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Singular Potentials and Regularization

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Let $V(r)$ be a potential repulsive and singular at the origin, and A a physical quantity (say, the scattering length) associated with it. Let $V(r,\alpha)$ be a regulated version of the potential $V(r)$; $V(r,\alpha)$ is nonsingular for $\alpha>0$ and coincides with $V(r)$ for $\alpha=0$. Let $A(\alpha)$ be the corresponding scattering length. It is usually assumed that $A(0)=A$. Simple counterexamples are presented, namely, cases when A and $A(0)$, although both well defined and finite, are different. The existence of these counterexamples sheds doubt on the validity of a theorem given by Khuri and Pais. This doubt is substantiated by noting that the proof of the theorem contains an unjustified exchange of two limiting processes.

1. INTRODUCTION

Recently there has been an upsurge of interest in the problem of scattering on singular potentials,[1–4] mainly as a testing ground for the validity of the Peratization technique introduced by Feinberg and Pais to deal with unrenormalizable field theories.[5] The program of Peratization may be briefly formulated as follows (we refer immediately to the potential problem, where the situation is much better understood than in field theory): Let $V(r)=gv(r)$ be a potential more singular at the origin than the centrifugal term, and repulsive in that neighborhood. It is well known that under these circumstances the scattering problem is well defined and all the quantities of physical interest exist. For simplicity we focus our attention on the scattering length A, as has been customary in these studies. This quantity, considered as a function of the coupling constant g, has a singularity at $g=0$, due to the singular nature of the potential. Thus the power expansion in g of A has a vanishing radius of convergence, and if one tries to compute its coefficients, for instance by perturbation theory, one generally stumbles into infinities.[6]

The technique of Peratization has been devised just to overcome this difficulty. It works as follows: First of all the potential is regularized, i.e., the one-parameter family of potentials $V(r,\alpha)=gv(r,\alpha)$ is introduced, with the properties (i) *that* $V(r,\alpha)$ *is nonsingular for* $\alpha>0$ and (ii) *that* $V(r,\alpha)$ *coincides with* $V(r)$ *when* α *vanishes.* Associated with this family of potentials there exists a corresponding one-parameter family of scattering lengths $A(\alpha)$. Thanks to condition (i) these are now analytic functions of g in the neighborhood of $g=0$, provided $\alpha>0$. Thus they have, for $\alpha>0$, the power expansion in g

$$A(\alpha)=\sum_{n=0}^{\infty} a_n(\alpha)g^n \tag{1.1}$$

with nonvanishing radius of convergence $\rho(\alpha)$. On the other hand, when α vanishes $\rho(\alpha)$ also shrinks to zero; moreover the coefficients $a_n(\alpha)$ diverge.[6]

At this point the instructions of the Peratization program become relevant. They direct us to investigate in detail the behavior of the coefficients $a_n(\alpha)$ as α vanishes, separate out the most singular terms, insert only this part of $a_n(\alpha)$ in the sum in Eq. (1.1), carry out the summation, and finally approach the limit $\alpha\to 0$. The procedure is deemed successful if the function A_p thus computed provides, in some sense which need not be discussed here, a reasonable approximation to the original scattering length A. Both cases in which Peratization succeeds and cases in which it fails are discussed in the literature.[3]

Presumably a precondition for the success of the Peratization scheme is that the complete scattering length $A(\alpha)$ corresponding to the regulated potential $V(r,\alpha)$ converge, as α vanishes, to the scattering length A corresponding to the original potential $V(r)$:

$$\lim_{\alpha\to 0} A(\alpha)=A\,. \tag{1.2}$$

[1] N. N. Khuri and A. Pais, Rev. Mod. Phys. **36**, 590 (1964).

