and these can in principle be determined from experiment. The optical potentials V_β and V_α have in fact played a very subsidiary role in our argument, and will be eliminated altogether in Sec. IID. The only essential assumption is the following, that the absorption from the entrance and exit channels into the competing channels through the actual interaction occurs slowly and smoothly throughout the interaction region. Over any limited range of r, the absorption can be neglected, and the actual wave functions in the α and β channels can be approximated as linear combinations of the free ingoing and outgoing wave functions. The WKB argument was used primarily to determine the proper normalization of those wave functions. The fact that the approximate matrix element appears naturally in a factored form is readily understood from this point of view. In an ordinary high-energy scattering problem, a wave which starts at $r=\infty$ as an incoming wave e^{-ikr} propagates inward through the potential, and acquires a phase δ_l by the time it reaches the angular-momentum barrier. The wave is reflected at the barrier, propagates outward through the potential, and arrives at $r=\infty$ shifted in phase by $2\delta_l$ relative to a free wave. In the present problem, the wave is shifted from channel β to channel α in a region near the angular-momentum barrier in which the kinematics of the problem are more important than the detailed dynamics. The incoming wave has already acquired the phase $\delta_l{}^\beta$; the transition occurs with the amplitude $B_{\alpha\beta}{}^l$ characteristic of the Born approximation; and the outgoing wave in channel α acquires the phase $\delta_l{}^\alpha$ of the final-state interaction in escaping from the interaction region.

C. High-Energy Approximation for $S_{\alpha\beta}{}^l$: Long-Range Interactions

The situation is considerably more complicated when the transition potential is of long range compared to the absorptive potential,

$$k_\alpha \gg \nu_\alpha \gg \mu, \quad k_\beta \gg \nu_\beta \gg \mu. \tag{B}$$

The main contributions to the transition amplitude in the distorted-wave Born approximation arise in this situation from radii in the range $\mu^{-1}>r>\nu^{-1}$. In this region, the wave functions $F_l(k,r)$ can be approximated fairly well by their asymptotic forms, Eq. (5). Incorporating the phase factors $e^{i\delta_l}$ which appear in Eq. (8) into the wave functions, we obtain

$$S_{\alpha\beta}{}^l \approx i(v_\alpha v_\beta)^{-1/2}\int_{r_0}^{\infty}[H_{l,-}{}^\alpha - S_{\alpha\alpha}{}^l H_{l,+}{}^\alpha] \times V_{\alpha\beta}[H_{l,-}{}^\beta - S_{\beta\beta}{}^l H_{l,+}{}^\beta]dr. \tag{20}$$

Here $H_{l,\pm}$ are the outgoing and ingoing wave solutions of Eqs. (1) for $V_{\alpha\beta}=0$, and $S_{\beta\beta}{}^l$ and $S_{\alpha\alpha}{}^l$ are the S-matrix elements for elastic scattering in the initial and final states.

The condition $k\gg\mu$ implies that the functions $H_{l,+}{}^\alpha H_{l,+}{}^\beta$ and $H_{l,-}{}^\alpha H_{l,-}{}^\beta$ undergo many oscillations in the region in which the transition potential is large. As a consequence, there are large cancellations in the corresponding integrals, and the results are of order $\mu/(k_\alpha+k_\beta)$ or $(k_\alpha-k_\beta)/(k_\alpha+k_\beta)$ relative to those obtained with the slowly varying functions $H_{l,+}{}^\alpha H_{l,-}{}^\beta$ and $H_{l,-}{}^\alpha H_{l,+}{}^\beta$. Although the latter are multiplied by factors $S_{\alpha\alpha}{}^l$ or $S_{\beta\beta}{}^l$ which are small for low partial waves, they yield the main contributions to $S_{\alpha\beta}{}^l$ for the medium and high partial waves. We note in addition that the $H_{l,\pm}{}^\alpha H_{l,\pm}{}^\beta$ terms contain factors $(-1)^l$ which are absent in the $H_{l,\pm}{}^\alpha H_{l,\mp}{}^\beta$ terms, and which lead to further cancellations in the complete scattering amplitude $M_{\alpha\beta}$. [The total contribution of the $H_{l,\pm}{}^\alpha H_{l,\mp}{}^\beta$ terms to $M_{\alpha\beta}$ is of order $\mu^2/(k_\alpha+k_\beta)^2$ or $(k_\alpha-k_\beta)^2/(k_\alpha+k_\beta)^2$ relative to the leading contributions. Furthermore, if the phase shifts δ_l do not have large real parts, the two contributions are 90° out of phase, and do not interfere in the reaction cross section.[33]] Provided that the elastic-scattering matrix elements for the initial and final systems are not too different, the $H_{l,\pm}{}^\alpha H_{l,\mp}{}^\beta$ terms in Eq. (20) can be approximated by the Born matrix elements. We will assume that this approximation is reasonable,[34] and obtain as an effective transition matrix element

$$S_{\alpha\beta}{}^l \approx \tfrac{1}{2}[S_{\alpha\alpha}{}^l B_{\alpha\beta}{}^l + B_{\alpha\beta}{}^l S_{\beta\beta}{}^l]. \tag{21}$$

[33] These conclusions have been checked by detailed calculation for a simple model.

[34] As justification for this assumption, we can cite only the empirical observation that most cross sections for the elastic scattering of elementary particles at high energies display rather similar forward diffraction peaks, and that the interaction radii corresponding to the peak widths do not differ drastically.

In addition to the terms noted previously, we have omitted terms of order $(S_{\alpha\alpha}{}^l - S_{\beta\beta}{}^l)/(S_{\alpha\alpha}{}^l + S_{\alpha\beta}{}^l)$ in this expression. The corrections are not serious, and Eq. (21) is substantially correct, for the medium to high partial waves, $l>k/\nu$. On the other hand, the individual matrix elements may be seriously in error for $l \lesssim k/\nu$; our argument asserts only that the total transition amplitude obtained using Eq. (21) is a good approximation to the true amplitude.

The result for $S_{\alpha\beta}{}^l$ given in Eq. (21) is the natural generalization of that obtained by Baker and Blankenbecler[11] neglecting final-state interactions ($S_{\alpha\alpha}{}^l=1$). Other generalizations are possible, for example, that given in Eq. (19), but these do not appear to be as reasonable. The present result, although based on potential scattering theory, provides some justification for, and illustrates fairly clearly the approximations involved in, the Baker-Blankenbecler approach to the relativistic problem, in particular, the roles of high energy and the long range of the basic exchange mechanism.

In an earlier discussion,[15] it was shown that Eq. (21) could be replaced in the two-channel case by Eq. (18). The two results are clearly equivalent for the higher partial waves (δ_l small), and are also equivalent for smaller values of l if, as we have assumed, $S_{\alpha\alpha}{}^l$ does not differ drastically from $S_{\beta\beta}{}^l$. However, the present procedure is more readily generalized to the case of many coupled channels.

D. Generalization to Many Coupled Channels

In the present section, we wish to repeat in somewhat less detail the arguments of the preceding sections, but to generalize our considerations to the case of many coupled channels. It is possible in this way to weaken or eliminate the assumptions which were made in Sec. IIA concerning the existence and properties of the optical potentials V_α and V_β. The results are also relevant to the case of particles with spin.

We will consider initially the case of many coupled two-body channels, and calculate the change in the scattering matrix which results from a perturbation which is of short range in the sense of condition (A). The calculation will be performed in the distorted-wave Born approximation using the WKB approximation for the wave functions. The result will therefore be a generalization of that given in Eq. (18) for the two-channel problem with complex absorptive potentials. The coupled-channel Schrödinger equations for a given energy and total angular momentum may be written in matrix form as

$$\left(\frac{d^2}{dr^2}+K^2-2M^{1/2}VM^{1/2}-\frac{L(L+1)}{r^2}\right)M^{-1/2}\Psi=0, \tag{22}$$

where K^2, M, and $L(L+1)$ are diagonal matrices with elements given by the channel momenta $k_\alpha{}^2$, the re-

duced masses m_α, and the orbital angular-momentum factors $l_\alpha(l_\alpha+1)$. The potential matrix V is real and symmetric. The set of wave functions is represented by the matrix Ψ with elements $\psi_{\alpha\beta}$, where $\psi_{\alpha\beta}$ is the wave function in channel α which results from an initial wave in channel β. The leading terms in the WKB expansion of Ψ are easily constructed. For convenience, we will introduce a real symmetric matrix Q^2,

$$Q^2=K^2-2M^{1/2}VM^{1/2}-(L+\tfrac{1}{2})^2/r^2, \tag{23}$$

and rewrite Eq. (22) in the form

$$[d^2/dr^2+Q^2]M^{-1/2}\Psi=0. \tag{24}$$

Substituting for $M^{-1/2}\Psi$ the ordered exponential

$$M^{-1/2}\Psi^{\pm}=P\exp\left[\pm i\int^r U^{\pm}(r)dr\right]$$
$$=\sum_{n=0}^{\infty}(\pm i)^n\int^r dr_1\int^{r_1}dr_2\cdots$$
$$\times\int^{r_{n-1}}dr_nU(r_1)U(r_2)\cdots U(r_n), \tag{25}$$

we obtain a nonlinear equation for the matrix $U(r)$,

$$\pm i(dU^{\pm}/dr)+Q^2-U^{\pm 2}=0. \tag{26}$$

The equation may be solved formally by an expansion in powers of $\hbar$ ($\hbar=1$ in our units),

$$U^{\pm}(r)=\sum_{n=0}U_n^{\pm}(r), \tag{27}$$

with

$$U_0^{\pm}(r)=Q^2(r), \tag{28}$$

and

$$\pm i\frac{dU^{\pm}_{n-1}}{dr}=\sum_{m=0}^{n}U_m^{\pm}U_{n-m}^{\pm},\quad n\geq 1. \tag{29}$$

We will consider only the leading terms, U_0 and U_1.

The matrix U_0 is evidently the square root of Q^2. Since Q^2 is real and symmetric, it may be diagonalized by a real orthogonal transformation,

$$Q^2=ODO^T,\quad O^T=O^\dagger=O^{-1},\quad D \text{ diagonal}. \tag{30}$$

The matrix $U_0=Q$ is then given by

$$Q=OD^{1/2}O^T,\quad D=D^{1/2}D^{1/2}, \tag{31}$$

where $D^{1/2}$ is the diagonal matrix with elements which are the square root of those of D. The matrix Q^2 approaches the positive diagonal matrix K^2 for $r\to\infty$, and the negative diagonal matrix $-(L+\frac{1}{2})^2/r^2$ for $r\to O$. Thus each matrix element d_i of D passes through zero at some point r_i, this point corresponding to the classical turning point in the single-channel problem. Since these points are in general different, it will be convenient to introduce a set of matrices Q_i, each with a definite turning point, and to write Q in the form

$$Q_i=OD_iO^T,\quad Q=\textstyle\sum_i Q_i, \tag{32}$$

where D_i is the matrix with the single nonvanishing element d_i at the ith location on the diagonal.

