Comments on Relativistic Supermultiplet Theories*

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It is proved that no well-behaved group of unitary operators on physical Hilbert space can have irreducible representations containing all spin states of the particles involved. It is shown that the recently proposed $U(12)$ symmetry evades this theorem by acting on the "generalized M function," rather than on the physical Hilbert space. Some other possible evasions are also discussed.

I. INTRODUCTION

DESPITE their apparent empirical success, the $SU(6)$ symmetry[1] and its offspring have evoked intense unpopularity. The trouble is generally ascribed to an incompatibility with Lorentz invariance, though there seems as yet to be no general agreement as to precisely where this difficulty lies.[2] Meanwhile, the Trieste[3] and Rockefeller Institute[4] groups have bravely developed a $U(12)$ symmetry which they claim to be perfectly well defined and compatible with Lorentz invariance. The purpose of the present note, written by an outsider, is to make this situation understandable to other outsiders, and perhaps even to insiders as well. I offer a very simple proof, that it is impossible to extend $SU(6)$ to a decent group of unitary operators on the physical Hilbert space (much less on quantum fields). However, this does not contradict the $U(12)$ work, because *the $U(12)$ symmetries do not act on physical particle states*, but rather on the purely formal indices on what may be called a "generalized M function." The recent realization of a conflict with unitarity seems to close off this loophole, thus forcing us to return to the search for the meaning of $SU(6)$. Some possible paths for this search are listed in Sec. V.

II. A PESSIMISTIC THEOREM

I assume that the group of all physical symmetries which commute with the four-momentum operators P^μ is the direct product of the translation group and some other group G. The latter group shares no members with the Poincaré group, and until recently we would all have assumed G to consist solely of intrinsic symmetries like $SU(3)$. In this case the physical irreducible representations of G would consist of multiplets of states with fixed momentum $\mathbf{p}$, fixed spin j, and fixed-spin z component σ.

It is the distinguishing and exciting feature of the new symmetries of the $SU(6)$ type that they act on spin as well as isospin, so that an irreducible representation of G consists of a supermultiplet of states $|\mathbf{p},n,\sigma\rangle$, where $\mathbf{p}$ is a fixed momentum, n runs over a set of particle names, and σ runs for each n over *all* the $2j_n+1$ possible values of the spin z component. (The supermultiplet may or may not include more than one value of j_n.) For $g\epsilon G$ we then have

$$g|\mathbf{p},n,\sigma\rangle=\sum_{n'\sigma'}\mathcal{G}_{n'\sigma',n\sigma}(g;\mathbf{p})|\mathbf{p},n',\sigma'\rangle, \tag{1}$$

with $\mathcal{G}$, an irreducible set of unitary matrices, satisfying the group property

$$\mathcal{G}(g_1;\mathbf{p})\mathcal{G}(g_2;\mathbf{p})=\mathcal{G}(g_1g_2;\mathbf{p}). \tag{2}$$

I will also tentatively assume that a given supermultiplet transforms under G according to equivalent representations for all different $\mathbf{p}$, so that there is a unitary matrix $\mathcal{B}(\mathbf{p})$ with

$$\mathcal{G}(g;\mathbf{p})=\mathcal{B}^{-1}(\mathbf{p})\mathcal{G}(g;0)\mathcal{B}(\mathbf{p}) \tag{3}$$

$$\mathcal{B}(0)\equiv 1. \tag{4}$$

The possibility of relaxing this assumption is discussed in Sec. V.