[2] N. Limić, Nuovo Cimento **26**, 581 (1962); E. Predazzi and T. Regge, *ibid.* **24**, 518 (1962); M. Giffon and E. Predazzi, *ibid.* **33**, 1374 (1964); G. Tiktopoulos and S. B. Treiman, Phys. Rev. **134**, B844 (1964); A. Pais and T. T. Wu, J. Math. Phys. **5**, 799 (1964); Phys. Rev. **134**, B1303 (1964); L. Bertocchi, S. Fubini, and G. Furlan, Nuovo Cimento **32**, 745 (1964); **35**, 633 (1965); H. Cornille and E. Predazzi, Phys. Letters **10**, 149 (1964); Nuovo Cimento **35**, 879 (1965); H. Cornille, *ibid.* **36**, 1316 (1965); H. Cornille and E.Predazzi, University of Chicago (unpublished); E. Del Giudice and E. Galzenati, University of Naples (unpublished); K. Meetz, Nuovo Cimento **34**, 690 (1964); J. M. Charap and N. Dombey, Phys. Letters **9**, 210 (1964); N. Dombey, University of Sussex (unpublished).

[3] A. Arbuzov, A. T. Filippov and O. A. Khrustalev, Phys. Letters **8**, 205 (1964); H. H. Aly, Riazuddin and A. H. Zimerman, Phys. Rev. **136**, B1174 (1964); T. T. Wu, Phys. Rev. **136**, B1176 (1964); M. A. Ahmed and D. B. Fairlie, University of Durham (unpublished); H. H. Aly, Riazuddin, and A. H. Zimerman, Nuovo Cimento **35**, 324 (1965); H. Cornille, CERN (unpublished); F. Calogero and M. Cassandro, Nuovo Cimento (to be published).

[4] F. Calogero and M. Cassandro, Nuovo Cimento **34**, 1712 (1964).

[5] G. Feinberg and A. Pais, Phys. Rev. **131**, 2724 (1963); **133**, B477 (1964); Y. Pwu and T. T. Wu, *ibid.* **133**, B1299 (1964).

[6] The possibility that the series diverges but its coefficients are finite exists (Ref. 4). This case is of no interest as regards Peratization.

It is reasonable to expect that this will happen generally in view of condition (ii) above. In fact a proof that it occurs, for the potentials $V(r)=gr^{-m}$ and for a special class of regulated potentials, has been published.[1] However, if the class of regulated potentials is unrestricted except for (i) and (ii) above, it is indeed possible to find examples which violate the condition Eq. (1.2). A remarkably simple instance is displayed in the following section, and is further discussed in a slightly more general case in Sec. 3. Although these examples do not fall within the class of regularized potentials considered by Khuri and Pais,[1] their existence sheds some doubt upon the validity of the Khuri-Pais theorem even for the cases in which it is supposed to hold. This doubt is substantiated in Sec. 4 by a scrutiny of the proof of the theorem, which reveals the unjustified exchange of two limiting processes. Some concluding remarks are collected in Sec. 5, and some mathematical details in an Appendix.

Units are chosen so that $\hbar=2m=1$, where m is the mass of the scattering particle.

2. A SIMPLE COUNTEREXAMPLE

We consider the potential

$$V(r)=gr^{-4},\quad g>0, \tag{2.1}$$

and the family of regulated potentials

$$V(r,\alpha)=gr^{-4}[\exp(-2\alpha/r)-\alpha g^{-1/2}\exp(-\alpha/r)]. \tag{2.2}$$

Clearly this family satisfies the conditions (i) and (ii) of the preceding section. Note however that the regulated potentials are not linear in g. To avoid any misunderstanding we emphasize that here, and always in the following, g and $g^{1/2}$ are positive.

The scattering length $A(\alpha)$ is easily obtained from the asymptotic behavior of the corresponding radial wave function

$$u(r,\alpha)=r\exp[-\alpha^{-1}g^{1/2}e^{-\alpha/r}]. \tag{2.3}$$

It turns out that $A(\alpha)$ is independent of α, and we find

$$A(\alpha)=g^{1/2}. \tag{2.4}$$

On the other hand, from the radial wave function corresponding to the original potential, Eq. (2.1), namely,

$$u(r)=r\exp[-g^{1/2}/r], \tag{2.5}$$

we find for the scattering length the well-known result

$$A=-g^{1/2}. \tag{2.6}$$

Thus the scattering length is negative, as was to be expected, and it has the opposite value to that obtained through the regulated potential, Eq. (2.4).