The matrix U_1 is more difficult to construct. From the equation

$$\pm i(dU_0^{\pm}/dr)=U_0U_1^{\pm}+U_1^{\pm}U_0, \tag{33}$$

it is readily seen that $U_1^{\pm}$ is the generator of the congruent transformation which carries the diagonal matrix $K=U_0(\infty)$ into Q,

$$U_0=Q=\left[P\exp\left(\mp i\int_\infty^r U_1^{\pm}dr\right)\right]$$
$$\times K\left[P\exp\left(\mp i\int_\infty^r U_1^{\pm}dr\right)\right]^T; \tag{34}$$

hence,

$$P\exp\left(\mp i\int_\infty^r U_1^{\pm}dr\right)=Q^{1/2}K^{-1/2}. \tag{35}$$

In the general case in which the eigenvalues of the matrix $Q^{1/2}K^{-1/2}$ are distinct, or if $K^{-1/2}$ commutes with $Q^{1/2}$, then $Q^{1/2}K^{-1/2}$ can be diagonalized by a similarity transformation W,

$$Q^{1/2}K^{-1/2}=W^{-1}PW,\quad P \text{ diagonal}. \tag{36}$$

Then

$$U_1^{\pm}(r)=\pm i(d/dr)[W^{-1}(\ln P)W]. \tag{37}$$

It is assumed in this construction that the ordered product of one or more operators evaluated at a common point is given by the completely symmetrized product. In general, the matrix U_1 does not commute with Q, and the ordered product in Eq. (25) is consequently rather complicated. However, if r is not too close to the turning points of Q, and if the momenta k_α are large, it is clear from Eqs. (35) and (23) that the local changes in the amplitude of $M^{-1/2}\Psi$ generated by U_1 are small. We shall therefore drop U_1 from further consideration (a similar approximation was made in Sec. IIB when the amplitude factor $k/\tilde{k}$ was set equal to unity). With this approximation, and the substitution of Eq. (32) into Eq. (25) we obtain for the approximate wave functions $\Psi^{\pm}$

$$\Psi^{\pm}(r)=M^{1/2}P\exp\left(\pm i\int^r Q(r)dr\right)$$
$$=M^{1/2}\sum_{n=0}^{\infty}\sum_{i,j\cdots}(\pm i)^n\int_{r_i}^r dr_1\int_{r_j}^{r_1}dr_2\cdots$$
$$\times\int_{r_k}^{r_{n-1}}dr_nQ_i(r_1)Q_j(r_2)\cdots Q_k(r_n). \tag{38}$$

The corresponding free wave functions are obtained by replacing Q by the diagonal matrix Q_0,

$$Q_0=[K^2-(L+\tfrac{1}{2})^2/r^2]^{1/2}. \tag{39}$$

If we note that

$$\int_{r_i}^{r} Q_{0,i}dr \to k_i r-(l_i+\tfrac{1}{2})\frac{\pi}{2}, \quad k_i r\to\infty, \tag{40}$$

it is evident that the free outgoing and incoming wave solutions to Eq. (22) analogous to the H's of Sec. IIA are

$$H_0^{\pm}(r)=\exp\left\{\pm i\left[\int^r Q_0(r)dr+\tfrac{1}{4}\pi\right]\right\}, \quad r>r_i, \tag{41}$$

while the standing wave solution regular at $r=0$ is given by

$$\Psi_0(r)=(1/2i)[H_0^{+}(r)-H_0^{-}(r)]. \tag{42}$$

For large values of r, the interacting wave functions $H^{\pm}(r)$ are related to the free wave functions by a unitary matrix A,

$$H^{-}(r)\to H_0^{-}(r)A^{-1}, \tag{43a}$$

$$H^{+}(r)\to H_0^{+}(r)A^{T}, \quad r\to\infty \tag{43b}$$

where

$$A=\lim_{r\to\infty} P\exp\left(i\int^r Qdr\right)\exp\left(-i\int^r Q_0dr\right)$$

$$=\lim_{r\to\infty} P\exp\left(i\int^r Qdr\right)e^{-iKr+i\Delta}, \quad \Delta=(L+\tfrac{1}{2})\tfrac{1}{2}\pi, \tag{44}$$

and

$$M^{1/2}H^{\pm}(r)=e^{\pm i\pi/4}\Psi^{\pm}(r). \tag{45}$$

On the other hand, for r close to, but outside, the turning points, and K^2 large compared to $2M^{1/2}VM^{1/2}$, the matrix $Q(r)$ may be approximated in Eq. (38) by $Q_0(r)$ [or better, by $Q(r)$ evaluated using an average value of $V(r)$], and $H^{\pm}(r)\approx H_0^{\pm}(r)$.

The interacting (matrix) wave function Ψ^{in} for boundary conditions specified by unit incoming flux in the diagonal components $\psi_{\alpha\alpha}$, and only outgoing waves at infinity in the off-diagonal components $\psi_{\alpha\beta}$, is easily constructed. Combining the $H^{\pm}$ functions so as to obtain the regular function Ψ_0 for small r, and multiplying on the right by A and on the left by $K^{-1/2}$ to enforce the proper boundary conditions and normalization for $r\to\infty$, we obtain

$$\Psi^{\text{in}}=(1/2i)v^{-1/2}[H^{-}(r)-H^{+}(r)]A \tag{46a}$$

$$\to(1/2i)v^{-1/2}[H_0^{-}(r)-H_0^{+}(r)A^{T}A], \quad r\to\infty, \tag{46b}$$

$$\to -v^{-1/2}\Psi_0(r)A, \quad r_i\leq r\lesssim r_i+\nu^{-1}. \tag{46c}$$

In this expression, v is the diagonal matrix formed from the particle velocities, $v_{\alpha\alpha}=k_\alpha/m_\alpha$. The S matrix for the coupled-channel scattering problem is given by the coefficient of H_0^{+} in the asymptotic form of Ψ^{in} for $r\to\infty$,

$$S=A^{T}A. \tag{47}$$

As expected from the time-reversal invariance of the problem, $S^{T}=S$. The wave function Ψ^{out} appropriate for the boundary conditions specified by unit outgoing flux in the diagonal components $\psi_{\alpha\alpha}$, and only incoming waves in the off-diagonal components, is given by a similar expression,

$$\Psi^{\text{out}}=(1/2i)v^{-1/2}[H^{-}(r)-H^{+}(r)]A^{*} \tag{48a}$$

$$\to(1/2i)v^{-1/2}[H_0^{-}(r)A^{-1}A^{*}-H_0^{+}(r)], \quad r\to\infty \tag{48b}$$

$$\to -v^{-1/2}\Psi_0(r)A^{*}, \quad r_i\leq r\lesssim r_i+\nu^{-1}. \tag{48c}$$

The change in the S matrix which results from the addition to V of a short-range perturbation V', $V'^{T}=V'$, can be calculated in the distorted-wave Born approximation using the foregoing WKB wave functions. Accurate values of the wave functions are required only for a limited range of r near the angular-momentum barrier, $r_i\lesssim r\lesssim r_i+\mu^{-1}$, where μ^{-1} is the characteristic range parameter for V'. For r in this limited range, Ψ^{in} and Ψ^{out} can be approximated as in Eqs. (46c) and (48c). Thus,

$$S'=-4i\int \Psi^{\text{out}\,\dagger}(r)V'(r)\Psi^{\text{in}}(r)dr\to A^{T}BA, \tag{49}$$

where B is the Born approximation for S' calculated with the free wave functions $\Psi_0(r)$,

$$B=-4iv^{-1/2}\left[\int \Psi_0^{\dagger}(r)V'(r)\Psi_0(r)dr\right]v^{-1/2}. \tag{50}$$

It is readily verified using the unitarity of A and the relation $B^{\dagger}=-B$ that the corrected S matrix,

$$S=A^{T}A+A^{T}BA, \tag{51}$$

is unitary to within terms of second order in V'.[35] That

[35] The question of the unitarity of the approximate scattering matrix has caused some diffculty in the past. If we write $S=1+iT$, the unitarity of S leads to the familiar relation

$$2\,\text{Im}T=T^{\dagger}T,$$

or more explicitly, for one of the transition amplitudes,

$$2\,\text{Im}T_{\alpha\beta}=T_{\alpha\alpha}{}^{*}T_{\alpha\beta}+T_{\alpha\beta}{}^{*}T_{\beta\beta}+\sum_{\gamma\neq\alpha,\beta}T_{\alpha\gamma}{}^{*}T_{\alpha\beta}.$$

The first two terms of this expression involve only the amplitude in question and the elastic scattering amplitudes in the initial and final states. It is tempting to retain only those terms, since we have assumed previously that the channels α and β are not coupled for $V_{\alpha\beta}'=0$, or that the coupling is slight enough to neglect. This procedure would be incorrect, as may be verified by writing out the foregoing expression explicitly using T from Eq. (51) or Eq. (58): The sum over the channels $\gamma\neq\alpha, b$ contains terms of the same order in V' as appear in the first two terms. The point is the rather simple one, that if, for $V'=0$, some matrix element $T_{\gamma\alpha}$ (or $T_{\gamma\beta}$) is nonzero, the complete wave function defined by the ingoing wave boundary condition in channel γ contains some component in channel α (or β), and the matrix element of $V_{\alpha\beta}'$ between the wave functions in channels γ and β (or γ and α) will not vanish for $V_{\alpha\beta}'\neq 0$. Neglect of this point led Dar and Toboc-man (Ref. 20) to conclude that the initial- and final-state interactions must be substantially the same for the low partial waves, and that $T_{\alpha\beta}$ should be real for the high partial waves. These conclusions are clearly unwarranted, as is that of Selleri (Ref. 8) that the unmodified single-particle-exchange models are formally consistent with the unitarity relations even in the presence of strongly absorptive initial- and final-state interactions.

S' should have the form given in Eq. (49) can be understood on the basis of an argument similar to that given at the end of Sec. IIB.