It will now be shown that the above assumptions (and in particular the irreducibility of $\mathcal{G}$) are inconsistent with Lorentz invariance, except of course in the trivial case where all particles have spin $j_n=0$. The Lorentz transformation properties of the states $|\mathbf{p},n,\sigma\rangle$ have been completely described by Wigner[5]; if Λ is the unitary operator corresponding to a Lorentz transformation $x^\mu\rightarrow\Lambda^\mu{}_\nu x^\nu$, then[6]

$$\Lambda|\mathbf{p},n,\sigma\rangle=\sum_{\sigma'}D_{\sigma'\sigma}{}^{(j_n)}[W(\Lambda,\mathbf{p})]|\mathbf{p},n,\sigma'\rangle. \tag{5}$$

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[1] F. Gürsey and L. A. Radicati, Phys. Rev. Letters **13**, 173 (1964); A. Pais, *ibid.* **13**, 175 (1964); B. Sakita, Phys. Rev. **136**, B1756 (1964); M. A. Bég, B. W. Lee, and A. Pais, Phys. Rev. Letters **13**, 514 (1964); K. Bardakci, J. M. Cornwall, P. G. O. Freund, and B. W. Lee, *ibid.* **13**, 698 (1964). The above is a random assortment of references, and can be supplemented by referring to the back pages of any recent issue of Phys. Rev. Letters.

[2] In this connection, see especially S. Coleman, Phys. Rev. **138**, B1262 (1965); L. Michel and B. Sakita, Ann. Inst. Henri Poincaré (to be published); L. Michel, Phys. Rev. **137**, B405 (1965); and Thomas F. Jordan (to be published). It is quite possible that the theorem proved here in Sec. II is contained in the results of any or all of these references, but I feel that the arguments given here are particularly transparent.

[3] R. Delbourgo, Abdus Salam, and J. Strathdee, Phys. Rev. **138**, B420 (1965).

[4] M. A. B. Bég and A. Pais, Phys. Rev. Letters **14**, 267 (1965); see also B. Sakita and K. Wali, *ibid.* **14**, 404 (1965).

[5] E. P. Wigner, Ann. Math. **40**, 149 (1939).

[6] The states $|\mathbf{p},n,\sigma\rangle$ will be normalized covariantly, so that a factor $\omega(\mathbf{p})$ appears in the scalar product but not in the transformation rule (5).

Here $D^{(j)}$ is the familiar $(2j+1)$-dimensional unitary representation of the rotation group, and $W(\Lambda,\mathbf{p})$ is the "Wigner rotation[5]" [see Eq. (28) below]. It will be convenient to write (5) as a transformation of the supermultiplet

$$\Lambda|\mathbf{p},n,\sigma\rangle=\sum_{n'\sigma'}\mathcal{L}_{n'\sigma',n\sigma}(\Lambda;\mathbf{p})|\Lambda\mathbf{p},n',\sigma'\rangle \tag{6}$$

with $\mathcal{L}$ the unitary matrix

$$\mathcal{L}_{n',\sigma',n\sigma}(\Lambda;\mathbf{p})\equiv\delta_{n'n}D_{\sigma'\sigma}{}^{(j_n)}[W(\Lambda,\mathbf{p})]. \tag{7}$$

If g is in G then so is $\Lambda g\Lambda^{-1}$, and we can readily calculate from (1) and (6) that

$$\mathcal{G}(\Lambda^{-1}g\Lambda;\mathbf{p})=\mathcal{L}^{-1}(\Lambda,\mathbf{p})\mathcal{G}(g;\Lambda\mathbf{p})\mathcal{L}(\Lambda,\mathbf{p}). \tag{8}$$

(Incidentally, this implies that $\mathcal{G}$ cannot be $\mathbf{p}$-independent, though I will not show this here.) Using (3) in (8) then gives

$$\mathcal{G}(\Lambda^{-1}g\Lambda;0)=\mathcal{E}^{-1}(\Lambda,\mathbf{p})\mathcal{G}(g;0)\mathcal{E}(\Lambda,\mathbf{p}), \tag{9}$$

where

$$\mathcal{E}(\Lambda,\mathbf{p})\equiv\mathcal{B}(\Lambda\mathbf{p})\mathcal{L}(\Lambda,\mathbf{p})\mathcal{B}^{-1}(\mathbf{p}). \tag{10}$$

