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Phase-Shift Analysis of Elastic Pion-Nucleon Scattering*

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A many-energy analysis of pion-nucleon scattering data in the energy range 20–340 MeV has been carried out by fitting phase shifts in S, P, D, and F states in gradient searches. Semiphenomenological analyses, with F-wave phase shifts calculated from approximate evaluations of pion-nucleon dispersion relations and D-wave phase shifts similarly obtained for the lower energy range, have been equally successful in fitting the data. Particular attention has been paid to the possibility of obtaining a unique set from all possible sets allowed by data. Several sets of phase shifts have been found. One of these, called SEMI IIM, is preferred by data and by comparison with dispersion-theoretical calculations. Comparisons with other recent analyses are made.

I. INTRODUCTION

THE ambiguities in determining phase shifts in scattering by analysis of data in a restricted range of energies have been successfully reduced by employing a many-energy treatment in the nucleon-nucleon case.[1] In both the Yale[2] and Livermore[3] analyses, contributions to the scattering matrix from higher angular-momentum states were obtained from the one-pion-exchange part of the interaction, so that only phase parameters of lower angular momentum ($l \leq 4$) were varied in the phenomenological searches. Many-energy pion-nucleon phase-shift analyses were undertaken before those for the two-nucleon case by Fermi *et al.*,[4] de Hoffmann *et al.*,[5] Bethe and de Hoffmann,[6] Orear,[7] Anderson *et al.*,[8] and Chiu and Lomon.[9] These investigations assumed that only S- and P-wave phase shifts were required, whereas more recent discussions by Yamanouchi and Fukuda,[10] Clementel and Villi,[11] Goodwin *et al.*[12] have indicated the importance of D waves, while Vik and Rugge[13] have definitely established their occurrence in a careful phase-shift analysis at 310 MeV. In each of these cases, several sets of phase shifts resulted from among which a single fit could not clearly be selected.

These observations, together with the recognition that a number of data had accumulated in recent years which had not all been available to earlier workers, suggested to the authors the possible usefulness of an analysis modeled on the Yale nucleon-nucleon calculation.[1,2] The features of this analysis to be employed were (1) the use of a large fraction of published data on pion-nucleon scattering in the whole elastic range of energies (data omissions will be noted later), (2) searching for the best set of phase shifts for lower angular-momentum states, (3) the employment of phase shifts in higher angular-momentum states as calculated from some model of the interaction expected on theoretical grounds to have a direct relevance for the problem.

Chew, Goldberger, Low, and Nambu (CGLN)[14] treated pion-nucleon scattering in terms of fixed momentum-transfer dispersion relations, and assumed that the imaginary part of the amplitudes was saturated by the $T=\frac{3}{2}$, $J=\frac{3}{2}$ resonance which had been one of the outstanding experimental features of the early investigations. Bowcock, Cottingham, and Lurié (BCL)[15] subsequently showed that effects of the $T=1$, $J=1$ resonance in the pion-pion system could also be incorporated in the pion-nucleon scattering matrix by an approximate treatment of the Mandelstam representation according to Cini and Fubini,[16] wherein only singularities close to the physically interesting ranges of the variables are taken into account. The BCL treatment will be used to provide the phase shifts in higher angular-momentum states for the semiphenomenological analyses to be described below.

While this work was in progress, a phenomenological analysis employing searches for S, P, D, and F waves, with inelastic contributions, was carried out by Roper[17]

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in the energy range up to 700 MeV and, after the present calculations were finished, Donnachie *et al.*[18] published predictions for P, D, F waves based on the detailed analysis of various parts of the pion-nucleon interaction with dispersion relations carried out by Hamilton *et al.*[19] On comparing this work with Roper's, one can say that in the common energy range one of the present families of phase shifts (namely, Family II) is in essential agreement with his. One set (SEMI IIM) is preferred by data and by comparison with dispersion-theoretical calculations, and further experiments which will be useful in confirming this selection among the various sets obtained will be pointed out. A further result of this analysis is the demonstration that the data allow the treatment of higher angular-momentum states by the method which has been employed, and since this reduces the number of searched phase shifts, their statistical uncertainties are relatively smaller. Finally, it is interesting to note that the relatively simple treatment of dispersion relations employed here leads to essentially the same D and F waves as the more elaborate work of Donnachie *et al.*

II. TREATMENT OF HIGHER ANGULAR-MOMENTUM STATES

In order to establish the notation and sketch the essentials of the approximations, it will be necessary to reproduce some of the results of Bowcock, Cottingham, and Lurié and of Chew *et al.* Lorentz invariance and isotopic spin symmetry require that one deal with four amplitudes, $A^{(\pm)}(s,t)$ and $B^{(\pm)}(s,t)$, where the variables in the zero-momentum pion-nucleon frame are

$$\begin{aligned} s&=-(q_1+p_1)^2=(E_p+\omega_q)^2\,,\\ t&=-(q_1+q_2)^2=-2\bar{q}^2(1-\cos\theta)\,,\\ \bar{s}&=-(q_1+p_2)^2=(E_p-\omega_q)^2-2\bar{q}^2(1+\cos\theta)\,. \end{aligned}\tag{1}$$

Here q_1, q_2 are the pion four-momenta, p_1, p_2 those of the nucleon, E_p, ω_q the nucleon and pion energies, respectively, and θ is the scattering angle. The variables are not all independent, since they satisfy

$$s+\bar{s}+t=2(M^2+1)\,,\tag{2}$$

where M is the nucleon mass in units of the pion mass, $\mu=1$. For the pion-pion channel, the "crossed channel" in the Mandelstam picture, the variables are given by

$$\begin{aligned} s&=-p^2-q^2+2pq\cos\varphi\,,\\ t&=4(q^2+\mu^2)=4(p^2+M^2)\,,\\ \bar{s}&=-p^2-q^2-2pq\cos\varphi\,, \end{aligned}\tag{3}$$

where φ is the relevant angle in this system.

The BCL expressions for the amplitudes, in the Cini-Fubini approximation, are

$$A^{(\pm)}(s,t)=\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\alpha_{\rm el}^{(\pm)}(s,t)\left\{\frac{1}{s'-s}\pm\frac{1}{s'-\bar{s}}\right\}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{a^{(\pm)}(t',s,\bar{s})}{t'-t}dt'+C_A^{(+)}\,,$$

$$B^{(\pm)}(s,t)=g_r^2\left(\frac{1}{M^2-s}\pm\frac{1}{M^2-\bar{s}}\right)+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\beta_{\rm el}^{(\pm)}(s',t)\left(\frac{1}{s'-s}\mp\frac{1}{s'-\bar{s}}\right)+\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{b^{(\pm)}(t',s,\bar{s})}{t'-t}dt+C_B^{(-)}\,.\tag{4}$$

In these equations, $\alpha_{\rm el}^{(\pm)}$ represent the elastic parts of the once-integrated spectral functions composing the amplitude $A(s,t)$, and similarly for $\beta_{\rm el}^{(\pm)}$ in $B(s,t)$. In the same manner, $a^{(\pm)}$ and $b^{(\pm)}$ come from the inelastic contribution of the spectral functions. The quantity g_r^2 is the renormalized pion-nucleon coupling constant. The constants $C_A^{(+)}$ and $C_B^{(-)}$ are supposed to contain effects of singularities in the spectral functions far away from the physical region of energies and to be determined by fitting to experiment. If they were truly constants, $C_A^{(-)}=C_B^{(+)}$ would be zero to satisfy crossing symmetry. Since they make no contribution to phase shifts for $l>1$, they are of no further interest. The extra term in $B^{\pm}(s,t)$ of Eq. (4) is the nucleon pole term. The $\alpha_{\rm el}$, $\beta_{\rm el}$ terms contain effects from the physical range of the s, $\bar{s}$ variables and thus are equivalent to the CGLN dispersion relations, as is readily verified. The effects of the pion-pion channel are correspondingly to be found in the a, b terms.

Before discussing the further approximations in Eqs. (4) which will be employed, the connection of these amplitudes with the pion-nucleon phase shifts may be indicated. This is established through the scattering amplitude $f(\theta)$, which, in turn, is considered in terms of its phase-shift expansion. Labeling by the subscript $l\pm$ a phase shift in the partial wave with angular momentum $J=l\pm\frac{1}{2}$, one defines

$$f_{l\pm}=e^{i\delta_{l\pm}}\sin\delta_{l\pm}/q\sim\delta_{l\pm}/q\,.\tag{5}$$

The states of total isotopic spin are $\frac{1}{2}$ and $\frac{3}{2}$, and if these are used as superscripts for identification, the phase shifts

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can be obtained from Eq. (5) and

$$
\begin{aligned}
f_{l\pm}{}^{1/2}=&\frac{E_p+M}{8\pi W}\{A_l{}^{(+)}+2A_l{}^{(-)}+(W-M)(B_l{}^{(+)}+2B_l{}^{(-)})\}\\
&+\frac{E_p-M}{8\pi W}\{-(A_{l\pm}{}^{(+)}+2A_{l\pm}{}^{(-)})+(W+M)(B_{l\pm}{}^{(+)}+2B_{l\pm}{}^{(-)})\}\,,\\
f_{l\pm}{}^{3/2}=&\frac{E_p+M}{8\pi W}\{A_l{}^{(+)}-A_l{}^{(-)}+(W-M)(B_l{}^{(+)}-B_l{}^{(-)})\}\\
&+\frac{E_p-M}{8\pi W}\{A_{l\pm}{}^{(-)}-A_{l\pm}{}^{(+)}+(W+M)(B_{l\pm}{}^{(+)}-B_{l\pm}{}^{(-)})\}\,.
\end{aligned}
\tag{6}
$$

In these formulas, W is the total energy in the zero-momentum frame, and thus is equal to $s^{1/2}$, while E_p is the nucleon energy in the same frame. The $A_l{}^{(\pm)}$, $B_l{}^{(\pm)}$, etc., are angular-momentum projections of the amplitudes:

$$
\begin{aligned}
A_l{}^{(\pm)}&=\frac{1}{2}\int_{-1}^{1}A^{(\pm)}(s,t)P_l(\cos\theta)d(\cos\theta)\,,\\
A_{l\pm}{}^{(\pm)}&=\frac{1}{2}\int_{-1}^{1}A^{(\pm)}(s,t)P_{l\pm1}(\cos\theta)d(\cos\theta)\,,
\end{aligned}
\tag{7}
$$

and similarly for $B_l{}^{(\pm)}$ and $B_{l\pm}{}^{(\pm)}$.

