connection, note that[10]

$$a_l=\max_x x\,|j_l(x)h_l^{(1)}(x)| \qquad (A2.1)$$

and

$$A_l=\max_x x^2|j_l(x)h_l^{(1)}(x)|\,. \qquad (A2.2)$$

We also have

$$b_l(l+1)=1/(2l+1)!!. \qquad (A2.3)$$

For large l, one can use the asymptotic formula of Watson and Nicholson,[20] according to which

$$H_\lambda^{(1)}(x)\sim(3)^{-1/2}wH_{1/3}^{(1)}(y)\exp(i\tfrac{1}{6}\pi) \qquad (A2.4)$$

as $l\rightarrow\infty$, where

$$\lambda\equiv l+\tfrac{1}{2}\,,$$

$$y=\tfrac{1}{3}\lambda w^3\,,$$

and

$$w^2=(x/\lambda)^2-1\,.$$

In writing this expression we used the fact that, for b_l, a_l, and A_l, the maximum is achieved for a value of y which tends to a constant as l tends to infinity; correspondingly, w tends to zero. We find that as $l\rightarrow\infty$

$$b_l(\nu)\sim0.845843(l+\tfrac{1}{2})^{1/6-\nu}(\nu<l+1)\,, \qquad (A2.5)$$

$$a_l\sim0.741397(l+\tfrac{1}{2})^{-2/3}\,, \qquad (A2.6)$$

and

$$A_l\sim0.741397(l+\tfrac{1}{2})^{1/3}\,. \qquad (A2.7)$$

[20] W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1954), Chap. 3, Sec. 3.

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Permutation Symmetry of Many-Particle Wave Functions*

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The symmetrization postulate (SP) states that wave functions are either completely symmetric or completely antisymmetric under permutations of identical particles. It is shown by one-dimensional counterexamples that SP is not demanded by the usual physical interpretation of the mathematical formalism of wave mechanics unless one makes use of further physical properties of real systems; the error in a standard proof of SP which ignores these properties is pointed out. It is then proved that SP is true for those systems of spinless particles which have the following properties: (a) probability densities are permutation-invariant, (b) allowable wave functions are continuous with continuous gradient, (c) the $3n$-dimensional configuration space is connected, (d) the Hamiltonian is symmetric, and (e) the nodes of allowed wave functions have certain properties. The counterexamples show that SP is not a necessary property of those systems which do not have property (c). The proof is extended to particles with internal degrees of freedom (including spin) by noting that any observable commutes with every permutation and making use of a particular observable acting only on internal variables. Extraneous mathematical assumptions, such as that of the existence of a "complete" set of commuting observables, criticized by Messiah and Greenberg, are not employed. Some comments are made on the conventional nature of the concept of identity for similar particles; the equivalence between the usual formulation in which different species of similar particles are treated as distinct, and that in which they are regarded as identical particles in different internal states, is emphasized.

1. INTRODUCTION

It is a well-known experimental fact that quantum-mechanical states of a system of identical elementary particles are either symmetrical (Bose-Einstein) or antisymmetrical (Fermi-Dirac) under permutations of the single-particle dynamical variables; more complicated permutation symmetries seem not to be realized in the real world. Messiah[1,2] calls this fact the *symmetrization postulate*. The pioneers in the development of quantum mechanics took this simply as an experimentally based fact. Thus, e.g., Dirac[3] states that: "Other more complicated kinds of symmetry are possible mathematically, but do not apply to any known particles." There were subsequent attempts, continuing up to the present time, to deduce the symmetrization postulate from other physical principles. One simple argument, found in many textbooks, runs as follows[4]: Let ψ be the Schrödinger wave function of a system of identical particles, let $P\psi$ be the wave function differing

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[1] A. Messiah, *Quantum Mechanics* (North-Holland Publishing Company, Amsterdam, 1962), Vol. II, p. 595.

[2] A. M. L. Messiah and O. Greenberg, Phys. Rev. **136**, B248 (1964).

[3] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1947), 3rd ed., p. 211.