In those situations in which we are primarily interested, the perturbation V' is just that part of V which connects some subsets of the incident and outgoing channels. We will assume as previously that the "indirect" contributions to the corresponding transition matrix are small, and that the transition potential is sufficiently weak that its contributions to S can be calculated in the distorted-wave Born approximation. The transition matrix is then given by Eq. (49), $S'\approx A^T BA$, and the solution of the problem is reduced to the determination of the matrix A. In general, this requires knowledge of the complete potential matrix V and the use of Eq. (44). However, A may be obtained directly from S if K commutes at least approximately with Q, and if we neglect the small effects of V' on the scattering in the other channels. Then

$$A=P\exp\left(i\int^{\infty}[Q-K]dr\right)e^{i\Delta}$$
$$=[e^{-i\Delta}Se^{-i\Delta}]^{1/2}e^{i\Delta}. \quad (52)$$

If S is diagonal in L, this result simplifies to one which has been suggested previously,[16,25] but not proved,

$$A=S^{1/2}, \quad [\Delta,S]=0. \quad (53)$$

The latter form is expected to be roughly correct for the elastic-scattering parts of S at high energies in the presence of many inelastic channels (spin-independent diffraction scattering). The additional assumption that those parts of S which connect the elastic and inelastic channels have random signs and magnitudes, hence contribute less to $S^{1/2}$ than the square roots of the elastic-scattering matrices, then yields the result which has been used in recent calculations.[16,18]

In the case of long-range transition potentials, we are primarily interested as in Sec. IIC in the asymptotic forms of the wave functions Ψ^{in} and Ψ^{out}. Using the results given in Eqs. (46b) and (48b) and the definition of S in Eq. (47), the change in S which results from a perturbation V' may be written approximately as

$$S'\approx i\int[H_0^-(r)-SH_0^+(r)]v^{-1/2}$$
$$\times V'(r)v^{-1/2}[H_0^-(r)-H_0^+(r)S]dr. \quad (54)$$

The range of integration lies outside the turning points and the region of strong interactions involving the unperturbed potential V. Although the contributions from small values of r could be evaluated using the WKB wave functions correct for that region, they are expected to be relatively unimportant if the potential V' is of sufficiently long range. We will not repeat the arguments of Sec. IIC in detail, but only note that by dropping terms of order μ/K, we obtain an effective partial-wave matrix element S' given by

$$S'\approx\tfrac{1}{2}[S_0B+BS_0]+\tfrac{1}{2}[S_0,\tilde{B}], \quad (55)$$

where S_0 is the S matrix for the unperturbed problem, B is the usual Born amplitude, and $\tilde{B}$ is given by

$$\tilde{B}=i\int[H_0^+(r)v^{-1/2}V(r)v^{-1/2}H_0^-(r)$$
$$-H_0^-(r)v^{-1/2}V(r)v^{-1/2}H_0^+(r)]dr. \quad (56)$$

The extra term involving $\tilde{B}$ is expected to be small compared to the leading term at high energies, and will be neglected. Then

$$S'\approx\tfrac{1}{2}[S_0B+BS_0]. \quad (57)$$

This matrix equation is the analog for the many-channel problem of the two-channel result given in Eq. (21). Since it is also a generalization of the Baker-Blankenbecler matrix element,[11] it is clear that the latter is valid only as an effective matrix element for long-range transition potentials. The complete S matrix,

$$S=S_0+\tfrac{1}{2}[S_0B+BS_0] \quad (58)$$

is clearly unitary to within terms of second order in V' (recall that $B^\dagger=-B$). To the same order, S_0 may be approximated by the (observed) scattering matrix S in the evaluation of S', Eq. (57) or (56).

Although the results for the transition matrix S' given in Eqs. (49) and (57) were derived for the case of nonrelativistic potential scattering, their form, and the assumptions used in their derivation, are sufficiently general that one may reasonably expect them to apply also to relativistic problems. In this connection, we note only that Eq. (57) can be derived in the relativistic theory using the methods of Baker and Blankenbecler,[11,36] and that the simple version of Eq. (49) in which the matrix A is evaluated as $S^{1/2}$ has been proposed independently by Omnes[25,37] on the basis of a dis-

[36] The "derivation" is not unique. Unitarity was enforced only on the right-hand, or physical cut, and the left-hand singularity structure of the analytically continued transition matrix element was ignored. This method is not reliable unless supplemented by independent arguments. The forms for S' given in Eqs. (49), (55), and (57) all yield S matrices which are unitary to the appropriate order in V'. As we have seen, the Baker-Blankenbecler matrix element is appropriate for high energies and long-range transition potentials, while that of Eq. (49) is appropriate when the transition potential is of short range. The justification for using one form or the other for S' derives from our detailed arguments about the wave functions, and not from the incidental, but necessary, fact that the corresponding S matrices are unitary.

[37] More assumptions than necessary were used by Omnes in the paper of Ref. 25. Specifically, the result $S'=e^{i\delta}Be^{i\delta}$ was obtained in the representation in which S is diagonal, $S=e^{2i\delta}$. Transformation of this result to the representation specified by the boundary conditions of Eq. (46b) yields Eq. (49) with A evaluated as in Eq. (53). However, it is clear from the definition of A, and, for example, from the special case given in Eq. (52) that the replacement of A by $S^{1/2}$ is not generally valid. The expansion performed by Omnes to obtain his final result, and the very detailed and somewhat suspect assumptions made about the eigen phase shifts, are not necessary.

persion relation argument. In the ensuing discussion, we will assume without further argument that this transposition of our results to the relativistic problem is correct; a more careful examination of this point would be of interest.

E. Generalized K-Matrix Approach

The approximate S matrices for the perturbed-scattering problem constructed in the preceding sections are unitary to within terms of second order in V'. However, they may still fail to satisfy the restrictions implied by the exact unitarity relations. For example, if contributions from channels other than those of interest are neglected, the exact relation

$$S^\dagger S=1 \tag{59}$$

can be reduced to the familiar inequality for the transition amplitudes S',

$$1\geq\sum_\beta|S_{\alpha\beta}'|^2, \quad S_{\alpha\beta}'=S_{\beta\alpha}'. \tag{60}$$

This inequality can be strengthened if the elastic-scattering amplitudes, or other inelastic amplitudes, are known. The bounds on the individual amplitudes implied by Eq. (60) are violated quite generally for the low partial waves in unmodified-single-particle exchange models, and may also be violated by the modified amplitudes for some reactions.[38] The simple distorted-wave Born approximation for the transition amplitudes is clearly inadequate in such cases, and our previous results must be modified to take some account of the expected self-damping of the large partial-wave amplitudes. (It may also be necessary to include some of the shorter range exchange processes in the discussion.) It is also desirable to develop a framework within which the mutual effects of several strong elastic or inelastic processes can be studied systematically. It may be possible in some cases to use the WKB estimate for S, but this appears to be rather complicated in general. An alternative approach to these problems is provided by the K-matrix method, in which the S matrix is written in the form[39]

$$S=(1+iK)(1-iK)^{-1}, \quad K^\dagger=K, \tag{61}$$

and is necessarily unitary. The crudest approximation for S is obtained by retaining only that part of K which connects the channels of interest, and evaluating the corresponding matrix elements in the Born approximation.

$$iK\approx\tfrac{1}{2}B. \tag{62}$$

This approximation has been considered recently[22] by Arnold and by Dietz and Pilkuhn. However, even if each matrix element is small enough to be calculated in the Born approximation, the totality of the transitions from the channels in question to other channels can lead to a significant distortion of the wave functions, and one should properly evaluate K in the distorted-wave Born approximation.

It is of interest with respect to the last point to re-examine our previous results using the K-matrix formalism. In the situation which we have considered S or equivalently, K was assumed to be known for part of the problem, and the effects of a small perturbation were calculated in the distorted-wave Born approximation. Assuming that S' and K' are small, their relation is easily determined,[40]

$$\begin{aligned} iK'&=(1+S)^{-1}2S'(1+S)^{-1}\\ &=(1-iK)\tfrac{1}{2}S'(1-iK). \end{aligned} \tag{63}$$

If S is substantially different from the unit matrix, K' will also differ substantially from the Born approximation estimate, Eq. (62). This effect of the competition from competing channels on the elements of K will persist even if the individual matrix elements are not small. It may be taken into account roughly in a self-consistent treatment of several coupled reactions by evaluating K as in Eq. (63), with S' given by whichever of Eqs. (49) or (57) is appropriate. Substitution of this result in Eq. (61) leads to a matrix equation for S which may be solved directly, or by iteration. Although there are some difficulties when the various interactions are of comparable range, and there is no clear choice between the short- or long-range approximations for S', the resulting expression for S should be considerably more accurate than that obtained using the Born approximation for K. (A more careful treatment of the problem using the N/D method may also be possible in some cases.) It would be of particular interest to apply this method to the study of such coupled processes as $p\pi(\pi)p\rho$, $pp(\pi)N^*N$, in which the main inelasticity is associated at moderate energies with a particular inelastic channel. One could hope in this manner to obtain not only the inelastic transition amplitude, but a reasonable estimate of the small-angle elastic-scattering amplitude including the main diffractive effects.

III. APPLICATIONS TO SINGLE-PARTICLE EXCHANGE PROCESSES

A. Spinless Particles

It will be convenient as a first application of our results to a relativistic problem to consider a single transi-

[38] This is apparently the case for single-pion-exchange reactions for energies below 2–3 BeV, for which the main inelasticity typically arises from the reaction in question. The damping of the large partial-wave amplitudes is more likely to result in these cases from multiple exchanges of a single pion between the initial and final systems, than from competition from other channels.

[39] In the language of the matrix N/D method, $K=\rho N/\mathrm{Re}D$. We will not attempt to solve the integral equations of the N/D method, but rather to approximate K directly.

[40] This relation provides the general connection between the K matrix and the distorted-wave Born approximation approaches for small perturbations, hence answers the questions with respect to their equivalence which were raised by Arnold in Ref. 22.

tion involving only spinless particles. The unmodified transition amplitude corresponding to the diagram in Fig. 1 is then

$$M_B(s,t)=\frac{B}{\mu^2-t}=\frac{B}{2pp'}\frac{1}{z-x}, \tag{64}$$

where μ is the mass of the exchanged particle, and s and t are respectively the square of the total energy in the center-of-mass system, and the square of the four-momentum transfer,

$$\begin{aligned} s&=(p_a+p_b)^2=(p_c+p_d)^2,\\ t&=(p_a-p_c)^2=(p_b-p_d)^2, \end{aligned} \tag{65}$$

and $x=\cos\theta=\hat{p}_a\cdot\hat{p}_c$. The quantity $z(s)$ will be a convenient variable in the ensuing analysis:

$$z=1+(\mu^2-t)/2pp'|_{\theta=0}=\frac{s^2-s(m_a^2+m_b^2+m_c^2+m_d^2-2\mu^2)+(m_a^2-m_b^2)(m_c^2-m_d^2)}{[(s-m_a^2-m_b^2)^2-4m_a^2m_b^2]^{1/2}[(s-m_c^2-m_d^2)^2-4m_c^2m_d^2]^{1/2}}. \tag{66}$$

The normalization is such that the differential reaction cross section is given by the square of the transition amplitude,

$$d\sigma_B/d\Omega=|M_B(s,t)|^2. \tag{67}$$

The factor B in Eq. (64) depends on the coupling constants and on the initial and final momenta. For a suitable definition of the coupling,

$$B=-\frac{i}{2}\frac{g_{ac}g_{bd}}{4\pi}\left(\frac{p'}{ps}\right)^{1/2}\mu^2, \tag{68}$$

where $p=|\mathbf{p}_a|=|\mathbf{p}_b|$ and $p'=|\mathbf{p}_c|=|\mathbf{p}_d|$.