In the physical region of elastic pion-nucleon scattering, $\alpha_{\mathrm{el}}{}^{(\pm)}=\mathrm{Im}A^{(\pm)}$, $\beta_{\mathrm{el}}{}^{(\pm)}=\mathrm{Im}B^{(\pm)}$, respectively. Following CGLN, it is assumed that these terms are dominated by the $T=\frac{3}{2}$, $J=\frac{3}{2}$ resonance, so that the partial wave f_{33} is the only one of consequence. Then

$$
\begin{aligned}
\alpha_{\mathrm{el}}{}^{(\pm)}&=\begin{pmatrix}2\\-1\end{pmatrix}\frac{4\pi}{3}(\mathrm{Im}f_{33})\\
&\quad\times\left[\frac{3(W'+M)}{E_p'+M}\cos\theta'+\frac{W'-M}{E_p'-M}\right],\\
\beta_{\mathrm{el}}{}^{(\pm)}&=\begin{pmatrix}2\\-1\end{pmatrix}\frac{4\pi}{3}(\mathrm{Im}f_{33})\left[\frac{3\cos\theta'}{E_p'+M}-\frac{1}{E_p'-M}\right],
\end{aligned}
\tag{8}
$$

where the factor 2 applies for $\alpha_{\mathrm{el}}{}^{(+)}$, $\beta_{\mathrm{el}}{}^{(+)}$ and the factor -1 for $\alpha_{\mathrm{el}}{}^{(-)}$, $\beta_{\mathrm{el}}{}^{(-)}$, respectively. The first terms of Eqs. (4) are then evaluated by expanding $1/(s'-s)$ in powers of $\cos\theta$, inserting Eqs. (8) and applying Eqs. (7). The result is a set of expansions, for arbitrary l, with known coefficients and integrals of the form

$$
\int\frac{a(s')\,\mathrm{Im}f_{33}(s')ds'}{[2q'^2+2(E_p'\omega_q'-E_p\omega_q)]^{n+1}}\,.
$$

Appendix I contains the complete series.

If the sharp resonance assumption of CGLN is employed, the results for $f_l{}^{1/2,3/2}$ and $f_{l\pm}{}^{1/2,3/2}$ obtained by CGLN for the P and D waves are reproduced. The pole terms in $B^{(\pm)}$ yield similar expansions, except there are no integrals on ds'. In the present work, the integrals are evaluated numerically employing a Breit-Wigner form for the resonance:

$$
e^{i\delta_{33}{}^1}\sin\delta_{33}{}^1=\Gamma/[(s_r-s')-i\Gamma]\,,
\tag{9}
$$

where $\Gamma=|q'|^3\gamma_{\pi N}$ is the half-width with the partial width $\gamma_{\pi N}$ and s_r is the square of the total energy of the resonance. Experimental values of the parameters s_r, $\gamma_{\pi N}$ will be used. The phase-shift notation is $\delta_{2T,2J}{}^l$, where l is the orbital and J the total angular momentum, and T the isotopic spin.

For the pion-pion contribution to Eqs. (4), BCL have made similar assumptions; namely, that the $J=1$, $T=1$ resonance dominates the integral. With this assumption, there are contributions only to $A^{(-)}$ and $B^{(-)}$. They use the helicity amplitudes of Jacob and Wick,[20] and also assume a Breit-Wigner form for the resonance:

$$
f_\pm{}^1=N_\pm/[(t_r-t)-i\gamma_{\pi\pi}q^3]\,,
\tag{10}
$$

where the subscripts $\pm$ refer to the helicities, t_r is the square of the energy at resonance, and $\gamma_{\pi\pi}$ is the partial width in the $\pi\pi$ channel. The possibility of adjusting t_r and $\gamma_{\pi\pi}$ is investigated, but the initial values are those of the ρ meson. The coefficients $N_\pm$ are related by BCL to the electromagnetic form factors of the nucleon via the theory of Frazer and Fulco[21]; the sensitivity of our results to variations in the constants will be considered, especially in view of questions concerning the completeness of the Frazer-Fulco treatment,[22] but the initial values will be adjusted in this way. The functions appearing in the last terms of Eq. (4) become, under these assumptions,

$$
\begin{aligned}
a^{(-)}&=(s-\bar{s})(3\pi/p'^2)\\
&\quad\times\{(M/\sqrt{2})\,\mathrm{Im}f_-{}^1(t')-\mathrm{Im}f_+{}^1(t')\}\,,\\
b^{(-)}&=(12\pi/\sqrt{2})\,\mathrm{Im}f_-{}^1(t')\,.
\end{aligned}
\tag{11}
$$

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TABLE I. Pion-nucleon phase shifts calculated with dispersion relations, in degrees.

State / E, MeV	D_{13}	D_{15}	D_{33}	D_{35}	F_{15}	F_{17}	F_{35}	F_{37}
100	0.0016	0.1049	−0.0047	−0.2694	0.0065	−0.0080	−0.0041	0.0157
200	0.1333	0.4001	−0.1582	−1.0709	0.0694	−0.0552	−0.0365	0.1069
310	0.5931	0.8880	−0.6530	−2.3932	0.2933	−0.1739	−0.1406	0.3275

Projections of angular-momentum states are again carried out with the aid of Eqs. (7), and series result with terms involving known coefficients and integrals of the form

$$\int \frac{f_{\pm}(t')}{(t'+2\mathbf{q}^2)^{n+1}}dt'.$$

The BCL results for the states they calculated are reproduced if the sharp resonance assumption is made; in this work the integrations were performed numerically. Appendix I gives the series for these contributions.

In the semiphenomenological searches to be described, phase shifts for $l=2$ to $l=9$ were obtained from the expansions described. For the $l=9$ states, only the first four terms were used, and for states of lower l all terms up to the order of this term were retained. The choice of $l=9$ to cut off the higher angular-momentum states was made on the basis of experience gained in the nucleon-nucleon problem,[1,2] where phase parameters smaller than 0.00005 rad could be safely neglected. Representative D and F wave phase shifts computed from the formulas of Appendix I are shown for a few energies in Table I.

III. DATA SELECTION

The three cases for which there are experimental data available are

$$\pi^+ + p \rightarrow \pi^+ + p, \tag{12a}$$

$$\pi^- + p \rightarrow \pi^- + p, \tag{12b}$$

$$\pi^- + p \rightarrow \pi^0 + n. \tag{12c}$$

Since charged particles are involved, Coulomb-scattering corrections have been incorporated in the phase-shift expansions of the scattering matrix (here $\boldsymbol{\sigma}$ is the nucleon spin operator and $\mathbf{n}$ is a unit vector normal to the scattering plane).

$$S=g(\theta)+h(\theta)(\boldsymbol{\sigma}\cdot\mathbf{n}), \tag{13}$$

in the usual way.[23] A specific formula appropriate directly for $\pi^+ + p$ elastic scattering is given by Breit and McIntosh.[24] Suitable modifications are introduced to take account of attraction instead of repulsion in cases (12b), (12c), the fact that the Coulomb force only acts in the incident channel for (12c), and to provide for the different isotopic-spin states occurring in the pion-nucleon system.

All data available in the literature up to July 1964 in the range 20- to 333-MeV pion laboratory energy which satisfied certain criteria were employed in the searches. The criteria were simply that cross section or recoil-nucleon polarization data be given directly in tabular form with assigned experimental errors, and be the latest values from a given author at a given angle and energy where duplication occurred. Data published in the form of scattering lengths, parametrized angular distributions, or without experimental errors were omitted. If errors were given as angular-dependent and angular-independent parts, respectively, they were added in quadrature.

Some 620 measurements were used, of which 244 were type (12a), 287 of type (12b) and 89 of type (12c). These include measurements of the polarization of the recoil nucleon, of which there are one of type (12c),[25] 9 of type (12b),[26] and 13 of type (12a).[27] The recent polarization measurements of Schultz *et al.*[28] performed with polarized targets have special bearing in selection among various sets and will be discussed separately. The possibility of systematic experimental errors of the level type which could unduly influence the search has been met by allowing the theoretical value of the observable to be multiplied by a factor adjusted to fit the data best at the energy in question. Effects of this type of adjustment have been studied recently in the nucleon-nucleon analysis.[29] In the present case, the

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[29] G. Breit, A. N. Christakis, M. H. Hull, Jr., H. M. Ruppel, and R. E. Seamon, in *Proceedings of the 12th Annual International Conference on High-Energy Physics, Dubna, 1964* (Moscow, 1965); G. Breit, A. N. Christakis, M. H. Hull, Jr., H. M. Ruppel, and R. E. Seamon, Bull. Am. Phys. Soc. **9**, 378 (1964). The special case corresponding to no measured value of the normalization constant has been used in the present work.

factor has been used on only four sets of data which satisfied the following criteria: During initial fitting efforts the data stood out as contributing unusually to the total error compared to contributions at neighboring energies, the differences from calculated values appeared to be angular independent for the set, and plots of data at several constant angles for a range of energies were studied for obvious deviations. The data of Ashkin *et al.*[30] at 150 and 170 MeV and of Goodwin *et al.*[12] at 230 and 290 MeV appeared, under these conditions, to require use of the factor. Since other data are available at the same or nearby energies, the effect of special treatment in these cases should not be large.

IV. PHASE-SHIFT SEARCHES

A. Many-Energy Searches

The method of the Yale nucleon-nucleon searches[2] has been employed for these analyses. A quantity D, giving a measure of the quality of fit to data, is minimized by calculating its gradient in the space of parameters employed to expand a function of the energy which provides a change of an input set of phase shifts. Here

$$D=\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\eta_i-y_i}{\Delta y_i}\right)^2, \tag{14}$$

where N is the total number of data (620), η_i the calculated value of the observable at given energy and angle, y_i its experimental value, and Δy_i the experimental uncertainty. The variation of D along the gradient direction is obtained and its minimum fitted parabolically. A new gradient is calculated at the minimum point, with the changed phase shifts corresponding to the minimum value acting as a new intermediate input set. This procedure is continued until successive gradients produce relatively insignificant improvements in D (an improvement of order $1/N$ is considered for statistical reasons to be a minimum useful change).

The starting sets employed all depend on the results of the careful analysis of the 310-MeV data by Rugge and Vik.[13] In their work, three satisfactory sets of phase shifts for S, P, D, F states were obtained and are given in their Table XII numbered in increasing order of D value: I, II, III. We shall refer to these as RV I, RV II, and RV III, respectively.

The initial searches were phenomenological, in which 14 phase shifts were allowed to vary in the method discussed. These will be called PHEN I, PHEN II, and PHEN III, respectively.