But (9) implies that $\mathcal{E}^{-1}(\Lambda_1\Lambda_2,\mathbf{p})\mathcal{E}(\Lambda_1,\mathbf{p})\mathcal{E}(\Lambda_2,\mathbf{p})$ commutes with $\mathcal{G}(g;0)$ for all g, and since the $\mathcal{G}(g;0)$ form an irreducible set, this matrix must be equal to a numerical factor η times the unit matrix, or

$$\mathcal{E}(\Lambda_1,\mathbf{p})\mathcal{E}(\Lambda_2,\mathbf{p})=\eta(\Lambda_1,\Lambda_2,\mathbf{p})\mathcal{E}(\Lambda_1\Lambda_2,\mathbf{p}). \tag{11}$$

Also, (10) shows that $\mathcal{E}$ is unitary, so

$$|\eta(\Lambda_1,\Lambda_2,\mathbf{p})|=1. \tag{12}$$

[In the same way we could show that $\mathcal{E}(\Lambda,\mathbf{p})$ is $\mathbf{p}$-independent, up to a phase.] Hence we have constructed from $\mathcal{B}$ and $\mathcal{L}$ a unitary, finite-dimensional ray representation $\mathcal{E}(\Lambda;\mathbf{p})$ of the homogeneous Lorentz group. *But there are not any such representations*, so our assumptions are inconsistent with Lorentz invariance.

Strictly speaking, there is *one* unitary finite-dimensional ray representation of the homogeneous Lorentz group, namely, the identity

$$\mathcal{E}(\Lambda,\mathbf{p})\equiv 1. \tag{13}$$

But it is easy to see that in this case all particles in the supermultiplet would have zero spin, for setting $\mathbf{p}=0$ and Λ equal to a rotation R in Eq. (10), we find in general that

$$\mathcal{E}_{n'\sigma',n\sigma}(R,0)=\mathcal{L}_{n,\sigma',n\sigma}(R,0)=\delta_{n'n}D_{\sigma'\sigma}{}^{(j_n)}[R]. \tag{14}$$

With all particles spinless, our group G reduces to an ordinary group of intrinsic symmetries.

It is perhaps worth repeating that an ordinary group like $SU(3)$ does not lead to the contradiction found here, because a set of states with different spin z components would not furnish an irreducible representation of such a group, so that (11) could not be deduced from (9).

III. FIELD THEORY

As an example of what goes wrong when we try to impose a symmetry of the type discussed in Sec. II, let us consider a theory containing a Heisenberg representation field $\psi_{ab}(x)$ which transforms according to the (A,B) representation of the homogeneous Lorentz group[7]:

$$\Lambda\psi_{ab}(x)\Lambda^{-1}=\mathfrak{D}_{a'b',ab}{}^{(A,B)}(\Lambda)\psi_{a'b'}(\Lambda x). \tag{15}$$

The indices a, a' and b, b' run by unit steps from $-A$ to $+A$ and $-B$ to $+B$. (For instance, if $A=\frac{1}{2}$, $B=0$ or $A=0$, $B=\frac{1}{2}$ then ψ is the part of the Dirac field with $\gamma_5=+1$ or $\gamma_5=-1$.) These are the most general irreducible field transformation laws.

Suppose there exists a symmetry g which acts on the spin components of ψ, but not on x:

$$g^{-1}\psi_{ab}(x)g=\Gamma_{ab,a'b'}\psi_{a'b'}(x). \tag{16}$$

[We omit $SU(3)$ indices.] It is then possible to compute precisely what g does to one-particle states. For Lorentz invariance and Eq. (15) tell us that

$$\langle 0|\psi_{ab}(x)|\mathbf{p},\lambda\rangle=N\sum_{a'b'}C_{AB}(j\lambda,a'b')\left(\frac{p+m}{\omega}\right)^{-a'}\left(\frac{p+m}{\omega}\right)^{b'}\times D_{aa'}{}^{(A)}[R(\hat{p})]D_{bb'}{}^{(B)}[R(\hat{p})]e^{ip\cdot x}. \tag{17}$$