It is also interesting to investigate what happens to the radial wave function $u(r,\alpha)$ as α vanishes. We find

$$u(r,\alpha)\xrightarrow[\alpha\to 0]{} C(\alpha)r\exp[g^{1/2}/r], \tag{2.7}$$

with

$$C(\alpha)=\exp(-g^{1/2}/\alpha). \tag{2.8}$$

Thus $u(r,\alpha)$ becomes the *irregular* solution of the (zero-energy S-wave) radial Schrödinger equation

$$u''(r)=V(r)u(r), \tag{2.9}$$

and it is multiplied by a constant factor which vanishes exponentially as α vanishes.

3. A GENERALIZATION OF THE PRECEDING EXAMPLE

In this section we consider a slightly more general family of regulated potentials, namely,

$$V(r,\alpha)=gr^{-4}[\exp(-2\alpha/r)-(2n+1)\alpha g^{-1/2}\exp(-\alpha/r)]. \tag{3.1}$$

For $\alpha=0$ these potentials reduce to that given in Eq. (2.1). The case treated in the preceding section corresponds to $n=0$.

The radial wave function corresponding to this potential is now (see the Appendix)

$$u(r,\alpha)=r\exp[-g^{1/2}\alpha^{-1}e^{-r/\alpha}]\times\Phi(-n,1;2\alpha^{-1}g^{1/2}e^{-\alpha/r}), \tag{3.2}$$

where $\Phi(a,c;z)$ is the confluent hypergeometric function.[7] From this we obtain (see the Appendix) the scattering length

$$A(\alpha)=g^{1/2}\left\{1-2\frac{\Phi'(-n,1;2\alpha^{-1}g^{1/2})}{\Phi(-n,1;2\alpha^{-1}g^{1/2})}\right\}. \tag{3.3}$$

Note that, for a general value of n, the function $A(\alpha)$ now does depend on α.

We then investigate the limit of vanishing α. If n is a non-negative integer, $\Phi(-n,1;z)$ becomes a (Laguerre) polynomial of degree n, so that

$$\Phi(-n,1;z)\underset{z\to\infty}{\rightarrow}\text{const}\times z^n$$

and the ratio Φ'/Φ in Eq. (3.3) vanishes proportionally to α. On the other hand, if n is not a non-negative integer, $\Phi(-n,1;z)\underset{z\to\infty}{\rightarrow}\text{const}\times z^{-n-1}e^z$ and therefore the ratio Φ'/Φ in Eq. (3.3) tends to unity as α vanishes. Thus we find

$$\lim_{\alpha\to 0}A(\alpha)=g^{1/2}\quad\text{if}\quad n=0,1,2,3\cdots, \tag{3.4a}$$

$$\lim_{\alpha\to 0}A(\alpha)=-g^{1/2}\quad\text{if}\quad n\neq 0,1,2,3\cdots. \tag{3.4b}$$

In conclusion we see that, unless n is a non-negative integer, the scattering length $A(\alpha)$ tends to the correct scattering length A [Eq. (2.6)]. If instead n is a

[7] Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I. The equations of this volume will be referred to by a capital B followed by their number.

non-negative integer, the scattering length $A(\alpha)$, corresponding to the regulated potential $V(r,\alpha)$, Eq. (3.1), tends to the negative of the correct value as α vanishes.