The starting set of phase shifts for the fit PHEN I was obtained by fitting the constant in an assumed low-energy behavior; namely, $\delta_l=\text{const}\times q^{2l+1}$, to the values determined by Orear,[7] Chiu and Lomon.[9] These were extrapolated to the RV I fit at 310 MeV in a smooth curve so that the causality conditions of Wigner[31] and Breit[32] were satisfied. The initial and final D values are shown in Table II. It is of special interest to note that for the state S_{11} (the notation for the state is $l_{2T,2J}$, with l being represented by its spectroscopic designation), the low-energy fits require positive values with a maximum of nearly 12° while RV I calls for a negative value at 310 MeV of $-6°$. The gradient searches carried out in this study confirmed the positive trend of the S_{11} phase shift at the lower energies, and produced an increase of as much as 1° at 80 MeV. At higher energies, however, the gradient searches carried the phase shift to smaller values, reducing it by nearly 9° at 240 MeV. A very large negative slope in the resulting graph of the phase shift versus energy in the vicinity of 220-MeV results, and the causality condition is violated. Attempts at rectifying this difficulty by adjusting other phase shifts along lines suggested by theoretical considerations but limited by requirements of fitting the data failed. One may conclude that, although the data at high energies allow a negative phase shift in the S_{11} state, such values cannot be made consistent with the apparent requirement that it be positive at low and intermediate energies. Further work with modifications of PHEN I, to be described later, led to other difficulties, in confirmation of this conclusion.

TABLE II. Initial and final D values of the searches.

Search	D initial	D final
PHEN I	9.7	2.4
PHEN II	2.9	2.2
PHEN III	5.3	3.1
SEMI I	3.3	2.8
SEMI II	2.5	2.4
SEMI III	3.4	3.0
SEMI ISP	11.1	no improvement
SEMI IISP	6.2	5.0
SEMI IIISP	6.5	3.9
SEMI IIM	5.0	2.4
SEMI IIISPM	3.9	2.7

When the low-energy behavior of the phase shifts of PHEN I could be smoothly joined to values at 310 MeV for sets RV II and RV III, this was done to utilize improvements in fits already achieved in constructing starting sets for PHEN II and PHEN III. The phase shifts for which this was possible were for states P_{33}, F_{35}, and F_{17} in RV II, and P_{33}, D_{35}, F_{35}, F_{37}, D_{15}, F_{17} in RV III. Otherwise, the same procedure employed for obtaining the starting set for PHEN I was followed. In particular, the problem with the phase shift in state S_{11} did not arise in extrapolating to the RV II and RV III values at 310 MeV since they are positive (10.9° and 5.0°, respectively). The relevant D values recorded in Table II, where PHEN II is seen to be the best fit obtained.

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[31] L. Eisenbud, thesis, Princeton University, 1948 (unpublished); E. P. Wigner, Phys. Rev. **98**, 145 (1955).

[32] G. Breit, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1959), Vol. 41, Part 1.

In the three cases just discussed, the F waves played an essential part in fitting the data at higher energies. This is consistent with the results of Rugge and Vik.[13] However, a principal purpose of this investigation was to see to what extent the higher angular-momentum states could be included by direct calculation on the assumptions outlined in Sec. II. Consequently, these semiphenomenological searches were undertaken, called SEMI I, SEMI II, SEMI III, respectively. The starting phase consisted of the final ones of sets PHEN I, II, III for S, P, D states. The F waves were replaced by values calculated with series discussed in Sec. II, and, in addition, phase shifts similarly obtained were included up to $l=9$. The initial D values for these searches shows the effect of this replacement (cf. Table II), and the final D values after searching S, P, D waves, show that fits essentially as good as the phenomenological ones were achieved. The values of theoretically calculated F waves range from the same order to ten times smaller than phenomenological ones at 310 MeV, and only in case PHEN II are the signs all the same.

A similar set of searches was made with the D waves also replaced by theoretical values and only S, P waves searched. The final phase shifts of PHEN I, II, III were used for starting values of S, P waves for these searches, denoted by SEMI ISP, SEMI IISP, SEMI IIISP, respectively. The D phase shifts of PHEN II have the same sign as the theoretical values at high energies. Two of the D phase shifts of this set are similar in size to the theoretical ones. The D wave replacements to produce the starting sets for SEMI ISP and SEMI IIISP are relatively drastic; both sign changes at high energies and changes in size by an order of magnitude are necessary. The data fit most affected by these replacements was SEMI I: It had a final D of 2.8 while SEMI ISP began with 11.1, and a reduced value could not be obtained by the procedures employed. SEMI IISP and SEMI IIISP could be improved by searching, but not sufficiently to suggest that all the theoretical S waves can be accommodated by the data in a satisfactory fit. It is significant in assessing this result that the principal contribution to the high D value comes from errors in fitting the 310-MeV data, where about 10% of the data lead to about 50% of the error. At least at high energies the inescapable conclusion appears to be that at least some of the D waves require a substantial phenomenological part. In the Family II searches, only D_{13} requires the sizeable phenomenological addition.

However, the evidence for this conclusion at lower energies is not so strong. The possibility that in the first third of the energy range investigated theoretical D waves could be used was only partially tested by the SEMI SP searches. Consequently, two further cases were constructed, with somewhat different parentage in order to allow for more varied treatment. The D waves of PHEN II were replaced up to 120 MeV by theoretical values and extrapolated smoothly up to the phenomenological values at 310 MeV. This caused major modifica-

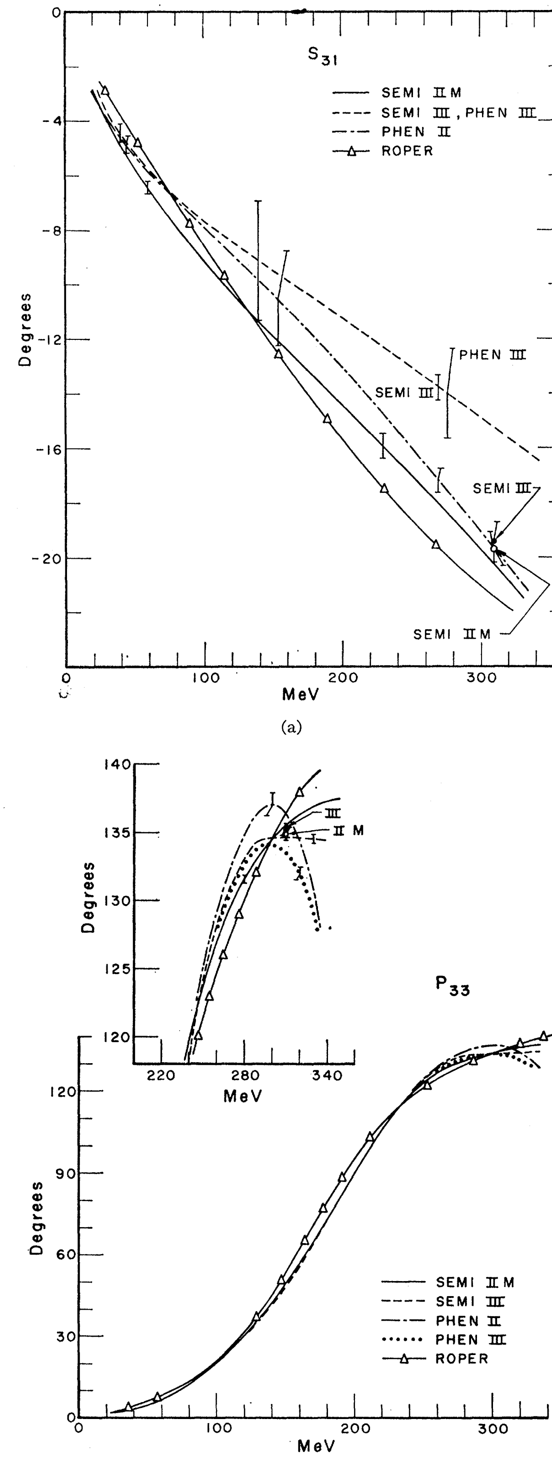
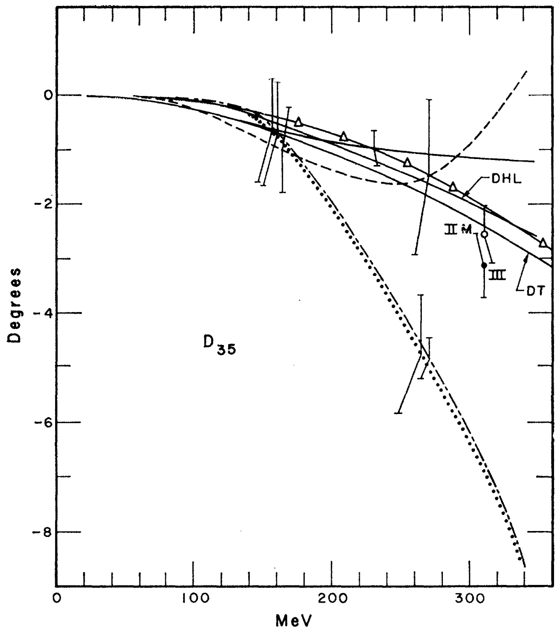
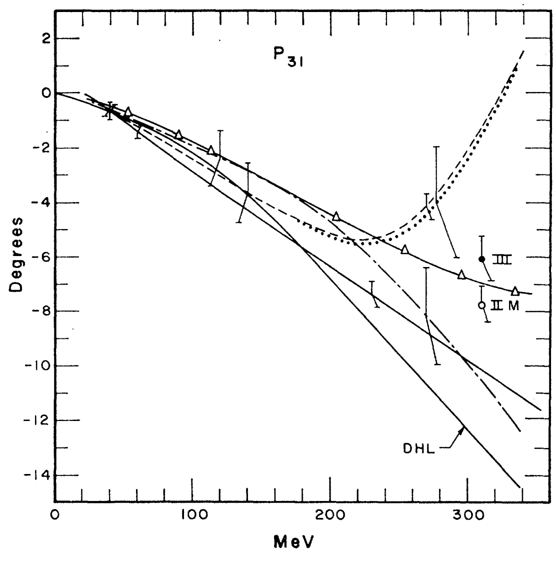


FIG. 1. (a) Phase shift S_{31} for family II, family III, and Roper's solution plotted against energy. The statistical uncertainty of the determination is indicated by error bars in this and other cases. Results of single-energy searches at 310 MeV for SEMI III and SEMI IIM are also shown. (b) Phase shift P_{33} for family II, family III, and Roper's solution plotted against energy. The statistical uncertainty and results of single-energy searches at 310 MeV for SEMI III and SEMI IIM are shown in the inset on enlarged scale at the upper left.