[4] See, e.g., E. M. Corson, *Perturbation Methods in the Quantum Mechanics of n-Electron Systems* (Blackie and Son, Ltd., Glasgow, 1951), p. 113.

from ψ only through a permutation P of the dynamical variables (arguments of ψ), and assume that the precise mathematical statement of the identity of the particles is that the configurational probability is permutation-invariant, i.e.,

$$|P\psi|^2=|\psi|^2. \tag{1}$$

Then it follows from (1) that

$$P\psi=c_P\psi \tag{2}$$

where c_P is a complex number of modulus unity. Applying a second permutation, one finds

$$PQ\psi=c_{PQ}\psi=c_Pc_Q\psi, \tag{3}$$

hence

$$c_{PQ}=c_Pc_Q. \tag{4}$$

But (4) is just the condition that the c_P form a scalar representation of the permutation group. It is well known that there are just two scalar representations, one being

$$c_P=1 \quad \text{for all} \quad P \tag{5}$$

for which ψ is completely symmetric, the other

$$\begin{aligned} c_P&=+1, \quad P \text{ even}\\ &=-1, \quad P \text{ odd}\end{aligned} \tag{6}$$

for which ψ is completely antisymmetric, Q.E.D.

However, this simple proof is in fact incorrect, and more sophisticated proofs[5–8] involve mathematical assumptions which are either in conflict with known physical principles or at least do not follow directly from such principles.[9] This is most clearly seen with the aid of counterexamples which help to locate the unwarranted assumptions in the incorrect proofs and suggest the additional physical information which must be incorporated into a correct proof.

[5] J. M. Jauch, Helv. Phys. Acta. **33**, 711 (1960).

[6] J. M. Jauch and B. Misra, Helv. Phys. Acta. **34**, 699 (1961).

[7] D. Pandres, Jr., J. Math. Phys. **3**, 305 (1962).

[8] A. Galindo, A. Morales, and R. Nuñez-Lagos, J. Math. Phys. **3**, 324 (1962).

[9] In these proofs the existence of a complete set of commuting observables is assumed; as pointed out by Greenberg and Messiah (Ref. 2), this assumption need not be true. Pandres (Ref. 7) makes the further assumption that any operator which commutes with every member of this set must be a multiple of the unit operator; this ignores the existence of superselection rules [G. C. Wick, A. S. Wightman, and E. P. Wigner, Phys. Rev. **88**, 101 (1952)] which follow from the fact that every true observable of a system of *identical* particles must be *symmetric* in single-particle variables. As a result, an operator can commute with all true observables and yet not be a multiple of the unit operator. This is the case, e.g., for an arbitrary permutation operator P. Another argument found in many textbooks omits (1), instead taking (2) as the starting point on the basis of the assertion that wave functions differing only through a permutation of identical particles must represent the same physical state (ray in Hilbert space) and hence can differ only by a constant factor of modulus unity. However, such an assertion is not supported by the physical interpretation, according to which all observable quantities are *bi*linear in wave functions, being expressible in terms of inner products, expectation values, and matrix elements. Hence the ray need not be permutation-invariant.

2. COUNTER EXAMPLES

Counter examples to the conclusions of the above "proof" are provided by a previously studied class of one-dimensional models.[10] We consider a system of n identical particles confined in a one-dimensional box $0\leqslant x\leqslant L$. These particles are assumed to have no internal degrees of freedom (not even spin); thus the configuration of a single particle is completely characterized by its one-dimensional position x, and a state of the n-particle system can be represented by a Schrödinger wave function $\psi(x_1\cdots x_n)$ with $0\leqslant x_j\leqslant L$, $1\leqslant j\leqslant n$. The boundary conditions on ψ are

$$\psi(x_1\cdots x_n)=0 \quad \text{if} \quad x_j=0 \quad \text{or} \quad L, \quad 1\leqslant j\leqslant n. \tag{7}$$

We furthermore assume that the particles are impenetrable with hard-core diameter a, in the sense that

$$\begin{aligned}\psi(x_1\cdots x_n)=0 \quad &\text{if} \quad |x_j-x_l|\leqslant a, \quad 1\leqslant j<l\leqslant n\\ &\text{or if} \quad x_j\leqslant \tfrac{1}{2}a \quad \text{or} \quad L-x_j\leqslant\tfrac{1}{2}a, \quad 1\leqslant j\leqslant n.\end{aligned} \tag{8}$$