The partial-wave expansion of $M_B(s,t)$ is readily obtained:

$$M_B(s,t)=\sum_{l=0}^{\infty}(2l+1)M_l^B(s)P_l(x), \tag{69}$$

where

$$M_l^B(s)=(B/2pp')Q_l(z), \tag{70}$$

with $Q_l(z)$ the Legendre function of the second kind. Assuming the validity in the relativistic theory of the results derived in Sec. II for the effects of rescattering in the initial and final states, the modified-transition amplitude is given by

$$M(s,t)=\sum_{l=0}^{\infty}(2l+1)M_l(s)P_l(x), \tag{71}$$

where

$$M_l(s)=A_l'^{T}M_l^B(s)A_l \tag{72}$$

for $\mu>\nu$, and

$$M_l(s)=\tfrac{1}{2}[S_l'M_l^B(s)+M_l^B(s)S_l] \tag{73}$$

for $\mu<\nu$. In these expressions, A'^T and A or S' and S are the elements of the corresponding matrices in the exit and entrance channels. To illustrate the effects of the absorption as simply as possible, we will consider the case of exponential diffraction scattering in the initial and final states, assuming also that S' and S are diagonal in the l representation, and that A may be approximated by $S^{1/2}$. The S-matrix elements in this case have the form

$$S_l\approx 1-e^{-(\nu^2/p^2)l(l+1)} \tag{74}$$

corresponding to an elastic-scattering amplitude

$$f(s,t)=(ip/2\nu^2)e^{t/4\nu^2}. \tag{75}$$

The low partial waves are nearly completely absorbed in this model. Although the complete scattering amplitude can now be constructed using the partial-wave expansion in Eq. (71), this procedure is rather cumbersome, and it is convenient to convert the sum into a Fourier-Bessel integral. The basis of the approximation is the well-known observation that the Legendre functions $P_l(x)$ and $Q_l(z)$ can be approximated for x or z near unity by Bessel functions,[41]

$$P_l(x)\to J_0([2(1-x)l(l+1)]^{1/2}), \tag{76}$$

$$Q_l(z)\to K_0([2(z-1)l(l+1)]^{1/2}). \tag{77}$$

With this identification of the functions, the series expansion for the transition amplitude in the Born approximation, Eq. (69), may be converted into the (exact) integral representation[42]

$$M_B(s,t)=(B/pp')\int_0^{\infty}J_0([2(1-x)]^{1/2}u)\times K_0([2(z-1)]^{1/2}u)udu. \tag{78}$$

The corresponding (approximate) expression for the modified amplitude $M(s,t)$ is readily constructed. In order to illustrate our main points without becoming overly involved in mathematical complications, we will use the same S-matrix factor, Eq. (74), for the initial and final states, but with p^2 or p'^2 replaced by pp' to take the difference in the momenta partially into ac-

[41] The hyperbolic Bessel functions $K_\nu(z)$ are used with the phase conventions of G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1958), Sec. 3.7.

[42] We could have used the Fourier-Bessel integral representation from the beginning (see, for example, R. Blankenbecler and M. L. Goldberger, Ref. 23, and Appendix B of the present paper). The approximation would then have appeared in the transition from the Fourier-Bessel weight function to be partial wave amplitudes.

count. Then

$$M(s,t)\to(B/pp')\int_0^\infty J_0([2(1-x)]^{1/2}u)\times K_0([2(z-1)]^{1/2}u)[1-e^{-\nu^2u^2/pp'}]udu. \quad (79)$$

The first term in this expression reproduces the Born amplitude; the second term represents the correction for initial- and final-state interactions. The integral representing the latter is readily cast in a form suitable for numerical evaluation. Using the integral representation for the hyperbolic Bessel function[43]

$$K_\lambda(z)=\tfrac{1}{2}(\tfrac{1}{2}z)^\lambda\int_0^\infty\frac{dt}{t^{\lambda+1}}e^{-t-z^2/4t}, \quad (80)$$

and noting the result[44]

$$\int_0^\infty J_\lambda(at)e^{-b^2t^2}t^{\lambda+1}dt=\frac{a^\lambda}{(2b^2)^{\lambda+1}}e^{-a^2/4b^2}, \quad (81)$$

we obtain after a simple change of variables

$$M(s,t)=\frac{B}{\mu^2-t}[1-L_0(s,t)], \quad (82)$$

where[45]

$$L_\lambda(s,t)=\int_0^\infty e^{-y+g(y)}\left[1+\frac{4\nu^2}{\mu^2-t}y\right]^{-\lambda-1}dy, \quad (83)$$

$$g(y)=\frac{2pp'(1-x)}{\mu^2-t}\times\frac{4\nu^2y^2}{\mu^2-t+4\nu^2y}. \quad (84)$$

The integral $L_\lambda(s,t)$ generally converges well, and can be evaluated numerically. However, we can readily obtain an asymptotic expansion for the result in the case that $4\nu^2/(\mu^2-t)\ll1$. Then

$$L_\lambda(s,t)\to1-\frac{4\nu^2}{\mu^2-t}\left[\lambda+1-\frac{4pp'(1-x)}{\mu^2-t}\right]+O(\nu^4), \quad (85)$$

and

$$M(s,t)\to\frac{B}{\mu^2-t}\frac{4\nu^2}{\mu^2-t}\left[1-\frac{4pp'(1-x)}{\mu^2-t}\right],\quad\frac{4\nu^2}{\mu^2-t}\ll1. \quad (86)$$

This formula displays clearly the diminution of the transition amplitude, and the sharpening of the angular distribution, expected as a result of the absorption in the entrance and exit channels, and suggests that conclusions drawn about the magnitude of coupling constants, the t dependence of possible form factors, and perhaps even the basic reaction mechanism, are likely to be incorrect when based on the usual unmodified single particle exchange models. An additional result is also of some interest. If we consider an extrapolation to the exchange pole at $t=\mu^2$ $(x=z)$, the absorption of the low partial waves becomes unimportant, and we recover the usual pole term plus finite corrections,

$$M(s,t)\to\frac{B}{\mu^2-t}-\frac{B}{4\nu^2}\times\int_0^\infty\frac{du}{1+u}\exp\{[-(\mu^2-t)-2pp'(z-1)u]\times[u/4\nu^2(1+u)]\}$$
$$\to\frac{B}{\mu^2-t}-\frac{B\sqrt{\pi}}{4\nu}[2pp'(z-1)]^{-1/2}, \quad (87)$$
$$2\nu^2/pp'(z-1)\ll1,\quad t\to\mu^2.$$

Although $M(s,t)$ assumes the correct form at the pole (this result is clearly general), it is evident from Eq. (86) that the Chew-Low extrapolation procedure,[46] which attempts to pick out of the cross section in the physical region the term which varies as $(\mu^2-t)^{-2}$, will fail in the presence of strong absorption. The term sought is absent, or much reduced in magnitude, in the modified cross section. This difficulty may well be responsible for the failure of the extrapolation procedure when applied to a number of high-energy reactions.

We note finally that a model proposed by Dar and Tobocman,[20] and applied to a number of reactions by Dar[21] is obtained as a simple consequence of our general results if we specialize the S-matrix factors to the form appropriate to the cutoff or black sphere model for the elastic scattering,

$$S_l=0,\quad l<L;\ S_l=1,\quad l>L;\ L=pR, \quad (88)$$

where R is the absorption radius. The effect of this change is to replace the factor in square brackets in Eq. (79) by a step function, $\theta(u-[L(L+1)]^{1/2})$. The resulting scattering amplitude is given by

$$M(s,t)=\frac{B}{\mu^2-t}\Big\{uK_0([2(z-1)]^{1/2}u)\times\frac{d}{du}J_0([2(1-x)]^{1/2}u)-uJ_0([2(1-x)]^{1/2}u)\times\frac{d}{du}K_0([2(z-1)]^{1/2}u)\Big\}_{u=[L(L+1)]^{1/2}}. \quad (89)$$

This formula displays the excessive diffraction structure typical of the sharp-cutoff model, and is less realistic

[43] Reference 41, Sec. 6.22.
[44] Reference 41, Sec. 13.3.
[45] The functions $L_\lambda(s,t)$ for $\lambda\neq0$ appear in the discussion of transitions involving particles with spin.
[46] G. F. Chew and F. E. Low, Phys. Rev. **113**, 1640 (1959).

in general than that obtained from the exponential model. Nevertheless, the dangers inherent in the uncritical application of single-particle exchange models to the description of high-energy phenomena are again clearly evident.

B. Particles with Spin

The generalizations necessary for the discussion of reactions which involve particles with spin lead to some complications, especially for high spins, but are generally straightforward. Using the helicity representation for angular momentum, the transition matrix for the reaction $a+b\to c+d$ can be written in the form[47]

$$M_{\lambda_a\lambda_b;\lambda_c\lambda_d}(s,t)=\sum_j(2j+1)M_{\lambda_c\lambda_d;\lambda_a\lambda_b}{}^j(s)\times d_{\lambda\mu}{}^j(x)e^{i(\lambda-\mu)\phi}, \quad (90)$$

$$\lambda=\lambda_a-\lambda_b, \quad \mu=\lambda_c-\lambda_d, \quad x=\cos\theta.$$

Our normalization is such that the differential reaction cross section for unpolarized particles is given by

$$d\sigma/d\Omega=(2s_a+1)^{-1}(2s_b+1)^{-1}\sum_{[\lambda]}|M_{\lambda_a\lambda_b;\lambda_c\lambda_d}(s,t)|^2. \quad (91)$$

In these expressions, $M^j=(2ip)^{-1}S^j$ is an element of the transition matrix for total angular momentum j, the λ label the helicities of the particles, and the d^j are the familiar representation coefficients for the rotation group.[48] If neither of the initial particles is polarized, Eq. (90) can be specialized to the case of scattering in the x-z plane, $\phi=0$. The relevant properties of the rotation coefficients d^j are summarized in Appendix A. We note only that these functions contain an over-all factor,

$$\left(\frac{1-x}{2}\right)^{\frac{1}{2}|\lambda-\mu|}\left(\frac{1+x}{2}\right)^{\frac{1}{2}|\lambda+\mu|} \quad (92)$$

which is independent of j. It is, therefore, clear that the matrix elements $M_{\lambda_c\lambda_d;\lambda_a\lambda_b}{}^B(s;t)$ calculated in the single-particle exchange approximation must have as their minimal angular dependence that arising from this factor and the propagator of the exchanged particle, $(\mu^2-t)^{-1}=[2pp'(z-x)]^{-1}$. In general,

$$M_{\lambda_c\lambda_d;\lambda_a\lambda_b}{}^B(s,t)=\left(\frac{1-x}{2}\right)^{\frac{1}{2}|\lambda-\mu|}\left(\frac{1+x}{2}\right)^{\frac{1}{2}|\lambda+\mu|}\times\left[\frac{B[\lambda]}{2pp'(z-x)}+\text{polynomial in } x\right], \quad (93)$$

where $B[\lambda]$ depends on the coupling constants and the particle momenta, and may be different for different helicity states. The variable z is defined in Eq. (66).