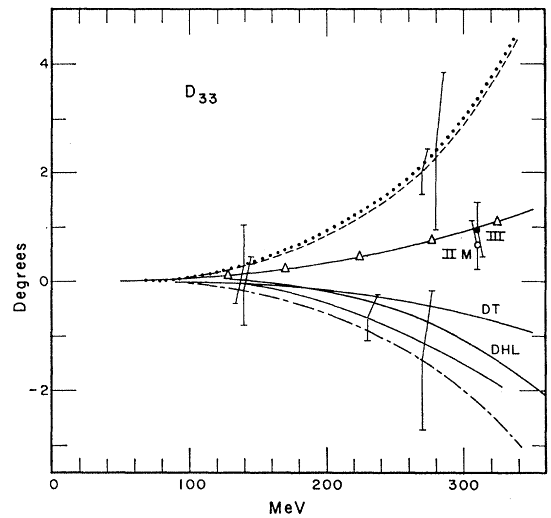
(a)



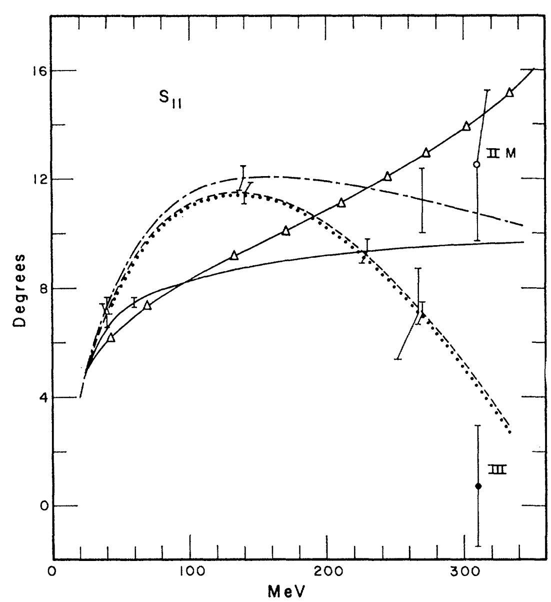
(b)

FIG. 2. (a) Phase shift D_{35} with error bars plotted against energy. Dispersion-theoretically calculated values obtained in this work (DT) and the solution of Ref. 18 (DHL) are also shown. Different sets of phase shifts are designated as in Fig. 1(b). Results of single-energy searches are labeled IIM and III for SEMI IIM and SEMI III, respectively. (b) Phase shift P_{31} with error bars plotted against energy. The solution of Ref. 18 (DHL) is also shown. Different sets of phase shifts are designated as in Fig. 1(b). Results of single-energy searches are labeled IIM and III for SEMI IIM and SEMI III, respectively.

tions in the phase shifts for states D_{13}, D_{15}, nontrivial ones for D_{35}, while D_{33} was relatively unchanged. It has been noted that all the D waves obtained in the PHEN II search have phase shifts with the same signs at high energies (as do the F waves) as theoretical values, which allows the extrapolations to be made in a relatively simple manner. Sign changes *are* introduced at intermeditate energies, however: Thus D_{15} in PHEN II is negative between 40 and 260 MeV, but is always positive in the adjustment discussed. The results of those modifications on the fit to data is to raise the final D value of PHEN II, 2.2, to the initial D value of SEMI IIM, namely, 5.0. After searching, the quite satisfactory D value of 2.4 was reached, and in comparison with the parent phenomenological case, PHEN



(a)



(b)

FIG. 3. (a) Phase shift D_{33} with error bars plotted against energy. All designations are as in Fig. 2(a). Results of single-energy searches are shown for the semiphenomenological fits and labeled IIM and III, accordingly. (b) Phase shift S_{11} with error bars plotted against energy. All designations are as in Fig. 1(b). Results of single-energy searches are displayed for the semiphenomenological fits and labeled IIM and III, accordingly.

II, it is notable that there were only 6 phase shifts searched below 120 MeV and 10 above. For the phenomenological cases, there are 6 searched below 80 MeV, 10 between 80 and 200 MeV, and 14 above 200 MeV.

The results of the SEMI IIM search are shown in Figs. 1–5 inclusive, where the error bars represent uncertainties calculated by the methods employed in the nucleon-nucleon fits[1]; i.e., energy regions are treated as a whole in estimating a common uncertainty for the region by means of the error matrix. The theoretical D waves are shown for comparison (Table I), and it is interesting to note that only D_{13} has a phase shift differing significantly from the theoretical values.

The starting point for SEMI IIISPM was, as the designation implies, the end of SEMI IIISP. Since the D waves were already theoretically calculated, further searching for this case involved releasing them for energies above 120 MeV (the change functions used in the searches are adjusted in such cases to go to zero smoothly at the cutoff energy). The same number of searched parameters is employed as in SEMI IIM. The D waves phase shifts of SEMI III are of opposite sign in the states D_{33} and D_{15} compared to the theoretical values. In this search, the movement in these states as the phase shifts were released above 120 MeV was again toward their SEMI III values, although the minimum value of D was attained before the old values were reached. This suggests essential incompatibility

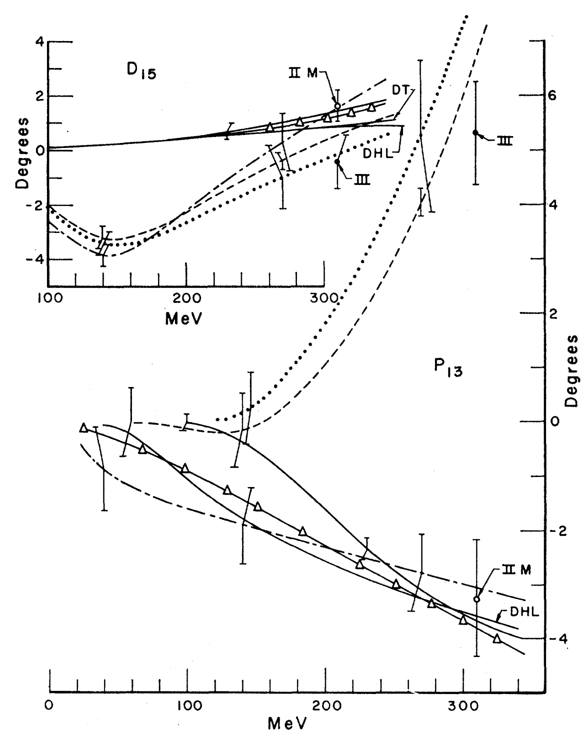


FIG. 4. Phase shifts D_{15} and P_{13} with error bars plotted against energy. All designations are as in Fig. 2(a). Results of single-energy searches are shown for the semiphenomenological fits and labeled IIM and III, accordingly.

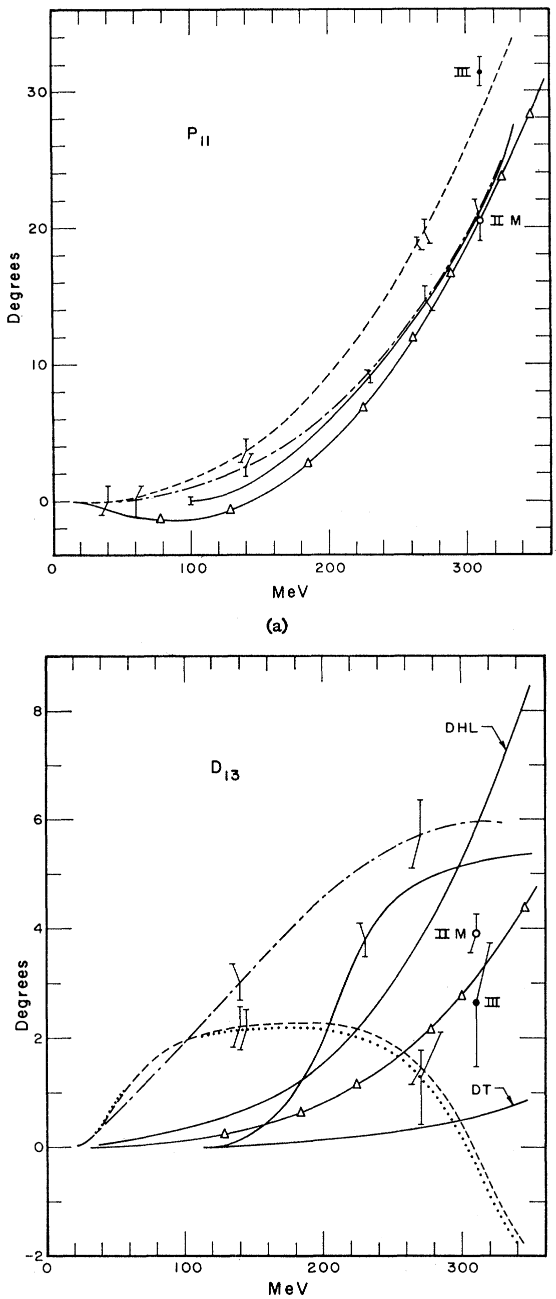


FIG. 5. (a) Phase shift P_{11} with error bars plotted against energy. All designations are as in Fig. 1(a). Results of single-energy searches are displayed for the semiphenomenological fits and labeled IIM and III, accordingly. (b) Phase shift D_{13} with error bars plotted against energy. All designations are as in Fig. 2(a). Results of single-energy searches are shown for the semiphenomenological fits and labeled IIM and III, accordingly.

of the Family III fits with these two theoretical D waves. Thus, a decision against SEMI III on the basis of the data would constitute some support for the theoretical calculations. Statistically, however, the data appear not quite able to make the selection. While SEMI III is

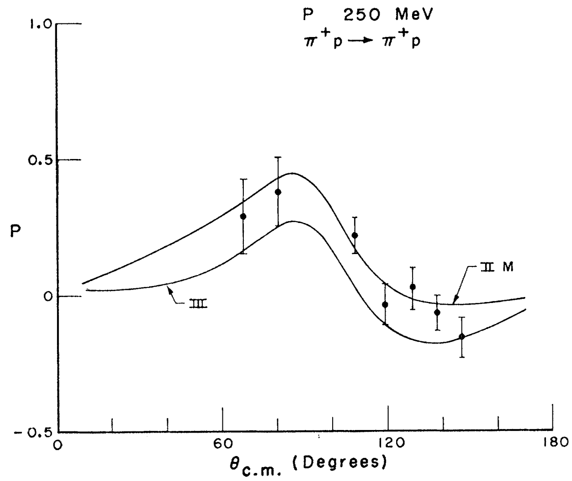


FIG. 6. Polarization at 250 MeV as predicted by the semiphenomenological solutions IIM and III. The experimental points are those of Ref. 28.