Here $|0\rangle$ is the exact vacuum, $|\mathbf{p},\lambda\rangle$ the exact one-particle state with momentum $\mathbf{p}$, spin j, and helicity λ, N a normalization constant, and C_{AB} the usual Clebsch-Gordan coefficient; $p=|\mathbf{p}|$, $\omega=(p^2+m^2)^{1/2}$, and $D^{(A)}$ as before is the ordinary $(2A+1)\times(2A+1)$ rotation matrix, with $R(\hat{p})$ the rotation that takes the z axis into the direction of $\mathbf{p}$. Since g leaves x unaltered it must take $|\mathbf{p},\lambda\rangle$ into a state of equal energy and momentum, which for a stable particle can be at most a linear combination of helicity states

$$g|\mathbf{p},\lambda\rangle=\sum_{\lambda'}\mathcal{G}_{\lambda'\lambda}(\mathbf{p})|\mathbf{p},\lambda'\rangle. \tag{18}$$

Similarly, g must leave the vacuum invariant. Inserting (18) and (16) in (17) allows us to solve for $\mathcal{G}$, and we find that

$$\begin{aligned}\mathcal{G}_{\lambda'\lambda}(\mathbf{p})=&\sum_{ab}\sum_{a'b'}\sum_{a''b''}\sum_{a'''b'''}C_{AB}(j\lambda',a'''b''')\\&\times[(p+m)/\omega]^{a'''}[(p+m)/\omega]^{-b'''}\\&\times D_{a'''a''}{}^{(A)}[R^{-1}(\hat{p})]D_{b'''b''}{}^{(B)}[R^{-1}(\hat{p})]\Gamma_{a''b'',a'b'}\\&\times D_{a'a}{}^{(A)}[R(\hat{p})]D_{b'b}{}^{(B)}[R(\hat{p})][(p+m)/\omega]^{-a}\\&\times[(p+m)/\omega]^{b}C_{AB}(j\lambda,ab).\end{aligned} \tag{19}$$

[7] The free fields with these transformation rules are constructed in Sec. VIII of S. Weinberg, Phys. Rev. **133**, B1318 (1964), and more explicitly in Vol. II of the lecture notes of the 1964 Brandeis Summer Institute of Theoretical Physics (to be published).

If g is a unitary operator then the matrices $\mathcal{G}_{\lambda'\lambda}$ and $\Gamma_{a''b'',a'b'}$ will both have to be unitary, but the presence of the factors $(p+m)/\omega$ in (19) makes this impossible (for Γ nontrivial). We see very clearly that there is no trouble in the nonrelativistic case $p\ll m$, where $(p+m)/\omega \simeq 1$, but that as we pass into the relativistic regime we lose the unitarity of our operator g.

However, the trouble here is not merely an inconsistency with the theory of free or interacting quantum fields (as has been sometimes claimed). Rather, as we have shown, it is simply impossible to interpret any symmetry that yields supermultiplets as a group of unitary operators on physical Hilbert space.

IV. THE 144-FOLD WAY OUT

The theorem of Sec. II can also be interpreted as saying that a Lorentz-invariant S matrix cannot be invariant under any group of unitary matrices of the supermultiplet type, i.e., whose irreducible pieces connect all spin states of the particles on which they act. But this theorem can be evaded by passing from the S matrix to a less directly physical object, the "generalized M function," which can be roughly described as the S matrix with its external-line wave functions omitted. In fact, this seems to be the path taken by Beg and Pais[4] and by Delbourgo, Salam, and Strathdee.[3] Before discussing their symmetry, it will be necessary to provide a short general explanation of how (and why) we introduce the M function.