The polynomial in Eq. (93) affects only the low-partial-wave amplitudes, and will henceforth be ignored: These amplitudes are certainly not given correctly by the single-particle-exchange model, modified or unmodified, and are expected in any case to be strongly suppressed in magnitude in the presence of strong absorption. (The omission of these "exceptional" terms, although convenient and reasonable, is of course not necessary.)

The single-particle-exchange amplitude is readily decomposed in a partial-wave expansion using the orthogonality relations for the rotation coefficients. The result reduces to the integral

$$\begin{aligned}M_{\lambda_c\lambda_d;\lambda_a\lambda_b}{}^{j,B}(s)&=\epsilon(\lambda,\mu)\frac{B[\lambda]}{4pp'}\int_{-1}^{1}\left(\frac{1-x}{2}\right)^{\frac{1}{2}(m'-m)}\times\left(\frac{1+x}{2}\right)^{\frac{1}{2}(m'+m)}d_{m'm}{}^j(x)\frac{dx}{z-x}\\&=\epsilon(\lambda,\mu)\frac{B[\lambda]}{2pp'}\left(\frac{z-1}{2}\right)^{\frac{1}{2}(m'-m)}\times\left(\frac{z+1}{2}\right)^{\frac{1}{2}(m'+m)}e_{m'm}{}^j(z), \quad (94)\end{aligned}$$

where

$$m'=\tfrac{1}{2}|\lambda+\mu|+\tfrac{1}{2}|\lambda-\mu|,$$
$$m=\tfrac{1}{2}|\lambda+\mu|-\tfrac{1}{2}|\lambda-\mu|, \quad j\geq m'\geq|m|,$$

and $\epsilon(\lambda,\mu)$ is a signature factor determined by the relative signs and magnitude of λ and μ and the symmetries of the d^j. The functions $e_{m'm}{}^j(z)$ are rotation coefficients of the second kind, related to the $d_{m'm}{}^j(x)$ as the functions $Q_l(z)$ are related to the ordinary Legendre functions $P_l(x)$. The properties of these functions which are relevant for present purposes are summarized in Appendix A. The functions e^j have been introduced independently, and their analytic properties studied in some detail, by Andrews and Gunson.[49] The functions e^j of low order are readily calculated from the integral representation in Eq. (94); the functions of higher order may be obtained from the recurrence relations given in Appendix A. The distorted-wave Born approximation for the matrix M^j can be expressed as before in terms of $M^{j,B}$, and the matrices A^j or S^j for the channels ab and cd. Thus, for short-range-exchange processes,[50]

$$M^j=A_{cd}{}^{j,T}M^{j,B}A_{ab}{}^j, \quad (95)$$

[47] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).
[48] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957), Chap. IV.
[49] M. Andrews and J. Gunson, J. Math. Phys. 5, 1391 (1964). The phase convention for the e^j used in this paper differs from that of the present paper by a factor $(-1)^{\frac{1}{2}(m'-m)}$.
[50] More generally, if we treat several different channels simultaneously, and use the transition amplitudes M^j instead of the S-matrix elements S^j, Eqs. (95) and (96) are replaced by

$$M^j=p^{-1}A^{j,T}pM^{j,B}A^j$$

and

$$M^j=\tfrac{1}{2}[p^{-1}S^jpM^{j,B}+M^{j,B}S^j],$$

where p is the diagonal matrix formed from the particle momenta.

while for long-range-exchange processes,

$$M^j=\tfrac{1}{2}[S_{cd}{}^jM^{j,B}+M^{j,B}S_{ab}{}^j]. \tag{96}$$

In practice, the matrices A^j are not known. However, under our previous assumptions (many open inelastic channels, with transition amplitudes with random signs and phases), it is probably reasonable to approximate the A's by the square roots of the corresponding elastic scattering matrices,

$$A_{ab}{}^j\approx(S_{ab}{}^j)^{1/2}=O_c{}^TD_{ab}{}^{1/2}O_c, \tag{97}$$

where O_c is a complex-orthogonal matrix, and D_{ab} is diagonal.

To illustrate the effects of absorption as simply as possible, we will consider the case in which the matrices $S_{ab}{}^j$ and $S_{cd}{}^j$ are diagonal in the helicity representation.[51] We will assume, furthermore, that $S_{ab}{}^j=S_{cd}{}^j=S^j\mathbf{1}$, where $\mathbf{1}$ is the unit matrix, and the factor S^j has the form appropriate to exponential diffraction scattering with complete absorption of the low partial waves, Eq. (74). More complicated assumptions for the behavior of $S_{ab}{}^j$ and $S_{cd}{}^j$ lead to qualitatively similar results. Finally, the partial-wave sum in Eq. (90) will be converted to a Fourier-Bessel integral[52] using the Bessel function approximations for the functions d^j and e^j discussed in Appendix A. After some manipulation, the modified transition amplitude can be cast in a form analogous to that found for spinless particles,

$$M_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s,t)=\left(\frac{1-x}{2}\right)^{\frac{1}{2}|\lambda-\mu|}\left(\frac{1+x}{2}\right)^{\frac{1}{2}|\lambda+\mu|}\times\frac{B[\lambda]}{2pp'(z-x)}[1-L_{|\lambda-\mu|}(s,t)]$$
$$=M'_{\lambda_c\lambda_d;\lambda_a\lambda_b}{}^B[1-L_{|\lambda-\mu|}], \tag{98}$$

where M'^B is the Born amplitude with the exceptional terms omitted, and $L_{|\lambda-\mu|}$ is the integral defined in Eq. (83). From the asymptotic form of $L_{|\lambda-\mu|}$ given in Eq. (85), it is clear that absorptive effects tend to suppress the transition amplitudes at large angles. Furthermore, the functions $L_{|\lambda-\mu|}$ are quite different for different values of $|\lambda-\mu|$, and the amplitudes for different helicity states are consequently changed by different amounts even though it was assumed that S^j was independent of the helicities. This has the consequence that the predictions of the modified single-particle-exchange model for spin-dependent quantities are in general quite different from the predictions of the unmodified model, thus providing a sensitive test of the dynamical assumptions involved. This point has been emphasized especially by Gottfried and Jackson.[30] For completeness, we note the generalization of the Dar-Tobocman[20] sharp cutoff model to the case of particles with spin,

$$M_{\lambda_c\lambda_d,\lambda_a\lambda_b}(s,t)$$
$$=M'_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^B(s,t)\left(\frac{z-1}{1-x}\right)^{\frac{1}{2}|\lambda-\mu|}\left\{uK_{|\lambda-\mu|}([2(z-1)]^{1/2}u)\times\frac{d}{du}J_{|\lambda-\mu|}([2(1-x)]^{1/2}u)-uJ_{|\lambda-\mu|}\times([2(1-x)]^{1/2}u)\times\frac{d}{du}K_{|\lambda-\mu|}([2(z-1)]^{1/2}u)\right\}_{u=J}. \tag{99}$$

We have assumed in this expression that the partial waves for $j<J$ are completely absorbed, while those for $j>J$ are unaffected by the presence of the other open channels. Although the factor in brackets may lead to considerable diffraction structure in the differential reaction cross section for definite λ and μ, this structure is largely washed out in the complete reaction cross section.

In order to illustrate the foregoing remarks quantitatively, we will conclude by considering briefly a rather

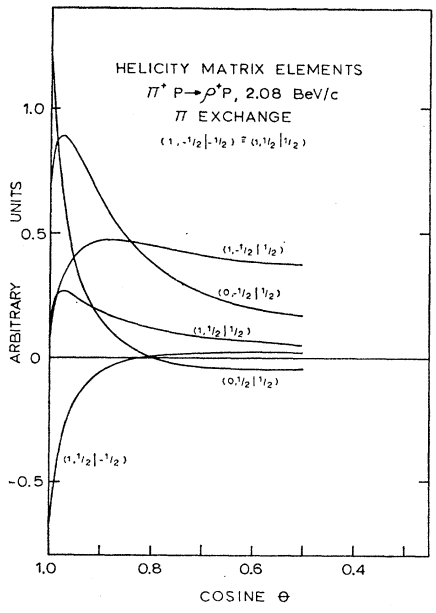

FIG. 2. The helicity amplitude for the modified single-pion-exchange model for the process $\pi^+p\to\rho^+p$ at 2.08 BeV/c. The amplitudes for all helicities reversed in sign are related to those shown as $(-\lambda_\rho,-\lambda_{p'}|-\lambda_p)=(-1)^{\lambda_\rho-\lambda_{p'}-\lambda_p}(\lambda_\rho,\lambda_{p'}|\lambda_p)$. The amplitude $(1,\frac{1}{2}|-\frac{1}{2})$ would vanish in the forward direction in the unmodified theory.

[51] This is probably not an unreasonable assumption in the high-energy region. The elastic scattering is then primarily diffractive in nature, and spin-dependent effects are expected to be small. Although it would perhaps be more plausible to assume that S is diagonal in l rather than j (the distance of closest approach, hence the amount of absorption, is determined by the former) the differences are of minor importance except for the lowest partial waves, and affect mainly the large angle reaction cross sections.