statistically a poorer fit than SEMI IIM (cf. Table II), the difference does not justify rejection of Family III fits on this basis alone. However, the recent polarization measurements of Schultz *et al.*[28] for positive pions scattered on protons at 250 MeV using polarized targets yielded seven pieces of data in a wide angular range. The experimental points and the corresponding theoretical predictions using phase shift sets of SEMI IIM and SEMI III are shown in Fig. 6. It should be emphasized that these fits of the polarization data have been obtained without further searching. The local normalized D values are 0.80 and 3.27 for SEMI IIM and SEMI III, respectively. This constitutes some evidence for preferring Family II solutions over those of Family III. These polarization data can also be fitted by the set PHEN II without further searching. It will be useful to obtain similar polarization measurements at this energy for the $T=\frac{1}{2}$ states by means of, for example, the scattering of negative pions on polarized hydrogen targets. Since the phase shifts of the various sets differ more in the $T=\frac{1}{2}$ states than the $T=\frac{3}{2}$ states, such data will be of value in selecting a unique set. In comparing the two families, especially in the states for $T=\frac{1}{2}$, Family III appears quite distinct from Family II fits, as Figs. 1–5 show, and the failure of even a "managed" search like SEMI IIISPM to bring them together is evidence that one is dealing with distinct fits.

Another test of the distinctness of the fits is to carry out the ξ variation suggested for the nucleon-nucleon searches by Breit.[2] In this test, the phase shifts of one set are smoothly changed into those of another by allowing a parameter ξ in the relation

$$\delta(\xi)=\delta(\mathrm{I})+\xi[\delta(\mathrm{II})-\delta(\mathrm{I})]$$

to vary between 0 and 1. The values of D for each value of ξ is plotted to exhibit an intermediate minimum or maximum, or a monotonic decrease or increase from one set to the other. Should an intermediate minimum occur, one may conclude that there exists a path in phase-shift space along which each set can change to a unique set. Should D vary monotonically between sets, one may conclude that the two sets are related, and that the one with lowest D represents the desired fit. For the cases at hand, the graph of D versus ξ goes through a maximum between each pair of solutions formed from PHEN I, II, and III. Since the dominant lower partial waves are the same as in these cases for SEMI I, II, III, respectively, the same result is expected for them. There is, therefore, no direct path between the cases in the multidimensional space. While this test does not rule out the possibility of one existing, the combined evidence of the gradient procedure and ξ variation both having failed to find such a path provides strong evidence that none exists.

The searches based on RV I have been shown to be unsatisfactory on the grounds cited, i.e., the behavior of the phase shift in the S_{11} state and the failure of the SEMI ISP search. On the basis of a small statistical superiority and on the strength of the 250-MeV polarization data,[28] one would probably choose the sets resulting from searches related to RV II over those related to RV III. These cases appear to provide distinct representations of the pion-nucleon scattering matrix in the energy range 20–300 MeV. In the next section, further evidence in favor of the sets related to RV II will be cited.

B. Comparison of Many-Energy Fits to Higher Energy Solution

Measurements of recoil-proton polarization in positive pion-nucleon scattering at 410 MeV have been recently performed by the Saclay group.[33] Since the many-energy analysis given in this paper has been performed under the assumption that the inelastic parameter is zero, a linear extrapolation is necessary in order to give an idea of the phase shifts at 410 MeV. Since, in addition, the Saclay analysis did not include higher partial waves, it is not expected that their results will be directly comparable to ours. However, it will be expected that it will be possible to identify these solutions with the corresponding Saclay results, as they ought to lie within the vicinity of each other. In Table III, a list of the isotopic spin-$\frac{3}{2}$ phase shifts at 410 MeV from the searches given in this paper and from Saclay has been compiled.

Comparison of columns I and IV of Table III shows that S_{31} and D_{35} partial waves lie within the statistical errors of each other. The P_{33} wave differs somewhat, but

[33] P. Bareyre, C. Bricman, M. J. Longo, G. Valladas, G. Villet, G. Bizard, J. Duchon, J. M. Fontaine, J. P. Patoy, J. Seguinot, and J. Yonnet, Phys. Rev. Letters **14**, 198 (1965); P. Bareyre, C. Bricman, J. Seguinot, and G. Villet, *ibid.* **14**, 201 (1965). We wish to thank Dr. G. Valladas for communicating to us the Saclay results before publication and for stimulating discussions.

TABLE III. Phase shifts at 410 MeV.

Phase shift \ Solution	SPD1 (Saclay) I	SPDF (Saclay) II	SEMI III III	SEMI IIM IV
S_{31}	$-26.7\ \pm 1$	$-26.5\ \pm 0.9$	$-18.9\ \pm 2.19$	$-25.6\ \pm 0.47$
P_{33}	$-32.0\ \pm 0.5$	$-31.4\ \pm 0.7$	$127.0\ \pm 0.51$	$139.5\ \pm 0.28$
D_{35}	$-2.3\ \pm 0.8$	-1.75 ± 1	-13.75 ± 0.95	-1.25 ± 0.36
P_{31}	-9.75 ± 1.2	$-11.1\ \pm 2$	$8.5\ \pm 1.12$	$-13.6\ \pm 0.48$
D_{33}	$1.4\ \pm 0.8$	$0.8\ \pm 1.2$	7.75 ± 0.73	$-3.2\ \pm 0.42$

some of this discrepancy can be accounted for by the manner of extrapolation, particularly since this is a large phase shift. Both the P_{31} and D_{33} phase shifts seem to deviate larger than the sum of their statistical errors. However, these partial waves are quite sensitive to the inclusion of higher partial waves, as evidenced by their movements when the F waves are included. Comparison of columns I and II of Table III, for instance, shows that the P_{31} and D_{33} phase shifts moved in a direction towards the solution in column IV. In fact, P_{31} of II has come to within the statistical error of P_{31} of IV.

The above evidences point to the possibility that the Saclay SPD1 and SPDF solutions are indeed identical to SEMI IIM. From continuity, therefore, SEMI IIM is preferred over SEMI III, in harmony with the conclusion arrived at in the previous section.

C. Searches at 310 MeV

The best selection of data available at this time in this energy range was obtained in the 310-MeV energy range.[34] Some 84 pieces of data have been published which satisfy our selection criteria at 307 and 310 MeV. These include at least one example of each type of scattering, both cross section and polarization. Thus one has the best chance of determining the scattering matrix at this energy, although the small angular range covered by the polarization measurements suggests that a unique set of phase shifts may not be obtainable. Rugge and Vik[13] used 66 pieces of data at 310 MeV including cross section for $\pi^-+p\rightarrow\pi^0+n$ represented by coefficients of a fitted angular distribution[35] (we prefer to include the 307-MeV data, where this reaction is represented by tabulated values with assigned errors). With F waves omitted, they obtained an S, P, D fit (solution 1, Table X of Ref. 13) with $D=1.08$. The inclusion of F waves, however, allowed three statistically satisfactory fits to be obtained, the RV I, II, III cited above, with $D=0.66$, 0.97, 1.07, respectively. The inclusion of data at lower and higher energies in the present work has made it impossible for the procedures employed here to accommodate the best single energy fit, RV I, principally because it includes the negative phase shift in the S_{11} state as already mentioned. For the other cases, on comparing their Tables X and XII, one sees that the inclusion of F waves has also introduced some instability in the fits: Though small in themselves, the presence of F waves has allowed large changes in some of the phase shifts for lower angular-momentum states. This is notable for states D_{33}, D_{35}, P_{13}, D_{15} where the phase shifts change sign for RV II, in P_{31} where the size is trebled, and especially in the state S_{11} where a 13° increase in value takes place, which causes a sign change as well.

Employing the calculated phase shifts for F waves and higher angular-momentum states up to $l=9$, one can test the possibility of eliminating the instability noted above and perhaps of obtaining a unique fit at this energy. To this end, the searched S, P, D phase shifts of fits SEMI IIM (at 310 MeV, SEMI II and SEMI IIM are essentially identical) and SEMI III were obtained at 310 MeV, and searched with the 84 pieces of data obtained at pion laboratory energies of 307 and 310 MeV. In addition, set RV II, with F waves replaced by dispersion theoretic values and higher waves up to $l=9$ included by the same means, was searched with the same data. Also, solution II, Table X of Rugge-Vik, which had no F waves, was investigated in the same way. The results of these searches are contained in Table IV. Although the cases are not exact parallels because there are some differences in the data selection, one may wish to compare the Rugge-Vik D-values. Thus their solution II, Table X, gives $D=1.9$, while RV II (with searched F waves) has $D=1.0$. The results for SEMI IIM and SEMI III are shown on Figs. 1–5 as separated points at 310 MeV. Error bars are obtained in the usual way,[2] and, as expected, they are larger for the single-energy searches than the curves because fewer data are involved in their determination. Two points stand out from Table IV and the figures: The data at 310 MeV appear not to be entirely consistent with that

[34] V. G. Zinov and S. M. Korenchenko, Zh. Eksperim. i Teor. Fiz. **33**, 1308 (1957); **38**, 1099 (1960) [English transls.: Soviet Phys.—JETP **6**, 1007 (1958); **11**, 794 (1960)]; A. I. Mukhin, E. B. Ozerov, B. Pontecorvo, *ibid.* **31**, 237 (1957) [English transl.: *ibid.* **4**, 371 (1956)]; J. H. Foote, O. Chamberlain, E. H. Rogers, H. M. Steiner, C. E. Wiegand, T. Ypsilantis, Phys. Rev. **122**, 948 (1961); J. H. Foote, O. Chamberlain, E. H. Rogers, and H. M. Steiner, *ibid.* **122**, 959 (1961); H. R. Rugge and O. T. Vik, *ibid.* **129**, 2300 (1963); E. H. Rogers, O. Chamberlain, J. H. Foote, H. M. Steiner, C. Wiegand, and T. Ypsilantis, Rev. Mod. Phys. **33**, 356 (1961); J. H. Foote, O. Chamberlain, E. H. Rogers, H. M. Steiner, C. Wiegand, and T. Ypsilantis, Phys. Rev. Letters **4**, 30 (1960); R. E. Hill, N. E. Booth, J. Esterling, D. L. Jenkins, N. H. Lipman, H. R. Rugge, and O. T. Vik, Bull. Am. Phys. Soc. **9**, 410 (1964).