Finally we discuss the limit of the radial wave function as α vanishes. We find

$$u(r,\alpha) \underset{\alpha\to 0}{\longrightarrow} C_n(\alpha) r \exp(g^{1/2}/r)\,, \qquad n=0,1,2\cdots, \tag{3.5a}$$

$$u(r,\alpha) \underset{\alpha\to 0}{\longrightarrow} \bar{C}_n(\alpha) r \exp(-g^{1/2}/r)\,, \quad n\neq 0,1,2\cdots, \tag{3.5b}$$

with

$$C_n(\alpha)=C_n\alpha^{-n}\exp(-g^{1/2}/\alpha)\,, \tag{3.6a}$$

$$\bar{C}_n(\alpha)=\bar{C}_n\alpha^{n+1}\exp(g^{1/2}/\alpha)\,, \tag{3.6b}$$

and C_n, $\bar{C}_n$ independent of α. We thus see that in all cases $u(r,\alpha)$ becomes, as α vanishes, a solution of the radial Schrödinger Eq. (2.9). However, if n is a non-negative integer, it becomes the irregular solution, multiplied by a constant which vanishes as α tends to zero; otherwise it becomes the regular solution, multiplied by a constant which diverges as α vanishes.

4. CRITIQUE OF THE KHURI-PAIS THEOREM

In this section we discuss the theorem given by Khuri and Pais,[1] which states that, independently of the type of regularization (within a certain class), $A(\alpha)$ converges to A as α vanishes. We refer in this discussion to the simple example of Sec. 2. We are aware that this example does not fall within the class of regularized potentials considered by Khuri and Pais and that therefore it does not provide a direct counter-example to their theorem. But we use this example only to pinpoint the weak spot in the proof of the theorem. As we show below, this lies in the exchange of two limiting processes. While we demonstrate explicitly that such an exchange is not permissible in the case of our example, it is generally true (including the cases considered by Khuri and Pais) that the exchange is unjustified. Thus the proof of the theorem is invalid, although of course its thesis may still hold in certain cases (in fact, in most cases).

First of all let us introduce two (properly normalized) independent solutions of the (S-wave zero-energy) radial Schrödinger equation with potential $V(r,\alpha)$, Eq. (2.2).

These are (see the Appendix)

$$\varphi_1(r,\alpha)=r\exp[(g^{1/2}/\alpha)(1-e^{-\alpha/r})]\,, \tag{4.1a}$$

$$\varphi_2(r,\alpha)=(r/\alpha)\exp[-(g^{1/2}/\alpha)(1+e^{-\alpha/r})]E(2x)\,, \tag{4.1b}$$

with

$$E(z)=\int_z^{2x_\infty} dt\, e^t/t\,, \tag{4.2}$$

and

$$x=(g^{1/2}/\alpha)e^{-\alpha/r}\,, \tag{4.3a}$$

$$x_\infty=g^{1/2}/\alpha. \tag{4.3b}$$

The first solution $\varphi_1(r,\alpha)$ is the radial wave function, normalized so that its asymptotic behavior is

$$\varphi_1(r,\alpha)=r+g^{1/2}+O(1/r). \tag{4.4a}$$

The second solution does not vanish at the origin and is chosen so that it converges asymptotically to unity:

$$\varphi_2(r,\alpha)=1+O(1/r)\,, \tag{4.4b}$$

as may be easily verified by expanding Eq. (4.2) with $z=2x_\infty-\epsilon$ as ϵ vanishes.

It should be emphasized that all these properties refer to the case $\alpha>0$. On the other hand in the limit of vanishing α we find

$$\varphi_1(r,0)=r\exp(g^{1/2}/r)\,, \tag{4.5a}$$

$$\varphi_2(r,0)=(4g)^{-1/2}r[\exp(g^{1/2}/r)-\exp(-g^{1/2}/r)]. \tag{4.5b}$$

The second equation is obtained by expanding $E(2x)$ as x_∞ and x diverge while their difference tends to $g^{1/2}/r$. The expansion is performed by partial integration.

We also indicate explicitly, for later reference, the behavior as r vanishes. We find

$$\varphi_1(r,\alpha)=r\exp(g^{1/2}/\alpha)[1+O(r)]\,, \tag{4.6}$$

and

$$\varphi_2(r,\alpha)=\exp(-g^{1/2}/\alpha)[1+O(r)]. \tag{4.7}$$

The second equation is obtained expanding $E(2x)$ in the neighborhood of $x=0$.