To define the generalized M function[8] in terms of the S matrix, the first step is to associate with each supermultiplet of *one-particle states* a representation $\mathfrak{D}_{N'N}(\Lambda)$ (usually nonunitary and reducible) of the homogeneous Lorentz group. The indices N do not directly correspond to particles in the supermultiplet; rather there are usually more indices than particles, and a particle n at rest with $J_z=\sigma$ will be associated with a "wave function" $u_N(n\sigma)$. The only immediate requirement on these wave functions is that when Λ is restricted to an ordinary rotation R they must satisfy the transformation law

$$\sum_N \mathfrak{D}_{NN'}(R)u_{N'}(n\sigma)=\sum_{\sigma'} u_N(n\sigma')D_{\sigma'\sigma}{}^{(j_n)}(R)\,, \quad (20)$$

where $D^{(j)}$ is, as before, the ordinary $(2j+1)\times(2j+1)$ rotation matrix. This condition tells us that for Λ restricted to the rotation group, $\mathfrak{D}(\Lambda)$ breaks up into irreducible pieces, each corresponding to one particular particle type n, and characterized by spin j_n. The "wave function in flight" for a particle n with $J_z=\sigma$ and momentum $\mathbf{p}$ can then be defined by

$$u_N(\mathbf{p}n\sigma)\equiv\sum_{N'} \mathfrak{D}_{NN'}(L(\mathbf{p}))u_{N'}(n\sigma)\,, \quad (21)$$

where $L(\mathbf{p})$ is the pure Lorentz transformation[9] that takes our particle from rest to momentum $\mathbf{p}$.

For example, the supermultiplet of quarks could be associated with a representation $\mathfrak{D}_{N'N}(\Lambda)$ defined by splitting the N index into a Dirac index α and a $SU(3)$ index n, with

$$\mathfrak{D}_{\alpha'n',\alpha n}(\Lambda)\equiv \mathcal{S}_{\alpha'\alpha}(\Lambda)\delta_{n'n}\,, \quad (22)$$

where $\mathcal{S}_{\alpha'\alpha}$ is the familiar four-dimensional $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ Dirac representation of the homogeneous Lorentz group. In this case we should find that

$$u_{\alpha n'}(\mathbf{p}n\sigma)=\delta_{n'n}u_\alpha(\mathbf{p}\sigma)\,, \quad (23)$$

where $u_\alpha(\mathbf{p}\sigma)$ is nothing but the usual Dirac wave function in momentum space, satisfying

$$(ip^\mu\gamma_\mu+m)u(\mathbf{p},\sigma)=0\,. \quad (24)$$

(Usually the wave functions are far more complicated than this.) It should be emphasized, both for this example and in general, that the wave functions $u_N(\mathbf{p}n\sigma)$ are *not* wave functions in the sense of representatives of a state in physical Hilbert space. They are purely formal objects, whose sole purpose in physics is to allow us to define free fields or M functions. (For complicated reasons it is possible to find the hydrogen fine structure by playing with Dirac wave functions as if they actually were state vectors, but this is an accident, and a misleading one.)

The generalized M function can now be introduced by the statement that the connected part of the S matrix takes the form

$$\begin{aligned}\langle \mathbf{p}_1n_1\sigma_1,\cdots|S|\mathbf{p}_2n_2\sigma_2,\cdots\rangle &= \sum_{N_1N_2\cdots} u_{N_1}{}^*(\mathbf{p}_1n_1\sigma_1)\cdots u_{N_2}(\mathbf{p}_2n_2\sigma_2)\cdots \\ &\quad\times M_{N_1\cdots,N_2\cdots}(\mathbf{p}_1\cdots,\mathbf{p}_2\cdots)\,. \quad (25)\end{aligned}$$

The . . . indicate that there may be other particles besides particle 1 in the final state, and other particles besides particle 2 in the initial state. The S matrix would naturally take the form (25) if we constructed the interaction out of free fields

$$\psi_N(x)=\sum_{n\sigma}\int d^3p\{a(\mathbf{p},n,\sigma)e^{ip\cdot x}u_N(\mathbf{p},n,\sigma) +\text{creation terms}\}\,. \quad (26)$$