Thus the configuration space $\mathfrak{C}$ consists of all ordered sets $x_1\cdots x_n$ satisfying

$$\begin{aligned}&\tfrac{1}{2}a\leqslant x_j\leqslant L-\tfrac{1}{2}a, \quad 1\leqslant j\leqslant n;\\ &|x_j-x_l|\geqslant a, \quad 1\leqslant j<l\leqslant n.\end{aligned} \tag{9}$$

If ψ_E is an energy eigenfunction with eigenvalue E, it satisfies a Schrödinger equation of the form

$$(T+V)\psi_E(x_1\cdots x_n)=E\psi(x_1\cdots x_n) \tag{10}$$

in $\mathfrak{C}$, where T is the kinetic and V the potential energy [not including the hard cores, which we treat by the subsidiary condition (8)]. V may be quite arbitrary; it need not be restricted to a sum of two-body interactions.

Let $\psi_E{}^B$ be a Bose solution of (10), (7), and (8), i.e., a solution which is completely symmetric under permutations P of the x_j. Then $\psi_E{}^B$ satisfies (1) trivially. Let us construct a function $\psi_E{}'$ by the prescription

$$\psi_E{}'(x_1\cdots x_n)=\psi_E{}^B(x_1\cdots x_n)S(x_1\cdots x_n), \tag{11}$$

where the function S is defined by

$$S(x_1\cdots x_n)=\prod_{j<l}{}' \operatorname{sgn}(x_j-x_l), \tag{12}$$

$\operatorname{sgn}(x)$ is the algebraic sign (±1) of x, and the prime in (12) implies that some of the $\frac{1}{2}n(n-1)$ pairs $j<l$ may be omitted from the product. If all pairs are omitted then $S\equiv1$, so $\psi_E{}'$ is identical with $\psi_E{}^B$; if all are included, we have the previously considered[10] case, in which $\psi_E{}'$ obeys Fermi kinematics[11] since $\psi_E{}^B$ is symmetrical (Bose kinematics[7]) and, in that special case, S is antisymmetrical. Except for these two trivial

[10] M. Girardeau, J. Math. Phys. **1**, 516 (1960).

[11] We prefer not to use the conventional but illogical terminology "statistics" in referring to a restriction on allowable states of the system which applies even (as here) to a pure state (purely mechanical description).

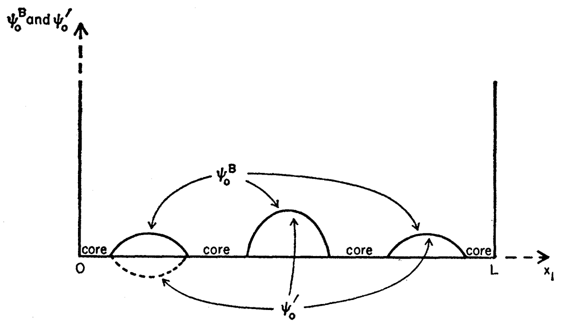

FIG. 1. ψ_0^B and ψ_0' (schematic) for the case $n=3$ and $S(x_1x_2x_3)=\mathrm{sgn}(x_1-x_2)$, plotted as a function of x_1 for fixed x_2 and x_3, with $x_2<x_3$.

cases, however, ψ_E' will have more complicated symmetry properties.[12] Nevertheless, it follows directly from (11) and (12) that ψ_E' satisfies (1). Furthermore, by the previously given arguments[10] ψ_E' satisfies the Schrödinger equation (10), the boundary and hard-core conditions (7) and (8), and the same regularity conditions as ψ_E^B. A simple example is shown in Fig. 1; since the ground Bose state ψ_0^B can be chosen to be real and non-negative, we have chosen that case to illustrate the general situation.