[52] The Fourier-Bessel integral approximation yields transition amplitudes which are somewhat too large at large angles, especially if $|\lambda-\mu|$ is large, but is sufficiently accurate for our purposes for x near 1.

simple, but representative reaction, $\pi^+p(\pi)\rho^+p$. This reaction has been studied experimentally at a number of energies; detailed theoretical analyses have been given by Gottfried and Jackson[17] and the present authors.[15,16] The basic transition matrix for the unmodified single-pion-exchange process can be obtained from the Feynman amplitude

$$[ig_{\pi NN}\bar{u}(p')\gamma_5 u(p)](\mu^2-t)^{-1}[ig_{\rho\pi\pi}\zeta_\rho\cdot(2p_\pi-p_\rho)] \quad (100)$$

if care is taken that the nucleon spinors and the ρ-meson spin vector ζ_ρ are defined with phases consistent with the helicity conventions of Jacob and Wick.[47] Alternative procedures which avoid this problem are based on the vertex function methods of Durand *et al.*[53] or the use of the crossed channel helicity amplitudes for the $j=0^-$ state and the Trueman-Wick crossing relations.[54] The resulting matrix M^B is as follows:

$$M^B(s,t)=\frac{g_{\rho\pi\pi}g_{\pi NN}}{8\pi pp'}\left[\frac{p'}{ps}\right]^{1/2}\frac{1}{z-x}$$
$$\times\begin{bmatrix} -A(1-x)^{1/2}(1+x) & -B(1-x)(1+x)^{1/2} \\ -B(1-x)(1+x)^{1/2} & A(1-x)^{1/2}(1+x) \\ C(1+x)^{1/2} & D(1-x)^{1/2} \\ D(1-x)^{1/2} & -C(1+x)^{1/2} \\ A(1-x)^{1/2}(1+x) & B(1-x)(1+x)^{1/2} \\ B(1-x)(1+x)^{1/2} & -A(1-x)^{1/2}(1+x) \end{bmatrix}. \quad (101)$$

The columns are to be labeled $\frac{1}{2}$, $-\frac{1}{2}$ for the proton helicity, left to right, and the rows, $1\frac{1}{2}$, $1-\frac{1}{2}$, $0\frac{1}{2}$, $0-\frac{1}{2}$, $-1\frac{1}{2}$, $-1-\frac{1}{2}$ from top to bottom, for the ρ meson and proton helicities. The functions A, B, C, and D can be expressed in terms of the particle momenta, energies, masses, and x as

$$\begin{aligned} A&=p[p_0p_0'-pp'-m_p^2]^{1/2},\\ B&=p[p_0p_0'+pp'-m_p^2]^{1/2},\\ C&=m_\rho^{-1}(p'p_{\pi 0}-pp_{\rho 0}x)[p_0p_0'-pp'-m_p^2]^{1/2},\\ D&=m_\rho^{-1}(p'p_{\pi 0}-pp_{\rho 0}x)[p_0p_0'+pp'-m_p^2]^{1/2}. \end{aligned} \quad (102)$$

It is evident from a comparison of the angular factors in M^B with those which must be present, Eq. (93), that all the matrix elements except those in the second and fifth rows contain exceptional terms of varying degrees of importance. We will consider as an example the $(1,\frac{1}{2};-\frac{1}{2})$ term in the first row. The angular factor which must be present on general grounds is $(1+x)^{1/2}\times(z-x)^{-1}$. However, the unmodified amplitude contains an addition factor $(1-x)$, and may be decomposed as follows:

$$(1-x)(1+x)^{1/2}(z-x)^{-1}$$
$$=(1+x)^{1/2}-(z-1)(1+x)^{1/2}(z-x)^{-1}. \quad (103)$$

The term $(1+x)^{1/2}$ in Eq. (103) is equal to

$$-\sqrt{2}d_{1/2,-1/2}{}^{1/2}(x),$$

and contributes only to the $j=\frac{1}{2}$ amplitude. The remaining term is of the normal form given in Eq. (93), and contributes to all partial-wave amplitudes. However, since $(z-1)$ is small at high energies, the normal matrix elements are much smaller than the exceptional term: The vanishing of the Born approximation for this amplitude in the forward direction, $x=1$, results from the destructive interference of the exceptional term and the remainder of the amplitude. Because we have not considered the shorter range parts of the interaction, we do not regard the exceptional term as significant, and will omit it in the remainder of the discussion. In any case, since the competition from other channels affects the $j=\frac{1}{2}$ states more strongly than states with higher values of j, modification of the Born amplitudes to account for initial- and final-state interactions will lead to a $(1,\frac{1}{2};-\frac{1}{2})$ amplitude which does not vanish in the forward direction, and is, in fact, quite sizeable there (Fig. 2). As may be seen from the transition matrix in Eq. (101) the only helicity state of the ρ which is populated in the unmodified model for $x=1$ is the state with $\lambda_\rho=0$. More generally, this is the only state which is populated in the ρ rest frame if the quantization axis is chosen along the direction of motion of the incoming

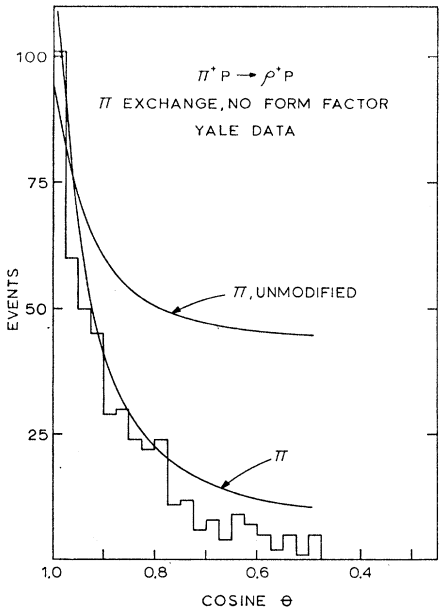

FIG. 3. Comparison of the predictions of the modified and unmodified single-pion-exchange models for the production angular distribution in the reaction $\pi^+p\to\rho^+p$ with the 2.08 BeV/c data of Ref. 55. The predictions are absolute.

[53] L. Durand, III, P. C. DeCelles, and R. B. Marr, Phys. Rev. **126**, 1882 (1962).

[54] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964).

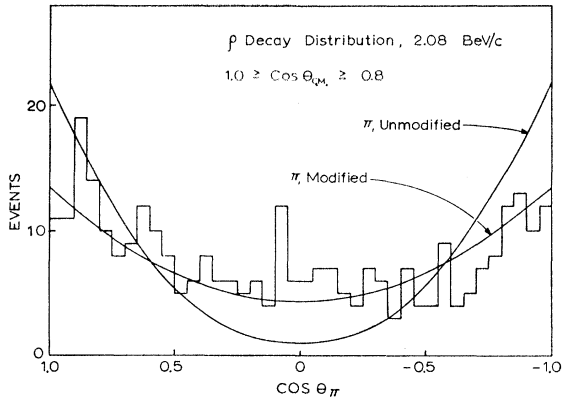

FIG. 4. Comparison of the predictions of the modified and unmodified single-pion exchange models for the decay angular distribution of the ρ^+ meson with the 2.08 BeV/c data of Ref. 55.

pion as seen in that frame. As a consequence, the angular distribution of the pions in the decay $\rho \to \pi\pi$ should display a $\cos^2\theta$ variation with respect to the direction of the incoming pion, and the decay should be azimuthally symmetric about that direction. These kinematic arguments are no longer valid in the modified theory, the corrections for initial- and final-state interactions removing the "accidental" cancellations which prevent the population of the states with $\lambda_\rho = \pm 1$ in the unmodified theory. The ρ decay distribution is consequently more complicated, and provides in principle a sensitive test of the theory.

The results obtained for the reaction $\pi^+ p \to \rho^+ p$ at 2.08 BeV/c assuming the reaction to proceed primarily through single pion exchange are shown in Figs. 2–4. The calculations were based on the simple model for the modified transition amplitudes given in Eq. (98). The functions $L_{|\lambda-\mu|}$ were evaluated by numerical integration using a value for the parameter ν, $\nu = 0.29$ BeV, taken from an exponential fit to the $\pi^+ p$ elastic-scattering cross section. The modified transition amplitudes are shown in Fig. 2. The size of the $(1, \frac{1}{2}; -\frac{1}{2})$ amplitude for $x \sim 1$ is particularly striking, since the amplitude vanishes as $(1-x)$ in the unmodified theory. The predicted ρ production angular distribution is compared to the prediction of the unmodified theory and the experimental data[55] in Fig. 3. The absorptive sharpening of the forward peak is clearly a major effect. Finally, the angular distribution of the decay pions as a function of the "$\pi\pi$ scattering angle" θ_π is shown in Fig. 4. The predicted and actual distributions should be compared with the $\cos^2\theta_\pi$ distribution expected for the unmodified-single-pion-exchange model. An improvement in the fit to the data is again evident. These results, and results for several other reactions, have been discussed in detail elsewhere.[16]

ACKNOWLEDGMENT

One of the authors (L.D.) would like to thank the Physics Division of the Aspen Institute for Humanistic Studies for the hospitality accorded him during the summer of 1964 when much of this work was completed.

APPENDIX A: PROPERTIES OF THE ROTATION COEFFICIENTS

A. General Properties

In this appendix, we wish to collect the relations among the rotation coefficients $d_{m'm}^{\,j}$ which have been useful in the present study, and to discuss some of the properties of the associated functions $e_{m'm}^{\,j}$. The ordinary rotation coefficients can be expressed in terms of hypergeometric functions, or, alternatively, Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$,[48,56] in the form

$$d_{m'm}^{\,j}(z) = (-1)^{m'-m}\left[\frac{(j+m')!(j-c)!}{(j+m)!(j-m')!}\right]^{1/2}\frac{1}{(m'-m)!}$$
$$\times\left(\frac{1-z}{2}\right)^{\frac{1}{2}(m'-m)}\left(\frac{1+z}{2}\right)^{\frac{1}{2}(m'+m)}{}_2F_1(-j+m',\, j+m'+1;\, m'-m+1;\, \tfrac{1}{2}(1-z))$$
$$= (-1)^{m'-m}\left[\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}\right]^{1/2}\left(\frac{1-z}{2}\right)^{\frac{1}{2}(m'-m)}\left(\frac{1+z}{2}\right)^{\frac{1}{2}(m'+m)}P_{j-m'}^{\,(m'-m,\,m'+m)}(z)\,. \tag{A1}$$

The indices j, m', and m are assumed to be such that none of the quantities $j \pm m$, $j \pm m'$, $m'-m$ is a negative integer. In normal applications, this requires that

$$j \geq m' \geq |m|\,. \tag{A2}$$

In general, the functions $d_{m'm}^{\,j}$ have branch points in the complex z plane at $z = \pm 1, \infty$; the branch cuts are customarily chosen so that the functions are real and single valued on the real axis for $|z| < 1$. The well-known symmetry properties of the rotation coefficients are easily derived. Thus, from the relation[57]

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,\, c-b;\, c;\, z)\,, \tag{A3}$$

[55] F. E. James and H. Kraybill, Proceedings of the 1964 International Conference on High Energy Physics at Dubna (to be published). The authors would like to thank Dr. James for supplying the data in Figs. 3 and 4.