We now construct the functions Ψ's introduced by Khuri and Pais.[1] They are

$$\Psi(r;\alpha)=\Psi_1(r;\alpha)+\Psi_2(r;\alpha)\,, \tag{4.8a}$$

$$\Psi_1(r;\alpha)=A(\alpha)/r\,, \tag{4.8b}$$

$$\Psi_2^{(1)}(r;\alpha)=[\varphi_1(r,\alpha)-g^{1/2}\varphi_2(r,\alpha)]/r\,, \tag{4.8c}$$

$$(1/r)+\Psi_2^{(2)}(r;\alpha)=\varphi_2(r,\alpha)/r. \tag{4.8d}$$

Let us then examine the definition of the scattering length $A(\alpha)$ given by Khuri and Pais.[1] One definition is given in Eq. (2.17) of their paper, and by an exchange in the priority of the two limiting processes $\alpha\to 0$, $\sigma\to 0$,[8] their Eq. (2.18) is derived. This exchange is not justified, and it is the weak point in the proof, as we show below. However, rather than discuss these equations, which require some integrations, we consider an equivalent definition of the scattering length, which is also proposed by Khuri and Pais.[1] It follows from their Eq. (2.12), namely,

$$\Psi_2(r;\alpha)=\Psi_2^{(1)}(r;\alpha)+A(\alpha)\Psi_2^{(2)}(r;\alpha)\,, \tag{4.9}$$

and from the requirement that $\Psi(r;\alpha)$, Eq. (4.8a), vanish in the origin. We thus find[8]

$$A(\alpha)=g^{1/2}-\lim_{\sigma\to 0}[\varphi_1(\sigma,\alpha)/\varphi_2(\sigma,\alpha)]. \tag{4.10}$$

[8] We write σ in place of r to adhere to the notation of Khuri and Pais.

The question at issue is now the possibility of exchanging the limits $\alpha \to 0$ and $\sigma \to 0$. This is clearly forbidden, because using the explicit expressions previously given, we find

$$\lim_{\alpha\to 0}\lim_{\sigma\to 0}[\varphi_1(\sigma,\alpha)/\varphi_2(\sigma,\alpha)]=0\,, \tag{4.11a}$$

$$\lim_{\sigma\to 0}\lim_{\alpha\to 0}[\varphi_1(\sigma,\alpha)/\varphi_2(\sigma,\alpha)]=2g^{1/2}. \tag{4.11b}$$

This result, which of course agrees with those of Sec. 2, settles the question of the breakdown of the Khuri-Pais proof.[1]

5. CONCLUDING REMARKS

We have shown, on the basis of explicit examples, that the procedure of first regularizing a singular potential, subsequently computing the physical quantities apposite to the regularized potential, and finally taking their limit as the regulating parameter vanishes, may lead to an incorrect result, namely, a result different from that corresponding to the original (unregularized) potential. This finding may be considered to shed some doubt on the validity of the Peratization approach to field theory. Actually the doubt it raises is more fundamental, because it refers to the regularization procedure, which is preliminary to Peratization. However, an overly pessimistic attitude should not be inferred, in our opinion, from the present results. For one thing, as emphasized by Khuri and Pais,[1] there is a major difference between a regulated potential and a regulated field theory, in that a regulated potential is still a *bona fide* potential, while a field theory regulated by means of a momentum-space cutoff is not a *bona fide* (i.e., relativistically invariant, local, probability-conserving) field theory. Moreover we have shown that in the potential case things *may* go the wrong way, but they need not do so.