In this case the M function would be the Fourier transform of the true-vacuum expectation value of the time-ordered product of Heisenberg representation fields ψ_{N_1}, $\psi_{N_2}{}^\dagger$, etc. But whether or not we believe in field theory (or even in supermultiplets, for that matter) the introduction of the M function in Eq. (25) is of great importance, because it allows us to state Lorentz

[8] M functions were introduced by H. Stapp, Phys. Rev. **125**, 2139 (1962), using the representation $(\frac{1}{2},0)$ for spin $j=\frac{1}{2}$. See also A. O. Barut, I. Muzinich, and D. N. Williams, Phys. Rev. **130**, 442 (1963).

[9] As far as I know, the word "boost" was first used with this meaning by A. S. Wightman in lectures at Princeton before 1957. But I could be wrong.

invariance as the invariance of M under a group of momentum-independent matrices:

$$M_{N_1\cdots,N_2\cdots}(\mathbf{p}_1\cdots,\mathbf{p}_2\cdots)=\sum_{N_1'N_2'\cdots}\mathfrak{D}_{N'_1N_1}{}^{(1)*}(\Lambda)\cdots\mathfrak{D}_{N_2'N_2}{}^{(2)}(\Lambda)\cdots\times M_{N_1'\cdots,N_2'\cdots}(\Lambda\mathbf{p}_1\cdots,\Lambda\mathbf{p}_2\cdots). \quad (27)$$

For we note from (21) that

$$\sum_N \mathfrak{D}_{N'N}(\Lambda)u_N(\mathbf{p}n\sigma)=\sum_{NN''}\mathfrak{D}_{N'N''}(L(\Lambda\mathbf{p}))\mathfrak{D}_{N''N}(L^{-1}(\Lambda\mathbf{p})\Lambda L(\mathbf{p}))u_N(n\sigma).$$

But $L^{-1}(\Lambda\mathbf{p})\Lambda L(\mathbf{p})$ is just the Wigner rotation

$$W(\Lambda,\mathbf{p})=L^{-1}(\Lambda\mathbf{p})\Lambda L(\mathbf{p}) \quad (28)$$

and so Eq. (20) shows that

$$\sum_N \mathfrak{D}_{N'N}(\Lambda)u_N(\mathbf{p}n\sigma)=\sum_{\sigma'}u_{N'}(\Lambda\mathbf{p},n,\sigma')D_{\sigma'\sigma}{}^{(j_n)}(W(\Lambda,\mathbf{p})). \quad (29)$$

Hence the transformation law (27) for the M function together with the transformation law (29) for the wave function give, when substituted into (25), the correct S-matrix transformation law

$$\langle\mathbf{p}_1n_1\sigma_1\cdots|S|\mathbf{p}_2n_2\sigma_2\cdots\rangle=\sum_{\sigma_1'\sigma_2'\cdots}D_{\sigma_1'\sigma_1}{}^{(j_1)*}(W(\Lambda,\mathbf{p}_1))\cdots\times D_{\sigma_2'\sigma_2}{}^{(j_2)}(W(\Lambda,\mathbf{p}_2))\cdots\times\langle\Lambda\mathbf{p}_1n_1\sigma_1'\cdots|S|\Lambda\mathbf{p}_2n_2\sigma_2'\cdots\rangle. \quad (30)$$

The great virtue of the M function lies in general in that it reduces the wretchedly complicated Poincaré invariance requirement (30) to a problem of finding Clebsch-Gordan coefficients for the *homogeneous* Lorentz group so as to satisfy (27).