This counter example clearly violates the conclusion of the "proof" given in Sec. 1, since we have found allowable wave functions ψ' satisfying (1), but nevertheless obeying neither Bose nor Fermi kinematics. The example studied clearly points out the error in the proof; it is simply the implicit assumption that c_P is independent of the configuration[13] $x_1\cdots x_n$. This assumption is not always correct; thus, e.g., in the example of Fig. 1, we have

$$\begin{aligned} c_I&=1,\quad c_{(12)}=-1,\\ c_{(13)}&=\mathrm{sgn}(x_3-x_2)/\mathrm{sgn}(x_1-x_2),\\ c_{(23)}&=\mathrm{sgn}(x_1-x_3)/\mathrm{sgn}(x_1-x_2),\\ c_{(123)}&=\mathrm{sgn}(x_2-x_3)/\mathrm{sgn}(x_1-x_2),\\ c_{(321)}&=\mathrm{sgn}(x_3-x_1)/\mathrm{sgn}(x_1-x_2), \end{aligned} \tag{13}$$

where I is the identity permutation, and the other five permutations are denoted in the usual cycle form. We use the usual notation: PQ means first Q, then P; (123) means $1\rightarrow2\rightarrow3\rightarrow1$. As soon as the c_P are allowed to be functions of the configuration, they no longer form a representation of the permutation group, so that the proof does not go through.

There are three important properties of these counterexamples which should be emphasized: The particles move in one dimension, they have hard cores, and they have no internal degrees of freedom. Of these three properties, the one-dimensionality is the most important; we shall find in the next section that as soon as it is removed, the counterexamples fail. Thus, the absence of elementary particles obeying other than Bose or Fermi kinematics is intimately related to the fact that the real world is not one-dimensional.

3. PROOF OF THE SYMMETRIZATION POSTULATE FOR SPINLESS PARTICLES IN THREE DIMENSIONS

If one attempts to extend the counterexample of the previous section to the case of particles moving in three dimensions, one encounters the immediate difficulty that when the x_j are replaced by vectors $\mathbf{r}_j$, there is no usable analog of the function $S(x_1\cdots x_n)$ defined in (12). This is brought out clearly by first defining $S(x_1\cdots x_n)$ in a more general fashion than (12), in order to emphasize the topological reason why no nontrivial analog $S(\mathbf{r}_1\cdots\mathbf{r}_n)$ exists. Note first that the configuration space $\mathcal{C}$ of n particles in one dimension is n-dimensional. Furthermore, each of the subspaces $x_j=x_l$ is $(n-1)$-dimensional and flat, and hence divides $\mathcal{C}$ into two disjoint regions. It follows that the set of all $\frac{1}{2}n(n-1)$ subspaces $x_j=x_l$, $1\leqslant j<l\leqslant n$, divides $\mathcal{C}$ into precisely $n!$ disjoint regions $\mathcal{C}_p$, $1\leqslant p\leqslant n!$, corresponding to the $n!$ different possible orderings of $x_1\cdots x_n$. If we now construct any function $S(x_1\cdots x_n)$ with the properties that it has the constant value $+1$ or -1 in each $\mathcal{C}_p$ but may jump from $+1$ to -1 at the boundary between any two $\mathcal{C}_p$ (i.e., on any surface $x_j=x_l$), then such an S can be used to construct a counterexample to the symmetrization postulate by the method of the previous section.[14] The reason the product $\psi'=\psi^BS$ [Eq. (11)] is continuous and satisfies the Schrödinger equation throughout $\mathcal{C}$ is simply that all discontinuities of S (the surfaces $x_j=x_l$) lie in the interior of the region where ψ^B vanishes identically because of the hard-core conditions (8), or equivalently, because of (9) all discontinuities of S lie outside $\mathcal{C}$.

One is tempted to try to extend this argument to a system of n particles with hard cores in three dimensions by defining a function $S(\mathbf{r}_1\cdots\mathbf{r}_n)$ which is either $+1$ or -1 at all points in $\mathcal{C}$, and jumps from $+1$ to -1 only on the subspaces $\mathbf{r}_j=\mathbf{r}_l$. Actually, however, no such nontrivial S exists. To see this, note first that for n particles in three dimensions, the configuration space is $3n$-dimensional. Then, since each subspace $\mathbf{r}_j=\mathbf{r}_l$ is $(3n-3)$-dimensional, such subspaces do *not* divide $\mathcal{C}$ into disjoint regions[15]; instead, if one deletes from $\mathcal{C}$ all

[12] In general, ψ_E' will not even belong to any single Young diagram, but will be a linear combination of functions belonging to various Young diagrams.

[13] In order to emphasize this point in (1)–(4), we have carefully been even more careless in notation than the original perpetrators of the proof.