It is in fact interesting to speculate on the underlying reasons for the failure of the regularization procedure to yield the correct result in some cases. One hint which may be enlightening is the observation that in the examples considered the regulated potential contained an attractive part. A physically reasonable conjecture would be to associate the breakdown with regulated potentials which are attractive at short distances. On the other hand, the example of Sec. 3 shows clearly that any attempt at a detailed physical understanding of the reasons for the anomalous behavior is doomed to failure, as there does not appear to be any possibility of ascribing a physical significance to the special values $n=0, 1, 2, 3\cdots$, of the parameter n entering in the definition of the regulated potential. Even aside from the question of the *physical* understanding of the breakdown of the regularization method, these examples also suggest that the purely *mathematical* problem of giving conditions on the unregularized and regularized potentials sufficient *and necessary* for the success of the regularization approach has no simple solution.

We also wish to comment on the question of Peratization itself. In the Introduction we stated that presumably a precondition for its success is the fact that the regularization procedure employed be itself successful. The noncommittal adverb "presumably" was employed to allow for the (very unlikely) possibility that, even when regularization fails, Peratization might work; or, more explicitly, that even when $A(\alpha)$ does not converge to A as α vanishes, A_p (defined as in Sec. 1) does coincide with A, or at least provides a good approximation to it. It is however trivial to check that this is not the case in the example of Sec. 2 (although in that case the use of a regularized potential which is nonlinear in the coupling constant is outside of the usual rules of the Peratization program).

Finally we mention the possibility of manufacturing, using the results of the Appendix, regularized potentials which are of the type considered by Khuri and Pais[1] and which might therefore provide direct counterexamples to their theorem. In fact, with the choice

$$g_2=g(1-y)\,,\quad g_1=gy\,, \tag{5.1}$$

where y is an arbitrary real constant, Eq. (A1) provides such a regularized version of the potential Eq. (2.1). Similarly the choice

$$g_2=-g/\alpha\,,\quad g_1=g/\alpha \tag{5.2}$$

yields a regularized version of the potential gr^{-5} which is also in the class considered by Khuri and Pais.[1] However, while the scattering lengths $A(\alpha)$ corresponding to these potentials may be obtained immediately from Eq. (A10), a complete discussion of their limit as α vanishes requires a more detailed analysis than we were able to find in the literature of the behavior of the confluent hypergeometric function $\Phi(a,c;z)$ as both a and z diverge.[9] On the basis of the preceding remarks we expect that the anomalous behavior, if it is present at all with these potentials, occurs only in the first case and only when y is negative.

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APPENDIX

In this Appendix we discuss the (S-wave zero-energy) radial wave function and the scattering length in the presence of the regular potential

$$V(r)=r^{-4}[g_2\exp(-2\alpha/r)+g_1\exp(-\alpha/r)]\,,\ \alpha>0\,. \tag{A1}$$

The radial wave function is that solution of the radial

[9] It is presumably always true, however, that the regularized function $u(r,\alpha)$ becomes, as α vanishes, a solution of the Schrödinger equation corresponding to the unregularized potential. However, only if it becomes the regular solution (i.e., vanishing in the origin) may we expect the regularization procedure to be successful.

Schrödinger equation

$$u''(r)=V(r)u(r) \tag{A2}$$

characterized (up to a normalization constant) by the boundary condition

$$u(0)=0. \tag{A3}$$

As may be verified by direct substitution, we have

$$u(r)=re^{-x}\Phi(\tfrac{1}{2}-k,\,1\,;2x)\,, \tag{A4}$$

with

$$x=(g_2^{1/2}/\alpha)\exp(-r/\alpha) \tag{A5}$$

and

$$k=-g_1g_2^{-1/2}/(2\alpha)\,. \tag{A6}$$

$\Phi(a,c\,;z)$ is the confluent hypergeometric function.[7]