We can now understand how to evade the theorem of Sec. II. Suppose that we try to impose on the M-function additional symmetry requirements of the form

$$M_{N_1\cdots N_2\cdots}(\mathbf{p}_1\cdots\mathbf{p}_2\cdots)=\mathcal{A}_{N_1'N_1}{}^{(1)*}(g)\cdots\mathcal{A}_{N_2'N_2}{}^{(2)}(g)\cdots\times M_{N_1'\cdots N_2'\cdots}(\mathbf{p}_1\cdots\mathbf{p}_2\cdots), \quad (31)$$

where the $\mathcal{A}(g)$ are various representations of some group G. In contrast with the situation described in Sec. II, a symmetry principle like Eq. (31) has the features:

(1) There is no need for the matrices $\mathcal{A}(g)$ to depend on momentum. If $\mathcal{A}(g)$ satisfies (31) then so does $\mathfrak{D}(\Lambda)\mathcal{A}(g)\mathfrak{D}^{-1}(\Lambda)$, but can be momentum independent because $\mathfrak{D}(\Lambda)$ is momentum independent.

(2) The matrices $\mathcal{A}(g)$ cannot be unitary (unless $\mathcal{A}\equiv 1$) because there are no finite-dimensional unitary representations of the homogeneous Lorentz group. But now there is no need for $\mathcal{A}(g)$ to be unitary, since it does not arise from a unitary operation on Hilbert space.

(3) There is no need for the spin and parity content of the representations $\mathfrak{D}(\Lambda)$ to be in exact correspondence with that of the particles in the corresponding supermultiplets. For instance, the Dirac $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of the homogeneous Lorentz group (including space inversion) contains $\frac{1}{2}^+$ and $\frac{1}{2}^-$ representations of the subgroup of rotations and space inversion, but it can be (and usually is) used to construct M functions for $\frac{1}{2}^+$ particles alone.

By using generalized M functions as the arena for symmetry principles we gain considerable freedom, but against this we must set several serious disadvantages:

(1) It is difficult to conceive how a dynamical theory could embody a symmetry of this type. *It should be realized that a condition like Eq. (31) can generally not even be stated as a linear relation among S-matrix elements.* For this to be possible, the quantities $\mathcal{A}(g)u(\mathbf{p},n,\sigma)$ would have to be expressible as a linear combination of wave functions $u(\mathbf{p},n',\sigma')$, and we can see in the example of the quark supermultiplet that they will generally not commute with $(ip^\mu\gamma_\mu+m)$ and hence will take us out of the space of quark "wave functions" satisfying Eq. (24). This typically happens when we define the M function using an irreducible representation $\mathfrak{D}(\Lambda)$ of the homogeneous Lorentz group which contains more spins and parities than does the supermultiplet.

(2) The unitarity relation for the S matrix becomes a set of nonlinear equations for the M function, which can be written symbolically as

$$\mathrm{Im}M=M\Sigma M^\dagger,$$

where Σ is a sum over σ's of a product of $uu^\dagger$'s for each particle. For instance, for each quark Σ contains a factor $(-i\not{p}+m)/2E$. But Σ will then not be $U(12)$ invariant, and we must entertain grave doubts whether any $U(12)$-invariant M function can ever yield a unitary S matrix. [A very recent article by Bég and Pais[10] shows that the quark-quark scattering matrix does indeed come out nonunitary, for just the above reason.]

(3) The future of such a radically new kind of symmetry principle will depend on the verdict of experiment. A recent letter[11] from Princeton shows that some specific predictions of $U(12)$ are in strong disagreement with experiment.

V. OTHER WAYS OUT

In closing, we list below some other possible ways of evading the limitations imposed by the theorem of Sec. II.

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[10] M. A. B. Bég and A. Pais, Phys. Rev. Letters **14**, 509, 576(E) (1965). The conflict with unitarity was apparently also realized by C. N. Yang, and by the team listed in Ref. 11.

[11] R. Blankenbecler, M. L. Goldberger, K. Johnson, and S. B. Treiman, Phys. Rev. Letters **14**, 518 (1965).