[35] J. C. Caris, R. W. Kenney, V. Perez-Mendez, and W. A. Perkins, III, Phys. Rev. **121**, 893 (1961).

TABLE IV. Results of 310-MeV single-energy searches.

Search designation \ State	S_{31}	P_{33}	D_{35}	P_{31}	D_{33}	S_{11}	P_{13}	D_{15}	P_{11}	D_{13}	D
Solution II,	−18.4	134.9	−4.1	−4.5	1.66	−3.01	6.3	0.03	29.2	3.26	
Table X	0.75	0.32	0.60	0.78	0.47	2.2	0.82	0.65	1.29	0.90	1.6
SEMI IIM	−19.7	134.9	−2.56	−7.78	−0.66	12.53	−3.26	1.63	20.53	3.92	
	0.61	0.33	0.53	0.66	0.45	2.79	1.09	0.57	1.51	0.37	1.4
SEMI III	−19.44	135.14	−3.12	−6.09	0.95	0.72	5.32	−0.41	31.4	2.67	
	0.73	0.35	0.62	0.80	0.50	2.25	0.94	0.93	1.06	1.15	1.7
RV II	−21.8	136.8	0.9	−11.5	−2.7	11.1	−3.6	0.4	23.1	5.8	3.1

at lower and higher energies, since the single-energy phase shifts differ in some cases from the curves by more than their uncertainties; and for $T=\frac{3}{2}$ states, the present procedure has led to just one set of phase shifts at 310 MeV. The single-energy searches brought every $T=\frac{3}{2}$ phase shift toward its partner for cases SEMI IIM and SEMI III, with only values for D_{33} finally remaining outside each other's uncertainties. At the same time, $T=\frac{1}{2}$ phase shifts were frequently moving apart, and in no case did values for any state finish as statistically indistinguishable. Even the search work on solution II, Table X of Rugge-Vik led to nearly the same $T=\frac{3}{2}$ phase shifts (again within uncertainties, except for D_{33}), while allowing quite different $T=\frac{1}{2}$ values. A ξ variation carried out between this case and SEMI IIM showed a maximum D value of 10.5 between the cases (i.e., at $\xi\sim\frac{1}{2}$), so there is emphatically no direct path between these fits: They are probably distinct.

The inability of the search procedure to produce a satisfactory fit at 310 MeV with phase shifts based on RV II is interesting. This same fit was used in constructing the starting phase shifts for SEMI II, which has turned out to be the best fit to all data and to the 307–310 MeV data individually. One observes again the effect of using data from several energies, and has another suggestion of some inconsistency between the 310-MeV data and that at other energies. Thus, in the SEMI IIM searches, the phase shift received a compromise adjustment which worsened the fit at 310 MeV, but allowed the other phase shifts to absorb some of the effect of replacing the R-V F waves by dispersion relation values (Table I). When the single-energy search based on SEMI IIM was then undertaken, the fit was in a region of phase shift space which allowed significant improvement. Without the help of other data, a similar region could not be found for the single-energy case started from RV II.

One can see whether the improvement gained in fitting the data at 310 MeV can be retained in the many energy fit by smoothly extrapolating the curves of Figs. 1–5 to the single-energy values. The D values of Table II for SEMI III and SEMI IIM are almost exactly reproduced by such a calculation, so that improvement in the fit at 310 MeV is offset by worsening at other energies. Here is another suggestion of the inconsistency of the 310-MeV data with that at other energies. The remedy is clearly new data, as will be discussed later.

D. Effect of Dispersion Relation Parameter Variations

In the work on analyses of nucleon-nucleon scattering, it has been possible roughly to determine (or confirm) the value of the pion-nucleon coupling constant, and to support the form of the one-pion exchange potential predicted by meson theory.[36] The same coupling constant, together with the parameters of the (3,3) resonance and the ρ meson enters the calculation of higher partial waves used in the present semiphenomenological fits. It will be of interest, therefore, to see whether the present searches can support values of these various parameters found by other, more direct, means.

The width and energy of the resonance in the state P_{33} appear so well fixed that they were not varied. The present work used initially values for the width and position derived from the total cross section data directly, giving $\gamma_{\pi N}=1.68$ and $E_{\pi N}=6.92\,\mu\, c^2$, respectively. Subsequent searching led to a stable P_{33}, mentioned above, consistent with these parameters. All contribution to the higher partial waves from the π-N channel were computed, therefore, with these parameters. The coupling constant, usually taken to be $g_0^2=14.0$ and related to the g_r^2 of Eq. (4) by a factor of

TABLE V. π-N channel contributions to phase shifts in degrees.

g_0^2	l	E	$T=\frac{1}{2}$, $J=l+\frac{1}{2}$	$T=\frac{1}{2}$, $J=l-\frac{1}{2}$	$T=\frac{3}{2}$, $J=l+\frac{1}{2}$	$T=\frac{3}{2}$, $J=l-\frac{1}{2}$
13.57	2	100 MeV	0.118	−0.103	−0.269	0.0461
14			0.123	−0.105	−0.278	0.0484
13.57	3		−0.0071	0.0026	0.0149	−0.0021
14			−0.0074	0.0027	0.0154	−0.0022
13.57	2	300 MeV	1.023	−1.257	−2.292	0.289
14			1.063	−1.267	−2.371	0.309
13.57	3		−0.133	0.0786	0.281	−0.0334
14			−0.138	0.0794	0.291	−0.0351

[36] G. Breit, M. H. Hull, Jr., K. E. Lassila, and H. M. Ruppel, Phys. Rev. Letters **5**, 274 (1960); G. Breit, M. H. Hull, Jr., F. A. McDonald, and H. M. Ruppel, in *Proceedings of the 1962 International Conference on High Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962).

4π, is not so well determined, and Table V shows the effect on π-N contributions to D and F waves at 100 and 300 MeV if reducing g_0^2 by 3% (the nucleon-nucleon work would actually allow a greater variation[36]). The changes are seen to be about the same in percentage as the change in g_0^2, and in the same direction for the absolute magnitude. This is just what Eq. (4) would lead one to expect when it is realized that the pole term in $B^{(\pm)}$ makes the dominant contribution at the energies in question from the π-N channel.

Contributions from the π-π channel depend on parameters $\gamma_{\pi\pi}$ and $t_{\pi\pi}$ of the resonance associated with the ρ meson. In Roos' table,[37] the values given are $\gamma_{\pi\pi}=0.3$ and $t_{\pi\pi}=29.0$. The third parameter may be taken to be one of the normalization constants $N_\pm$ of Eq. (10). However, BCL were followed in choosing initially a value of C_1 related to $N_\pm$ by

$$C_1=(M/p_{\pi\pi}^2)((E_{\pi\pi}^2/\sqrt{2}M)N_-+N_+), \tag{15}$$

where $p_{\pi\pi}$, $E_{\pi\pi}$ are the momentum and energy at resonance, respectively. The value $C_1=-0.88$ corresponds to more recent fitting of electromagnetic form factors in the BCL manner. In the BCL work, a quantity

$$C_2=(1/2p_{\pi\pi}^2)((M/\sqrt{2})N_--N_+) \tag{16}$$

and the relation $C_2/C_1=g/M$, where g is the gyromagnetic ratio of the nucleon $(=1.83)$, allow the required determination of N_+ and N_-. It would be possible to consider N_+, N_- as completely arbitrary, but, as an initial effort, only C_1 was varied. The results are shown in Table VI. When a parameter is not mentioned, it was

TABLE VI. π-π channel contributions to the phase shifts in degrees at 300 MeV.

Parameter values	l	$T=\frac{1}{2}$, $J=l+\frac{1}{2}$	$T=\frac{1}{2}$, $J=l-\frac{1}{2}$	$T=\frac{3}{2}$, $J=l+\frac{1}{2}$	$T=\frac{3}{2}$, $J=l-\frac{1}{2}$
all initial	2	−0.226	1.803	0.113	−0.902
$t_{\pi\pi}=35$		−0.185	1.403	0.092	−0.702
$\gamma_{\pi\pi}=0.36$		−0.255	2.107	0.127	−1.054
$C_1=-0.58$		−0.135	1.079	0.067	−0.539
all initial	3	−0.0216	0.185	0.0108	−0.0923
$t_{\pi\pi}=35$		−0.0155	0.129	0.0078	−0.0643
$\gamma_{\pi\pi}=0.36$		−0.0241	0.219	0.0120	−0.110
$C_1=-0.58$		−0.0129	0.110	0.0065	−0.0552

given its initial value, i.e., $\gamma_{\pi\pi}=0.3$, $t_{\pi\pi}=29.0$, $C_1=-0.88$. At 100 MeV, these phase shifts are about 10 times smaller, and so are not shown. The variations on the parameters shown are, in the case of $\gamma_{\pi\pi}$ and $t_{\pi\pi}$, larger than experimental uncertainties for the ρ meson allow.

It remains now to see the effect of this type of changes in the constants on the fit to data. Using SEMI IIM for a standard, one obtains $D=2.416$ with the initial values of $\gamma_{\pi\pi}$, $t_{\pi\pi}$, and C_1. If one now selects new values of the parameters with two criteria, (1) effects of parameter changes on phase shifts is cumulative, and (2) changes in phase shifts such as to take them toward the phenomenological values, the choice becomes $C_1=-1.0$, $t_{\pi\pi}=26$, $\gamma_{\pi\pi}=0.33$. The resulting fit to data with SEMI IIM gives $D=2.406$. The change is about $6/N$, where N is the total number of data, and so could be statistically significant. However, with three parameters involved, the chances of putting useful limits on their values by this means appear small at this time. The accumulation of more data between 240 and 300 MeV could improve the possibilities in such a calculation quite markedly.

[37] M. Roos, Rev. Mod. Phys. **35**, 314 (1963).

V. CONCLUSIONS

For the acceptable solutions of Family II and Family III, the sets with the least number of phase shifts varied are SEMI IIM and SEMI IIISPM, where only six phase shifts are varied below 120 MeV, and only ten above that energy. The main difference between them is found in the $T=\frac{1}{2}$ phase shifts, as study of Figs. 1–5 will confirm. Although evidences cited earlier seem to favor SEMI IIM over SEMI IIISPM, it will be useful to confirm this selection by further experimental data. The obvious gaps in our knowledge of the relevant experimental quantities are easily indicated. A frequency plot of number of data as function of energy shows deficiencies (1) at lower energies (below 80 MeV), (2) between 170 and 210 MeV, (3) and between 250 and 290 MeV. Charge exchange data are least available. The usefulness of selected data in distinguishing fits is clearly shown in Fig. 7: The fit SEMI I, eliminated for theoretical reasons, is made untenable also by the forward-angle polarization measurement of charge-exchange scattering by Hill *et al.*[38] The 307-MeV data for the cross section[39] in the same process gives further support in its forward angle behavior.