A second independent solution of the radial Schrödinger equation (A2) [but not of Eq. (A3)] may be easily obtained, if $k\neq\frac{1}{2}$, substituting in Eq. (A4) the function $\Psi(\frac{1}{2}-k,\,1\,;2x)$, defined by Eq. (B6.7(13)), in place of the function $\Phi(\frac{1}{2}-k,\,1\,;2x)$. In the special case $k=\frac{1}{2}$ (which corresponds to the example discussed in Secs. 2 and 4) a second solution may be obtained by considering directly the equation for the confluent hypergeometric function, Eq. (B6.1(2)), in the special case $a=0$, $c=1$. We find in this case that a second solution of the radial Schrödinger equation is

$$w(r)=re^{-x}E(2x)\,, \tag{A7}$$

where

$$E(z)=\int^z dt\,e^t/t\,. \tag{A8}$$

These functions are used in the discussion of Sec. 4.

The scattering length A is defined by the asymptotic behavior of the radial wave function $u(r)$, Eq. (A4), through

$$u(r)\underset{r\to\infty}{\longrightarrow}\text{const}\times[r+A+O(1/r)]\,. \tag{A9}$$

We obtain

$$A=-g_2^{1/2}\left\{2\frac{\Phi'(\frac{1}{2}-k,\,1\,;2x_\infty)}{\phi(\frac{1}{2}-k,\,1\,;2x_\infty)}-1\right\}, \tag{A10}$$

with

$$x_\infty=g_2^{1/2}/\alpha\,. \tag{A11}$$

Here, as everywhere else, the prime denotes differentiation with respect to the last argument.

Note that, because the potential Eq. (A1) is regular, the scattering length A must be an analytic function of both coupling constants g_i, $i=1$, 2, in the neighborhood of $g_i=0$. This is evident by inspection as regards the dependence upon g_1, because $\Phi(a,c\,;z)$ is an entire function of a. Since $\Phi(a,c\,;z)$ is also analytic in z, it is also evident that A is analytic in $g_2^{1/2}$, except possibly for an essential singularity due to the fact that as g_2 vanishes k diverges. It is however easily seen that this essential singularity cancels out in the ratio Φ'/Φ, and in fact using Eq. (B6.9.1 (18)) one recovers in this limit the correct scattering length due to the potential $V(r)=g_1r^{-4}\exp(-\alpha/r)$. [Actually a simpler way to obtain the quantities relevant to this case is to set $g_1=0$, which implies $k=0$, and use Eq. (B6.9.1(10)) to transform the confluent hypergeometric function into a Bessel function; and then perform the substitution $g_2\to g_1$, $\alpha\to\alpha/2$.]

It remains to be shown that A is actually analytic in g_2 and not only in $g_2^{1/2}$. To do this it is sufficient to show that A is an even function of $g^{1/2}$. In fact, using the Kummer transformation Eq. (B6.3(7)), one may rewrite the equation for the scattering length in the form

$$A=-g_2^{1/2}\left\{\frac{\Phi'(\frac{1}{2}-k,\,1\,;2x_\infty)}{\Phi(\frac{1}{2}-k,\,1\,;2x_\infty)}-\frac{\Phi'(\frac{1}{2}+k,\,1\,;-2x_\infty)}{\Phi(\frac{1}{2}+k,\,1\,;-2x_\infty)}\right\}, \tag{A12}$$

which displays this property.

Finally we mention that, if we consider the constants $g_1(\alpha)$ and $g_2(\alpha)$ as functions of α, then the potential Eq. (A1) provides a regularized version of the potential Eq. (2.1) provided the condition

$$g_1(0)+g_2(0)=g \tag{A13}$$

holds. The cases studied in this paper are of this type. Another possibility is the choice of two functions $g_1(\alpha)$, $g_2(\alpha)$ with the properties

$$g_1(\alpha)+g_2(\alpha)\underset{\alpha\to 0}{\longrightarrow}g\,, \tag{A14a}$$

$$g_1(\alpha)+2g_2(\alpha)\underset{\alpha\to 0}{\longrightarrow}-f/\alpha\,, \tag{A14b}$$

for in such a case the potential (A1) provides a regularized version of the potential

$$V(r)=gr^{-4}+fr^{-5}. \tag{A15}$$