[14] There are $2^{n!-1}$ distinct S's (including the trivial completely symmetrical and completely antisymmetrical S's, which do not give counterexamples), since the sign of S may without loss of generality be chosen positive in one sector $\mathcal{C}_p$, but may then be assigned at will in each of the remaining $n!-1$ sectors. Only $2^{\frac{1}{2}n(n-1)}$ of these S's are of the form (12).

[15] A boundary space of $3n-1$ dimensions is necessary in order to divide the $3n$-dimensional space into disjoint regions.

$\frac{1}{2}n(n-1)$ subspaces $\mathbf{r}_j=\mathbf{r}_l$, $1\leqslant j<l\leqslant n$, the remainder of $\mathcal{C}$ is still *connected*. Hence the only functions S which are $+1$ or -1 at every point in $\mathcal{C}$ and have discontinuities at most on the subspaces $\mathbf{r}_j=\mathbf{r}_l$ are the two trivial cases $S\equiv 1$ and $S\equiv -1$ throughout $\mathcal{C}$.

This suggests that a correct proof of the symmetrization postulate should involve topological arguments related to the three-dimensional nature of physical space. To this end, consider a system of n identical particles enclosed in any connected three-dimensional region $\mathcal{R}$. As in Sec. 2, these particles are assumed to have no internal degrees of freedom (not even spin), but may or may not have hard cores in the sense

$$\psi(\mathbf{r}_1\cdots\mathbf{r}_n)=0 \quad\text{if}\quad |\mathbf{r}_j-\mathbf{r}_l|\leqslant a,\quad 1\leqslant j<l\leqslant n, \tag{14}$$

where ψ is the Schrödinger wave function of the system. If there are no hard cores, ψ is required to vanish if any $\mathbf{r}_j$ is on the boundary of $\mathcal{R}$, whereas, if there are hard cores, ψ must vanish if any $\mathbf{r}_j$ is within a distance $\frac{1}{2}a$ of the boundary. Finally, we take the shape and size of $\mathcal{R}$ to be such that the configuration space $\mathcal{C}$ is connected.[16]

We are now ready to proceed with the proof. As in Sec. 1, we take as our *defining* property of *identical* particles the requirement that configurational probability densities be permutation-invariant,[17] i.e.,

$$|P\psi(\mathbf{r}_1\cdots\mathbf{r}_n)|^2=|\psi(\mathbf{r}_1\cdots\mathbf{r}_n)|^2 \tag{15}$$

for all admissible wave functions ψ and all permutations P. Upon replacing ψ by $\psi+i\varphi$ and $\psi-i\varphi$ and subtracting, one finds

$$(P\varphi)^*P\psi-(P\psi)^*P\varphi=\varphi^*\psi-\psi^*\varphi. \tag{16}$$

Replacing ψ by $i\psi$ in (16) and adding, one finds

$$[P\varphi(\mathbf{r}_1\cdots\mathbf{r}_n)]^*P\psi(\mathbf{r}_1\cdots\mathbf{r}_n)=\varphi^*(\mathbf{r}_1\cdots\mathbf{r}_n)\psi(\mathbf{r}_1\cdots\mathbf{r}_n). \tag{17}$$

Thus permutation invariance of probability densities $|\psi|^2$ implies that of transition densities $\varphi^*\psi$.

It follows from (15) that

$$P\psi(\mathbf{r}_1\cdots\mathbf{r}_n)=c_P(\mathbf{r}_1\cdots\mathbf{r}_n)\psi(\mathbf{r}_1\cdots\mathbf{r}_n) \tag{18}$$

with

$$|c_P(\mathbf{r}_1\cdots\mathbf{r}_n)|=1. \tag{19}$$

However, (4) is not in general valid if the c_P are allowed to depend on $\mathbf{r}_1\cdots\mathbf{r}_n$; instead, one finds, noting that P permutes both the arguments of ψ and those of c_Q,

$$c_{PQ}(\mathbf{r}_1\cdots\mathbf{r}_n)=c_P(\mathbf{r}_1\cdots\mathbf{r}_n)c_Q(P\mathbf{r}_1\cdots P\mathbf{r}_n) \tag{20}$$

in an obvious notation. As a result, the group-theoretical argument leading to (5) and (6) fails, so that a different method of proof must be used.