[56] G. Szegö, *Orthogonal Polynomials* (American Mathematical Society Colloquium Publications, New York, 1939), Vol. 23.

[57] *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Company, New York, 1953), Vol. 1, Chap. 2.

we find that

$$d_{m'm}{}^{j}(z)=d_{-m,-m'}{}^{j}(z). \tag{A4}$$

The other relation which we shall need may be deduced from the observation that ${}_2F_1(a,b;c;z)/\Gamma(c)$ is an entire function of the parameters a, b, c. We may consequently continue the function analytically from the positive integer values of c appropriate to Eq. (A1) to negative integer values. An examination of the resulting series yields the result

$${}_2F_1(a,b;c;z)/\Gamma(c)=z^{1-c}[\Gamma(1+a-c)\Gamma(1+b-c)/\Gamma(a)\Gamma(b)\Gamma(2-c)]{}_2F_1(1+a-c,\,1+b-c;\,2-c;\,z),\quad c=\text{integer}. \tag{A5}$$

Applying this relation to Eq. (A1), and continuing in m' and m, $m'\to m$, $m\to m'$, we obtain the second symmetry relation

$$d_{m'm}{}^{j}(z)=(-1)^{m'-m}d_{mm'}{}^{j}(z), \tag{A6}$$

hence, from Eq. (A4),

$$d_{m',m}{}^{j}(z)=(-1)^{m'-m}d_{-m',-m}{}^{j}(z)=(-1)^{m'-m}d_{m,m'}{}^{j}(z). \tag{A7}$$

The rotation coefficients of the second kind, $e_{m'm}{}^{j}(z)$, may be defined in terms of the Jacobi functions of the second kind,[56] $Q_n{}^{(\alpha,\beta)}$, following Eq. (A1):

$$\begin{aligned}e_{m'm}{}^{j}(z)&=(-1)^{m'-m}\left[\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}\right]^{1/2}\left(\frac{z-1}{2}\right)^{\frac{1}{2}(m'-m)}\left(\frac{z+1}{2}\right)^{\frac{1}{2}(m'+m)}Q_{j-m'}{}^{(m'-m,m'+m)}(z)\\&=\tfrac{1}{2}(-1)^{m'-m}[(j+m')!(j-m')!(j+m)!(j-m)!]^{1/2}/(2j+1)!\\&\quad\times\left(\frac{z-1}{2}\right)^{-j-1+\frac{1}{2}(m'+m)}\left(\frac{z+1}{2}\right)^{-\frac{1}{2}(m'+m)}{}_2F_1\left(j-m'+1,\,j-m+1;\,2j+2;\,\frac{2}{1-z}\right).\end{aligned} \tag{A8}$$

The phase conventions are such that $e_{m'm}{}^{j}(z)$ is real and single valued on the real axis for $z>1$. [This convention differs from that of Andrews and Gunson,[49] the functions $e_{m'm}{}^{j}$ differing by a factor $(-1)^{\frac{1}{2}(m'-m)}$.] It is clear from Eq. (A8), and the symmetry of the hypergeometric function in its first two indices, that

$$e_{m'm}{}^{j}(z)=e_{mm'}{}^{j}(z). \tag{A9}$$

Furthermore, from Eqs. (A3) and (A8),

$$e_{-m',-m}{}^{j}(z)=e_{m'm}{}^{j}(z). \tag{A10}$$

It should be emphasized that these symmetry relations for the $e_{m'm}{}^{j}$ differ from those for the $d_{m'm}{}^{j}$, the difference resulting from our choice of phase conventions.

An integral representation for the $e_{m'm}{}^{j}$ is readily derived starting with that for the Jacobi functions of the second kind,[51]

$$Q_n{}^{(\alpha,\beta)}(z)=\tfrac{1}{2}(z-1)^{-\alpha}(z+1)^{-\beta}\times\int_{-1}^{1}\frac{dx}{z-x}(1-x)^{\alpha}(1+x)^{\beta}P_n{}^{(\alpha,\beta)}(x),\quad\alpha,\beta>0. \tag{A11}$$

Substituting from Eqs. (A1) and (A8), we obtain the desired result,

$$e_{m'm}{}^{j}(z)=\tfrac{1}{2}(z-1)^{-\frac{1}{2}(m'-m)}(z+1)^{-\frac{1}{2}(m'+m)}\times\int_{-1}^{1}\frac{dx}{z-x}(1-x)^{\frac{1}{2}(m'-m)}(1+x)^{\frac{1}{2}(m'+m)}d_{m'm}{}^{j}(x),\quad j\geq m'\geq|m|. \tag{A12}$$

This integral, which was needed in Eq. (94), displays clearly the analytic structure of the functions $e_{m'm}{}^{j}$.

We note also the corresponding expansion theorem,

$$\left(\frac{1-x}{z-1}\right)^{\frac{1}{2}(m'-m)}\left(\frac{1+x}{z+1}\right)^{\frac{1}{2}(m'+m)}\frac{1}{z-x}=\sum_{j=0}^{\infty}(2j+1)d_{m'm}{}^{j}(x)e_{m'm}{}^{j}(z) \tag{A13}$$

which follows from the orthogonality of the $d_{m'm}{}^{j}$,[48]

$$\int_{-1}^{1}d_{m'm}{}^{j}(x)d_{m'm}{}^{j'}(x)dx=\frac{2}{2j+1}\delta_{jj'}. \tag{A14}$$

Recurrence relations for the rotation coefficients may be derived with some manipulation from the well-known recurrence relations for the Jacobi functions.[56] For the $d_{m'm}{}^{j}$, we obtain

$$\begin{aligned}&j[(j-m'+1)(j+m'+1)(j-m+1)(j+m+1)]^{1/2}\\&\quad\times d_{m'm}{}^{j+1}(z)=(2j+1)[j(j+1)z-mm']d_{m'm}{}^{j}(z)\\&\quad-(j+1)[(j-m')(j+m')(j-m)(j+m)]^{1/2}d_{m'm}{}^{j-1}(z);\end{aligned} \tag{A15}$$

$$\begin{aligned}&j[j+m'+1]^{1/2}[1-z^2]^{1/2}d_{m'+1,m}{}^{j}(z)\\&\quad=[j-m]^{1/2}[jz+m]d_{m'm}{}^{j}(z)\\&\quad-[(j+m')(j+m)(j-m)]^{1/2}d_{m'm}{}^{j-1}(z),\end{aligned} \tag{A16}$$

$$\begin{aligned}&j(1-z^2)\frac{d}{dz}d_{m'm}{}^{j}(z)=-[j^2z-mm']d_{m'm}{}^{j}(z)\\&\quad+[(j+m')(j-m')(j+m)(j-m)]^{1/2}d_{m'm}{}^{j-1}(z).\end{aligned} \tag{A17}$$

The first relation may be used to increase the value of j, the second, together with the symmetry relations in Eq. (A7), to increase the values of m' or m. The functions of low order are of course readily calculable using the hypergeometric series in Eq. (A1). The corresponding recurrence relations for the $e_{m'm}{}^j$ differ from the above only in the replacement of $[1-z^2]^{1/2}$ by $[z^2-1]^{1/2}$ in Eq. (A16). However, the recurrence relations in this case cease to be valid when any of the factors $(j\pm m')$, $(j\pm m)$ vanishes, the correct relations involving extra terms. It should be recalled also that the symmetry properties of the $e_{m'm}{}^j$ and the $d_{m'm}{}^j$ are different.

The analytic properties of the rotation coefficients $d_{m'm}{}^j(z)$ and $e_{m'm}{}^j(z)$ as functions of z and j have been explored in detail by Andrews and Gunson,[49] and will not be considered here (the simpler properties of these functions have also been noted by Kibble[58] and by Charap and Squires[59]).

B. Approximation by Bessel Functions

It is well known that the ordinary Legendre functions and the Jacobi polynomials for arguments near unity can be approximated by Bessel functions.[56] Similar approximations are readily derived for the rotation coefficients. Thus from Eq. (A1),

$$d_{m'm}{}^j(z)=(-1)^{m'-m}\times\left[\frac{(j+m')(j-m)!}{(j+m)!(j-m')!}\right]^{1/2}\frac{1}{(m'-m)!}\left(\frac{1-z}{2}\right)^{\frac{1}{2}(m'-m)}\times\left\{1-\frac{[j,m',m]}{m'-m+1}\frac{1-z}{2}+O\left(\left(\frac{1-z}{2}\right)^2\right)\right\},\quad\text{(A18)}$$

where

$$[j,m'm]=j(j+1)-\tfrac{1}{2}m'(m'+1)-\tfrac{1}{2}m(m-1),\quad\text{(A19)}$$

and we have expanded the factor $[(1+z)/2]^{\frac{1}{2}(m'+m)}$ in powers of $(1-z)/2$. Identification of the first two terms of the series with the corresponding terms in the expansion of the Bessel function $J_\nu(x)$,

$$J_\nu(x)=\sum_{r=0}^{\infty}(-1)^r\frac{1}{\Gamma(r+1)\Gamma(\nu+r+1)}\left(\frac{x}{2}\right)^{\nu+2r}\quad\text{(A20)}$$

yields the approximation

$$d_{m'm}{}^j(z)\to(-1)^{m'-m}\left[\frac{(j+m')!(j-m)!}{(j+m)!(j-m')!}\right]^{1/2}\times[j,m',m]^{-\frac{1}{2}(m'-m)}J_{m'-m}(\{2(1-z)[j,m'm]\}^{1/2}),\quad|(1-z)/2|\ll1.\quad\text{(A21)}$$

This result is exact to second order in $(1-z)/2$; if $j\gg m'$, $|m|$, the approximate expression is valid even if this quantity is not small. The numerical factor in Eq. (A21) is close to unity in magnitude. For large values of j, we obtain the simpler result

$$d_{m'm}{}^j(z)\to(-1)^{m'-m}J_{m'-m}\times([2j(j+1)(1-z)]^{1/2}),\quad j\gg m',|m|,\quad\text{(A22)}$$

or, somewhat more accurately if $j\gg m$, and $(1-z)/2$ is not small,

$$d_{m'm}{}^j(z)\to(-1)^{m'-m}((1+z)/2)^{\frac{1}{2}(m'+m)}\times J_{m'-m}([2j(j+1)(1-z)]^{1/2}).\quad\text{(A23)}$$