A similar single measurement to distinguish the remaining Family II and Family III fits is not so easy to suggest. The dominance of effects of the phase shift in the P_{33} state near 300 MeV tends to make their predictions similar at high energies over the whole angular range. The negative value of the S_{11} phase shift in the Family I fits strongly distinguishes their predictions at 300 MeV and eliminates them. At this moment, it would appear that further measurements, both cross section and polarization for all three processes, at one or two selected energies (say 200 and 250 MeV) would be most helpful in determining the elastic scattering matrix. As mentioned earlier, the recent polarization measurements at 250 MeV[28] already constitute some evidence for preferring Family II to Family III.

The comparison of D values in Table II and data graphs in Figs. 7–9 show that the higher partial waves predicted from the approximate dispersion relations are compatible with the data. However, their role is not

[38] R. E. Hill *et al.*, last reference in footnote 34.

[39] V. G. Zinov and S. M. Korenchenko, first two references in footnote 34.

decisive, and they cannot be said, from this work, to be required. While it is suggestive that a better single-energy fit could be found with theoretical higher partial waves than without (compare the Rugge-Vik solution II, Table X, with Table IV), this is not conclusive since at least as much improvement can be obtained by varying the F waves as Rugge and Vik have done.

The D wave phase shifts, although relatively small even at 300 MeV, are found in the phenomenological searches[13] to be larger than dispersion relation values (Table I). However, the search SEMI IIM has shown that only in the state D_{13} does the data demand this difference (Fig. 5). The Family III fits constitute an alternative in which the D waves are quite different. In both cases, theoretical values are allowed by available data below 120 MeV. This parallels the nucleon-nucleon

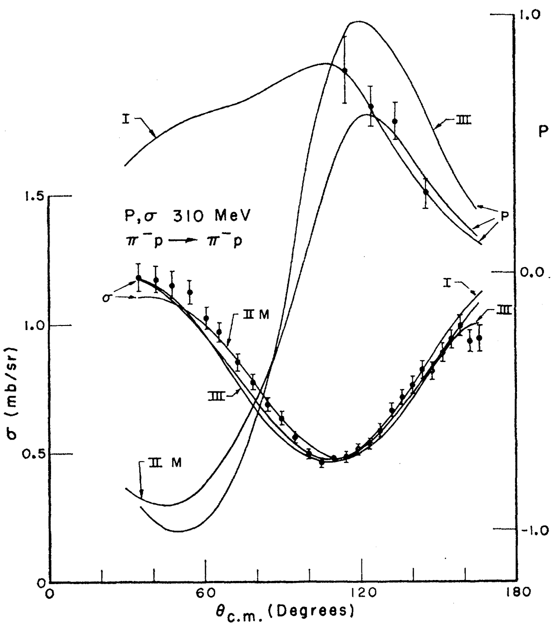


FIG. 7. Differential cross section σ and polarization P for negative-pion elastic scattering at 310 MeV as predicted by semi-phenomenological solutions I, IIM, and III. The experimental points are those of Rugge and Vik (Ref. 34).

experience,[1–4] where phase parameters in relatively low angular-momentum states are compatible with their theoretical values from the one pion-exchange interaction at low energies. The employment of fewer varied phase shifts in the SEMI IIM search leads to reduced uncertainties, as expected, in the fitted values. One may conclude that it is not only consistent to employ theoretical higher partial waves (including D waves except for the D_{13} state), but that simply on a statistical basis such a procedure is helpful in reducing uncertainties of the phase shifts treated phenomenologically. The variation in the $T=\frac{3}{2}$ phase shifts is considerably reduced by this procedure also. Accumulation of data, especially in π^- scattering, as already suggested, should assist one in

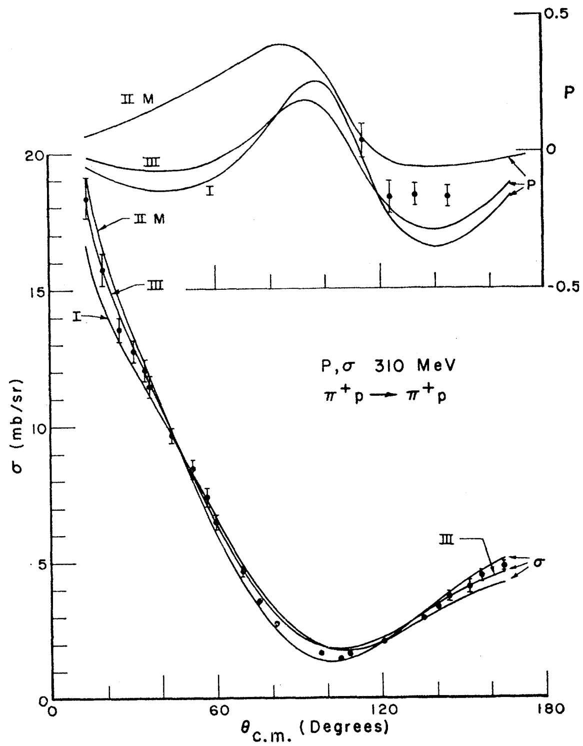


FIG. 8. Differential cross section σ and polarization P for positive-pion elastic scattering at 310 MeV as predicted by semi-phenomenological solutions I, IIM, and III. The experimental points are those of Foote *et al.* (Ref. 34).

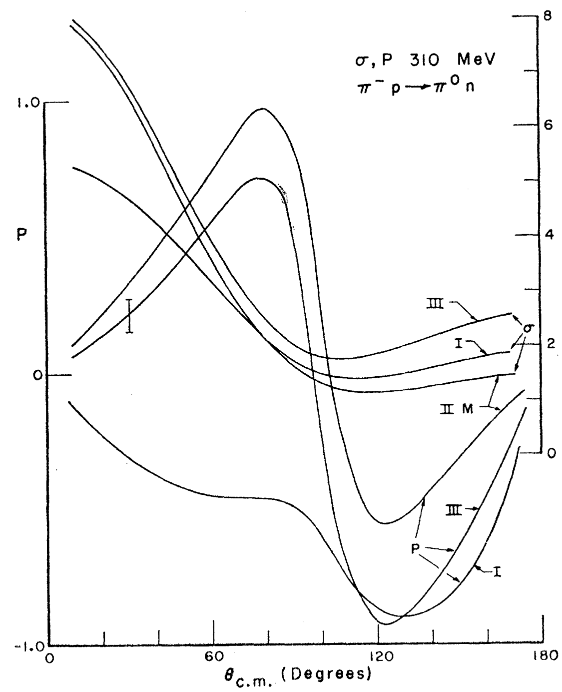


FIG. 9. Differential cross section σ and polarization P for charge-exchange scattering at 310 MeV as predicted by semiphenomenological solutions I, IIM, and III. The experimental point is that of Hill *et al.* (Ref. 38).

confirming the uniqueness of the fit as well as in answering some questions about the theoretical treatment.

In comparing results with Roper's phenomenological fit,[17] one finds general and rather close agreement with SEMI IIM, except for D_{33} where Roper's phase shift is of opposite sign (in agreement with SEMI III). There are differences in size of the phase shift in the P_{13} state, and there is a difference of 7 MeV in the energy at which the phase shift in the P_{33} state goes through resonance. There is, of course, no evidence either way in this energy range on Roper's proposed resonance in P_{11} state.[40]

The culmination of the detailed discussion of the pion-nucleon interaction with dispersion relations carried out by Hamilton and co-workers[19] is the prediction by Donnachie *et al.*[18] of P, D, and F waves in the elastic energy range. Of the P waves, the phase shift for the P_{11} state presented special problems, and is considered uncertain by Donnachie *et al.* Neither of their approximations is in agreement with SEMI IIM, as one sees in Fig. 5. Their P_{31} phase shift agrees well with SEMI IIM, except near 300 MeV, where it is outside both uncertainties. For the P_{13} state, it is at low energies that disagreement occurs, but not outside the uncertainties. Phenomenological information in the form of scattering lengths enters their evaluation of the P_{31} and P_{13} phase shifts.

The D waves are especially interesting in comparison with both our theoretical values (Table I) and with SEMI IIM. For D_{15}, D_{33}, and D_{35}, the two theoretical calculations yield nearly identical results, and both agree well with the values obtained by the limited search above 120 MeV (Figs. 2–4). In the state D_{13}, the calculation by Donnachie *et al.* agrees very well with the searched values, which, in turn, are much greater above 160 MeV than our theoretical ones. The evaluation of this phase shift included rescattering effects from high-energy resonances,[18] which are a little more uncertain than other contributions, and were not approximated by us. The fit SEMI IIM appears to provide support for their calculation (Fig. 5).

Among the F waves, only F_{15} differs appreciably from our theoretical values (Fig. 10), and again this involved in the work of Donnachie *et al.* a rescattering contribution we did not estimate (this is true also of F_{37}, but the contribution is small, and agreement is good). It is gratifying that our simpler calculation, with much the same input as the more detailed work of Donnachie *et al.*, provides an adequate approximation to phase shifts in the elastic energy range where rescattering is not important. The present work has provided formulas for arbitrary angular momentum (cf. Appendix), which may be useful in calculating the smaller phase shifts at higher energies for states in which rescattering effects are small.

[40] L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. **138**, B190 (1965).

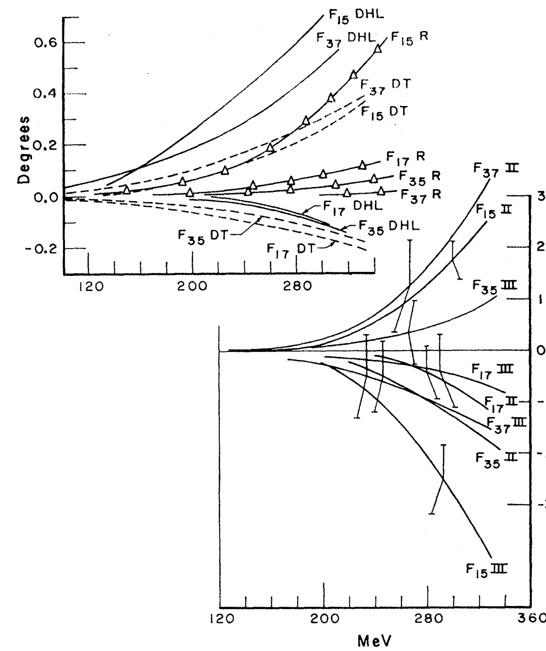


FIG. 10. The F waves plotted against energy. DHL designates solutions of Ref. 18; R, Roper's solution (Ref. 17) and DT, dispersion-theoretically calculated values obtained in this work. The F waves of phenomenological solutions with error bars are shown at lower right.