We shall first dispense with the possibility that c_P might depend on the wave function ψ on which P acts. By (19), one has

$$c_{P,\psi}^*(\mathbf{r}_1\cdots\mathbf{r}_n)=1/c_{P,\psi}(\mathbf{r}_1\cdots\mathbf{r}_n), \tag{21}$$

where the extra subscript ψ emphasizes the possible dependence of c_P on ψ. Then it follows from (17) and (18) that

$$\left[\frac{c_{P,\psi}(\mathbf{r}_1\cdots\mathbf{r}_n)}{c_{P,\varphi}(\mathbf{r}_1\cdots\mathbf{r}_n)}-1\right]\varphi^*(\mathbf{r}_1\cdots\mathbf{r}_n)\psi(\mathbf{r}_1\cdots\mathbf{r}_n)=0. \tag{22}$$

Hence for any admissible ψ and φ,

$$c_{P,\psi}(\mathbf{r}_1\cdots\mathbf{r}_n)=c_{P,\varphi}(\mathbf{r}_1\cdots\mathbf{r}_n) \tag{23}$$

except possibly at points where ψ or φ vanishes. But at points where, say, ψ vanishes, so must $P\psi$, because of (15). Hence at such points (18) is true for *any* c_P, so there is no loss of generality if we also assume (23) to be true at such points. Thus (23) is true throughout the configuration space $\mathcal{C}$, i.e., c_P is indeed independent of the wave function on which P acts.

Next we show that c_P must be real. To see this, one merely has to recall that any wave function ψ belonging to a nondegenerate energy level is necessarily real apart from a constant phase factor, which can be chosen so that ψ is real.[18] Letting P act on such a ψ, one concludes

[16] By definition, $\mathcal{C}$ is said to be *connected* if and only if for every two points in $\mathcal{C}$, there is a *continuous path* lying entirely in $\mathcal{C}$ connecting these two points. Let $Q=(\mathbf{r}_1\cdots\mathbf{r}_n)$ be any point in $\mathcal{C}$, and define the distance between any two such points to be

$$|Q_1-Q_2|\equiv[\sum_{j=1}^{n}(\mathbf{r}_{j1}-\mathbf{r}_{j2})^2]^{1/2}.$$

A *path* in $\mathcal{C}$ is a parametrization $Q(t)$ by a real parameter t, i.e., a one-one mapping from some interval $t_1\leqslant t\leqslant t_2$ onto some region of $\mathcal{C}$. A *continuous path* is a path with the property

$$\lim_{t\to t_0}|Q(t)-Q(t_0)|=0$$

for every t_0 satisfying $t_1\leqslant t_0\leqslant t_2$. If the three-dimensional region $\mathcal{R}$ is itself connected and the particles have no hard cores, then it is trivial to prove that the $3n$-dimensional configuration space $\mathcal{C}$ is also connected. However, the situation is much more complicated if there are hard cores. Thus, e.g., it is then easy to make $\mathcal{C}$ disconnected even for a connected $\mathcal{R}$ by trapping one or more hard spheres in a region of $\mathcal{R}$ which is connected to the remainder of $\mathcal{R}$ only by a channel of diameter $<a$. If such pathological shapes of $\mathcal{R}$ are excluded by requiring that it be convex, e.g., a cube or sphere, then $\mathcal{C}$ is certainly connected if the density [(number of hard spheres)/(volume of $\mathcal{R}$)] is low enough, although the author is not aware of a rigorous proof. However, even for convex $\mathcal{R}$ there may be a critical density beyond which $\mathcal{C}$ is disconnected. Thus, e.g., the cubic close-packed arrangement of spheres in a cubical $\mathcal{R}$ cannot be distorted into the hexagonal close-packed arrangement which occupies the same volume, and it is not known whether or not there is an irregular arrangement of the spheres with density higher than that of the cubic or hexagonal close-packed arrangements; see, e.g., H. F. Blichfeldt, Math. Ann. **101**, 605 (1929).

[17] Messiah and Greenberg (Ref. 2) adopt a more general definition of identical particles in which (15) is given up. We shall not consider such a generalization here.