An approximate expression for the function $e_{m'm}{}^j(z)$ in terms of the hyperbolic Bessel function[41] $K_{m'-m}$ is readily derived by using the foregoing results in Eq. (A12). Alternatively, one can compare the expansion of $e_{m',m}{}^j(z)$ in powers of $(z-1)/2$ with the series expansion of $K_\nu(x)$. The resulting approximation, valid for small values of $|(z-1)/2|$ or large values of j, is

$$e_{m'm}{}^j(z)\to(-1)^{m'-m}\times\left[\frac{(j+m)!(j-m')!}{(j+m')!(j-m)!}\right]^{1/2}[j,m,m']^{\frac{1}{2}(m'-m)}\times K_{m'-m}(\{2(z-1)[j,m,m']\}^{1/2})\quad\text{(A24)}$$

$$\to(-1)^{m'-m}K_{m'-m}([2j(j+1)(z-1)]^{1/2}),\quad j\gg m',|m|.\quad\text{(A25)}$$

A somewhat better approximation if $(z-1)/2$ is not small, but j is large, is given by

$$e_{m'm}{}^j(z)\to(-1)^{m'-m}\left(\frac{2}{z+1}\right)^{\frac{1}{2}(m'+m)}\times K_{m'-m}(\{2j(j+1)(z-1)\}^{1/2}).\quad\text{(A26)}$$

The approximations for $d_{m'm}{}^j$ and $e_{m'm}{}^j$ fail unless $j|(z-1)/2|\ll1$. However, if $|(z-1)/2|$ is small, the functions $e_{m'm}{}^j(z)$ will be quite small for $j\sim|2/(z-1)|$, and the effect of the approximation on the partial-wave expansion in Eq. (A13) will be negligible.

The utility of the Bessel function approximations for the rotation coefficients derives in part from the ease with which the magnitude of the functions may be estimated, and in part from the fact that j appears in either approximation in the argument of a continuous function rather than as a discrete parameter. For example, if $(1-x)/2$ and $(z-1)/2$ are small, so that many terms contribute significantly to the series in Eq. (A13), we may convert the series to an integral

[58] T. W. B. Kibble, Phys. Rev. **131**, 2282 (1963).
[59] J. M. Charap and E. J. Squires, Ann. Phys. (N. Y.) **20**, 145 (1962); **21**, 8 (1963).

over j, and obtain the approximate expression

$$\sum_{j=0}^{\infty}(2j+1)d_{m'm}{}^{j}(x)e_{m'm}{}^{j}(z)\to\left(\frac{1+x}{z+1}\right)^{\frac{1}{2}(m'+m)}\times\int_{m'}^{\infty}J_{m'-m}(\{2j(j+1)(1-x)\}^{1/2})\times K_{m'-m}(\{2j(j+1)(z-1)\}^{1/2})d[j(j+1)]\approx\left(\frac{1-x}{z-1}\right)^{\frac{1}{2}(m'-m)}\left(\frac{1+x}{z+1}\right)^{\frac{1}{2}(m'+m)}\frac{1}{z-x},\quad\text{(A27)}$$

where we have used the approximations of Eqs. (A23) and (A26) for the rotation coefficients. In the last step in Eq. (A27), we have assumed that the arguments of the Bessel functions are sufficiently small for $j=m'$ that the lower limit of integration can be replaced by $j=0$. We have used similar approximations to convert the partial-wave expansions in Sec. III into Fourier-Bessel integrals. The corresponding approximation in the K-matrix approach yields a rather intractable result: Even if all the elements of K are calculated in Born approximation, the resulting expression involves the factor $2iK[1-iK]^{-1}$, hence, integrals which contain several Bessel functions.

APPENDIX B: FOURIER-BESSEL REPRESENTATIONS FOR SCATTERING AMPLITUDES

We have indicated at the end of Appendix A how the Bessel function approximations for the rotation coefficients of the first and second kind can be used to approximate partial-wave expansions by Fourier-Bessel integrals. In the present Appendix, we will invert the procedure, and consider directly the Fourier-Bessel representation for scattering amplitudes, and the exact relation of this representation to the partial-wave expansions. The case of spinless particles has been considered in detail by Blankenbecler and Goldberger[23]; the reader is referred to that paper for discussions of the utility of the Fourier-Bessel representation in the high-energy limit, and of the analytic properties of the weight functions.

The basis of our treatment of the Fourier-Bessel representation is the assumption that the transition amplitude is of the form

$$M_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s,t)=\left(\frac{1-x}{2}\right)^{\frac{1}{2}|\lambda-\mu|}\times\left(\frac{1+x}{2}\right)^{\frac{1}{2}|\lambda+\mu|}A_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s,t),\quad\text{(B1)}$$

where the amplitude A may be a sum of terms which satisfy dispersion relations in the Mandelstam variables t and u (the properties of A as a function of s will not be considered in this paper).[60] We will consider only the t dependence; the u-dependent terms may be treated by similar methods. Thus, we will assume that

$$A_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s,t)=\int_{t_0}^{\infty}\tilde{a}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s,t')\frac{dt'}{t'-t}=\int_{z_0}^{\infty}a_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s,z)\frac{dz}{z-x},\quad\text{(B2)}$$

where z is given by Eq. (35) with $\mu^2=t'$. The partial-wave amplitudes may be calculated using Eq. (A12),

$$M_{\lambda_c\lambda_d;\lambda_a\lambda_b}{}^{j}(s)=\epsilon(\lambda,\mu)\int_{z_0}^{\infty}\left(\frac{z-1}{2}\right)^{\frac{1}{2}|\lambda-\mu|}\left(\frac{z+1}{2}\right)^{\frac{1}{2}|\lambda+\mu|}e_{\lambda\mu}{}^{j}(z)a_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s,z)dz.\quad\text{(B3)}$$

The signature factor $\epsilon(\lambda,\mu)$ is determined by the relative magnitudes and signs of λ and μ, and the symmetry properties of the rotation coefficients Eq. (A7).

A Fourier-Bessel representation for M is obtained by rewriting the denominator in Eq. (B2) using the integral representation

$$\frac{1}{z-x}=2\left(\frac{z-1}{1-x}\right)^{\nu/2}\times\int_0^{\infty}J_{\nu}([2(1-x)]^{1/2}u)K_{\nu}([2(z-1)]^{1/2}u)udu,\quad\text{(B4)}$$

and then inverting the order of the integrations. Introducing a weight function $h(s,u)$, and specializing to the index $\nu=|\lambda-\mu|$, we obtain the desired result,

$$M_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s,t)=2\left(\frac{1+x}{2}\right)^{\frac{1}{2}|\lambda+\mu|}\times\int_0^{\infty}J_{|\lambda-\mu|}([2(1-x)]^{1/2}u)h_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s,u)udu,\quad\text{(B5)}$$

with

$$h_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s,u)=\int_{z_0}^{\infty}\left(\frac{z-1}{2}\right)^{\frac{1}{2}|\lambda-\mu|}\times K_{|\lambda-\mu|}([2(z-1)]^{1/2}u)a_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s,z)dz.\quad\text{(B6)}$$

This result represents a generalization of the Blankenbecler-Goldberger[23] representation to the case of particles with spin. Alternative representations may be obtained by introducing the Fourier-Bessel integral for the denominator $(t'-t)$ rather than $(z-x)$, or by using Bessel functions of a different order. However, the order

[60] The analyticity of the helicity amplitudes in s is discussed, for example, by Y. Hara, Phys. Rev. **136**, B507 (1964).

we have used, $\nu=|\lambda-\mu|$, is certainly the most natural choice. The use of $(z-x)$ in the denominator also has the advantage from the point of view of the present paper of providing the closest connection between the Fourier-Bessel integral representation and the partial-wave expansion. On the other hand, the very complicated dependence of z on s [Eq. (66)] makes this choice of the denominator very awkward if one wishes to study the regions of analyticity of M or h considered as functions of s; the Mandelstam denominator $(t'-t)$ leads to much simpler results in this case.

The partial-wave amplitudes are readily estimated by identifying the Bessel function in Eqs. (B5) with the rotation coefficient $d_{\lambda\mu}{}^{j}(x)$ in the small angle approximation, Eq. (A23).

$$M_{\lambda_c\lambda_d;\lambda_a\lambda_b}{}^{j}(s)\approx(-1)^{|\lambda-\mu|}\times\epsilon(\lambda,\mu)h_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s,[j(j+1)]^{1/2}). \quad \text{(B7)}$$

The exact partial-wave amplitudes given in Eq. (B3) can also be expressed in terms of h. In this calculation, one encounters the integral

$$\frac{1}{2}\int_{-1}^{1}\left(\frac{1+x}{2}\right)^{\frac{1}{2}(m'+m)}J_{m'-m}([2(1-x)]^{1/2}u)d_{m'm}{}^{j}(x)dx = (-1)^{m'-m}\,2^{\frac{1}{2}(m'+m)}\left[\frac{(j+m')!(j+m)!}{(j-m')!(j-m)!}\right]^{1/2}u^{-m'-m-1}J_{2j+1}(2u), \quad \text{(B8)}$$

which may be evaluated by expanding the Bessel function in its Taylor series, introducing the Rodrigues formula for the rotation coefficients,[56]

$$\left(\frac{1-x}{2}\right)^{\frac{1}{2}(m'-m)}\left(\frac{1+x}{2}\right)^{\frac{1}{2}(m'+m)}d_{m'm}{}^{j}(x) = (-1)^{j-m}\,2^{-j}\left[\frac{(j+m')!}{(j+m)!(j-m)!(j-m')!}\right]^{1/2}\left(\frac{d}{dx}\right)^{j-m'}[(1-x)^{j-m}(1+x)^{j+m}], \quad \text{(B9)}$$

and integrating term-by-term. Using Eq. (B8), we obtain the exact result

$$M_{\lambda_c\lambda_d,\lambda_a\lambda_b}{}^{j}(s)=(-1)^{m'-m}\epsilon(\lambda,\mu)2^{\frac{1}{2}(m'+m+2)}\left[\frac{(j+m')!(j+m)!}{(j-m')!(j-m)!}\right]^{1/2}\int_0^\infty J_{2j+1}(2u)h_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s,u)u^{-m'-m}du, \quad \text{(B10)}$$

$$m'=\tfrac{1}{2}|\lambda+\mu|+\tfrac{1}{2}|\lambda-\mu|\,,\quad m=\tfrac{1}{2}|\lambda+\mu|-\tfrac{1}{2}|\lambda-\mu|\,.$$

The approximate expression in Eq. (B7) is recovered in the large j limit: The main contributions to the integral then arise from values of u near $(j+\frac{1}{2})$.