The support for the theoretical calculations provided by SEMI IIM suggests that Family III fits be eliminated, thus achieving a unique set of phase shifts in the elastic energy range (Donnachie *et al.* also prefer RV II at 300 MeV).

The polarization measurement at 250 MeV,[28] as well as comparison with the Saclay single-energy searches at 410 MeV,[33] lend strong support for eliminating Family III fits. However, the present work indicates that it will be desirable to have available some further measurements at about 200 and 250 MeV to eliminate SEMI III definitively on statistical grounds.

ACKNOWLEDGMENTS

The authors are indebted to Dr. H. M. Ruppel for valuable help in machine operations and programming.

APPENDIX

The predictions for the phase shifts for $l \geq 2$ in the present work may be obtained by evaluating the following series. Employing Eqs. (5) and (6) of the text to relate the phase shifts to the invariant amplitudes A and B, one projects out the contributions for each angular-momentum state and calculates separately the parts coming from the πN resonance and the $\pi\pi$ resonance. These two contributions are called A and ΔA

and B and ΔB, respectively. One finds

$$A_l{}^{\pm}=\binom{2}{-1}\left[\pm\left(\sum_{n=l-1,l+1,\ldots}(-)^n\lambda^n 2^l\frac{(n+1)![(n+l+1)/2]!}{(n+l+2)![(n+1-l)/2]!}\int\frac{a_1(s')ds'}{\zeta^{n+1}(s')}\right.\right.$$
$$\left.\left.+\sum_{n=l,l+2,\ldots}(-)^n\lambda^n 2^l\frac{n![(n+l)/2]!}{(n+l+1)![(n-l)/2]!}\int\frac{a_2(s')ds'}{\zeta^{n+1}(s')}\right)\right],\quad \text{(A1)}$$

$$A_{l+}{}^{\pm}=\binom{2}{1}\left[\pm\left(\sum_{n=l,l+2,\ldots}(-)^n\lambda^n 2^{l+1}\frac{(n+1)![(n+2+l)/2]!}{(n+l+3)![(n-l)/2]!}\int\frac{a_1(s')ds'}{\zeta^{n+1}(s')}\right.\right.$$
$$\left.\left.+\sum_{n=l+1,l+3,\ldots}(-)^n\lambda^n 2^{l+1}\frac{n![(n+l+1)/2]!}{(n+l+2)![(n-l-1)/2]!}\int\frac{a_2(s')ds'}{\zeta^{n+1}(s')}\right)\right],\quad \text{(A2)}$$

and

$$A_{l-}{}^{\pm}=\binom{2}{-1}\left[\left(\frac{1}{3}\int_{(M+1)^2}^{\infty}\frac{a_1(s')ds'}{s'-s}\delta_{l,2}\pm\sum_{n=l-2,l,\ldots}(-)^n\lambda^n 2^{l-1}\frac{(n+1)![(n+l)/2]!}{(n+1+l)![(n-l+2)/2]!}\int\frac{a_1(s')ds'}{\zeta^{n+1}(s')}\right.\right.$$
$$\left.\left.+\sum_{n=l-1,l+1,\ldots}(-)^n\lambda^n 2^{l-1}\frac{n![(n+l-1)/2]!}{(n+l)![(n-l+1)/2]!}\int\frac{a_2(s')ds'}{\zeta^{n+1}(s')}\right)\right],\quad \text{(A3)}$$

where

$$\zeta(s')=2\mathbf{q}'^2+2(E_p'\omega_q'-E_p\omega_q),\qquad \lambda=2\mathbf{q}^2,\quad \text{(A4)}$$

and

$$a_1(s')=[\tfrac{4}{3}\,\mathrm{Im}f_{33}(s')(3(W'+M)/(E'+M)\mathbf{q}'^2)]\mathbf{q}^2,$$
$$a_2(s')=\tfrac{4}{3}\,\mathrm{Im}f_{33}(s')\left(\frac{3(W'+M)}{E'+M}\frac{\mathbf{q}^2-\mathbf{q}'^2}{\mathbf{q}'^2}+\frac{W'-M}{E'-M}\right).\quad \text{(A5)}$$

The resonance amplitude is assumed to be

$$\mathrm{Im}f_{33}{}^P(s')=\frac{\gamma_{\pi N}{}^2|\mathbf{q}|^5}{[(s_r-s')^2+(\gamma_{\pi N}|\mathbf{q}|^3)^2]}.\quad \text{(A6)}$$

The terms in B contain contributions of exactly the same form as Eqs. (A1)–(A3) with $a_1(s')$ and $a_2(s')$ replaced, respectively, by

$$b_1(s')=[\tfrac{4}{3}\,\mathrm{Im}f_{33}(s')(3/E'+M)(1/\mathbf{q}'^2)]\mathbf{q}^2,$$

and

$$b_2(s')=\tfrac{4}{3}\,\mathrm{Im}f_{33}(s')\left(\frac{3}{E'+M}\frac{\mathbf{q}^2-\mathbf{q}'^2}{\mathbf{q}'^2}-\frac{1}{E'-M}\right).\quad \text{(A7)}$$

In addition, the nucleon pole term enters and must be added. This makes a contribution

$$B_{l\,\mathrm{pole}}{}^{\pm}=g_r{}^2\left\{\frac{1}{M^2-s}\delta_{l,0}\mp\sum_{n=l,l+2,\ldots}(-)^n\left[\frac{(2\mathbf{q}^2)^n}{\bar{g}^{n+1}(M^2)}\right]\frac{2^l n![(n+l)/2]!}{(n+l+1)![(n-l)/2]!}\right\},\quad \text{(A8)}$$

$$B_{l+\,\mathrm{pole}}{}^{\pm}=\mp g_r{}^2\sum_{n=l+1,l+3,\ldots}(-)^n\frac{(2\mathbf{q}^2)^n}{\bar{g}^{n+1}(M^2)}\frac{2^{l+1}n![(n+l+1)/2]!}{(n+l+2)![(n-l-1)/2]!},\quad \text{(A9)}$$

and

$$B_{l-\,\mathrm{pole}}{}^{\pm}=g_r{}^2\left\{\mp\sum_{n=l-1,l+1,\ldots}(-)^n\frac{(2\mathbf{q}^2)^n}{\bar{g}^{n+1}(M^2)}\frac{2^{l-1}n![(n+l-1)/2]!}{(n+l)![(n-l+1)/2]!}\right\}.\quad \text{(A10)}$$

In these series,

$$\bar{g}=2E_p\omega_q-1,\quad \text{(A11)}$$

and $g_r{}^2=4\pi g_0{}^2=175.93$ corresponds to $g_0{}^2=14$ for the renormalized coupling constant.

The contributions from the π-π channel may be written

$$\Delta A_l^{(-)}=\frac{3}{2}\left(\frac{MN_-}{\sqrt{2}}-N_+\right)\int_{4\mu^2}^{\infty} f(t')dt'[I_1(t')+I_2(t')], \tag{A12}$$

$$\Delta A_{l\pm}^{(-)}=\frac{3}{2}\left(\frac{MN_-}{\sqrt{2}}-N_+\right)\int_{4\mu^2}^{\infty} f(t')dt'[I_1^{\pm}(t')+I_2^{\pm}(t')], \tag{A13}$$

$$\Delta B_l^{(-)}=\frac{12N_-}{2\sqrt{2}}\int_{4\mu^2}^{\infty} g(t')dt'I_1(t')/a_{1\pi}, \tag{A14}$$

$$\Delta B_{l\pm}^{(-)}=\frac{12N_-}{2\sqrt{2}}\int_{4\mu^2}^{\infty} g(t')dt'I_1^{\pm}(t')/a_{1\pi}, \tag{A15}$$

where

$$f(t')=\frac{\gamma_{\pi\pi}[(t'/4)-1]^{3/2}}{[(t'/4)-M^2][(tr-t')^2+\gamma_{\pi\pi}^2[(t'/4)-1]^3}, \tag{A16}$$

$$g(t')=\frac{\gamma_{\pi\pi}[(t'/4)-1]^{3/2}}{(tr-t')^2+\gamma_{\pi\pi}^2[(t'/4)-1]^3}. \tag{A17}$$

The integrals $I_{1,2}$, etc., appearing in Eqs. (A12)–(A15) are expanded as series

$$I_1=a_{1\pi}\sum_{n=l,l+2}^{\infty}(-)^n\frac{1}{d_\pi}\left(\frac{c_\pi}{d_\pi}\right)^n\frac{2^{l+1}n![(n+l)/2]!}{(n+l+1)![(n-l)/2]!}, \tag{A18}$$

$$I_2=a_{2\pi}\sum_{n=l-1,l+1}^{\infty}(-)^n\frac{1}{d_\pi}\left(\frac{c_\pi}{d_\pi}\right)^n\frac{2^{l+1}(n+1)![(n+1+l)/2]!}{(n+l+2)![(n+1-l)/2]!}, \tag{A19}$$

$$I_1^{\pm}=a_{1\pi}\sum_{n=l\pm1,(l\pm1+2),\cdots}^{\infty}(-)^n\frac{1}{d_\pi}\left(\frac{c_\pi}{d_\pi}\right)^n 2\times2^{l\pm1}\frac{n![(n+l\pm1)/2]!}{(n+1+l\pm1)![(n-l\mp1)/2]!}, \tag{A20}$$

$$I_2^{\pm}=a_{2\pi}\sum_{n=l\pm1-1,(l\pm1+1),\cdots}^{\infty}(-)^n\frac{1}{d_\pi}\left(\frac{c_\pi}{d_\pi}\right)^n 2\times2^{l\pm1}\frac{(n+1)![(n+1+l\pm1)/2]!}{(n+2+l\pm1)![(n+1-l\mp1)/2]!}, \tag{A21}$$

where

$$a_{1\pi}=4E_p\omega_q+2\mathbf{q}^2,\quad a_{2\pi}=2\mathbf{q}^2,\quad d_\pi=t'+2\mathbf{q}^2,\quad c_\pi=-2\mathbf{q}^2. \tag{A22}$$