(1) *Groups with many inequivalent representations.* If a group G of the type described in Sec. II has only one irreducible representation of a given dimensionality and spin-parity content, then the assumption stated in Eq. (3) is unavoidable. If it has more than one such irreducible representation then we might try dividing momentum space into different equivalence classes, $\mathbf{p}$ being equivalent to $\mathbf{p}'$ if the representations $\mathcal{G}(g;\mathbf{p})$ and $\mathcal{G}(g;\mathbf{p}')$ are equivalent. (In order to avoid the contradiction found in Sec. II it would be necessary to suppose that for each $\mathbf{p}$ there is a Lorentz transformation Λ such that $\mathbf{p}$ is not equivalent to $\Lambda\mathbf{p}$.) It would be particularly repulsive to imagine momentum space divided discontinuously into a finite number of such equivalence classes, so the most natural possibility would be for G to have an infinite number of inequivalent representations of the same dimensionality and spin-parity content. (Each $\mathbf{p}$ might then be an equivalence class by itself.)

This is possible if G is not semi-simple. For instance, it might be that the group of all symmetries commuting with P^μ contains the translations as an invariant Abelian subgroup, but does *not* factor into the direct product of the translation subgroup and some other group. Or there might exist invariant Abelian subgroups of G consisting of new kinds of translation operators.[12] Neither possibility seems to offer much promise.

(2) *Nonunitary S-matrix symmetries.* We might try to impose a symmetry condition on the S matrix itself instead of the "generalized M function," but without insisting that the symmetry matrices be unitary. We would then find a representation $\mathcal{E}(\Lambda;\mathbf{p})$ of the homogeneous Lorentz group just as in Sec. II, but there would now be no reason for it to be unitary and hence no contradiction.

The trouble here is that the spin-parity content of this representation would have to be identical with that of the particle supermultiplet. [For spin this is a direct consequence of Eq. (14); the argument is identical with respect to parity.] But the representations of the homogeneous Lorentz group (including space inversion) have an undesirable parity content; for every half-integer spin particle there must appear another of opposite parity. [For instance, the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ Dirac representation contains $\frac{1}{2}^+$ and $\frac{1}{2}^-$ parts.] Thus alternative (2) leads us to a parity doubling, either through the introduction of new baryon states, or by the inclusion of antibaryons in the same supermultiplet as baryons. We have already remarked in Sec. IV that a symmetry defined on the M function escapes this difficulty because there is no one-to-one correspondence between the particle states and the vectors on which the symmetry acts. Also, the S matrix probably must be nonunitary if subject to nonunitary symmetry requirements.

(It is not impossible that the *weak* interactions could possess a nonunitary S-matrix symmetry of this type. If we exclude space inversions, then the homogeneous Lorentz group has non-parity-doubled representations, like the two-component $(\frac{1}{2},0)$ representation, which contains spin $\frac{1}{2}$ just once. Insisting on a strict nonunitary $GL(6)$ symmetry for the S matrix would yield such an $\mathcal{E}(\Lambda,\mathbf{p})$, and would predict a V-minus-A structure for the four-quark matrix elements.)

(3) *Inexact symmetries.* Conceivably we should not even try (as we have been doing) to imagine a universe with exact supermultiplet-generating symmetries which approximates the real universe [as we do for approximate symmetries like $SU(3)$], but should derive intrinsically approximate relations in some other way, perhaps from current commutation rules.[13]

(4) *Specific dynamical models.* Finally, it may be that the new supermultiplet theories are really just like their Wignerian archetype, in that they arise from specific bound-state models of elementary-particle structure, and simply tell us nothing at all about scattering processes with large relative velocities. I personally favor this last possibility, but time will tell.

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[12] T. Fulton and J. Wess (to be published).

[13] R. P. Feynman, M. Gell-Mann, and G. Zweig, Phys. Rev. Letters **13**, 678 (1964).