[18] This is true provided that the Hamiltonian is real (not merely Hermitian), as is the case in Schrödinger representation in the absence of external magnetic fields and other magnetic interactions (e.g., spin-spin and spin-orbit). Even in the presence of magnetic interactions, one is at liberty to work with the complete set of eigenstates of a fictitious Hamiltonian obtained by omitting the magnetic interactions; then the proof goes through as before.

immediately that c_P is also real. Then by (19),

$$c_P(\mathbf{r}_1\cdots\mathbf{r}_n)=\pm 1. \qquad (24)$$

It remains to be shown that c_P cannot have discontinuities as a function of $\mathbf{r}_1\cdots\mathbf{r}_n$, where it jumps from $+1$ to -1. In the proof we shall use the facts that the region $\mathfrak{R}$ within which the n identical particles are contained is connected and three-dimensional, so that the $3n$-dimensional configuration space $\mathfrak{C}$ is connected, that admissible wave functions $\psi(\mathbf{r}_1\cdots\mathbf{r}_n)$ are continuous with continuous $3n$-dimensional gradient throughout the interior of $\mathfrak{R}$, and that the Hamiltonian is a symmetrical function of single-particle variables; we shall also use certain information concerning the nature of the nodes of the wave functions. From the symmetry of the Hamiltonian H it follows that

$$[H,P]=0 \qquad (25)$$

for every permutation P, and hence that if ψ_E is an energy eigenfunction with eigenvalue E, then so must $P\psi_E$ be; then by (18), the same must hold for $c_P\psi_E$. In view of (24) and the presence of the kinetic-energy terms $-(\hbar^2/2m)\nabla_j^2$ in H, this can be true only if the discontinuities of c_P (if it has any) occur only at points where *both* ψ_E and its $3n$-dimensional gradient vanish.[19] Furthermore, since the c_P have been shown to be independent of the state on which P acts, c_P can be discontinuous only at points where *all* ψ_E and their gradients vanish. One reaches the same conclusion by recalling that allowable wave functions must be continuous with continuous gradient throughout the interior of the configuration space $\mathfrak{C}$; if this is true of ψ, it must also be true of $P\psi$, hence of $c_P\psi$.

At what points can *all* ψ_E and their gradients vanish? As a preliminary it is helpful to first introduce a classification of nodes. Any nodes present at the same points in all wave functions are, in our definition, *kinematical nodes;* they arise from constraints or symmetry properties common to all wave functions. We shall call *all* other nodes (i.e., nodes not shared by *all* wave functions of the system) *orthogonality nodes;* this name is justified by the fact that in a representation in which all members of a set of wave functions (e.g., the energy eigenfunctions ψ_E) are real, these nodes arise from the requirement of orthogonality of different members of the set. It is clear from the discussion of the preceding paragraph that discontinuities of c_P cannot occur on orthogonality nodes; hence we pass immediately to further consideration of the kinematical nodes. These may be further subdivided into *constraint nodes*, e.g., the regions of configuration space where hard-sphere wave functions vanish because of overlap,[20,21] and *symmetry nodes*, which arise from symmetry properties common to all wave functions. The most familiar example of symmetry nodes is that of nodes in many-fermion wave functions at $\mathbf{r}_j=\mathbf{r}_l$ which arise from the requirement of antisymmetry.[22] For this example it is clear that the (real) wave functions change sign at a symmetry node, and hence do not have vanishing gradient there. It would appear[23] that this is a general property of symmetry nodes; hence c_P cannot have discontinuities on symmetry nodes.

We are left, therefore, only with the constraint nodes. For particles without hard cores contained in a connected region $\mathfrak{C}$ there are no constraint nodes, so that *c_P cannot have discontinuities anywhere.* For particles with hard cores of diameter a, the constraint nodes consist of the $3n$-dimensional configuration subspaces ("rods")

$$|\mathbf{r}_j-\mathbf{r}_l|\leqslant a, \quad 1\leqslant j<l\leqslant n. \qquad (26)$$

Within each of these nodal regions every allowable wave function vanishes along with its $3n$-dimensional gradient, so that if ψ and its gradient are continuous, then so is *$c_P\psi$ within this nodal region* (both vanish there) even if c_P jumps from $+1$ to -1 within this region. We shall show, however, that if c_P has such a discontinuity, then $c_P\psi$ will necessarily be discontinuous *somewhere else*, outside the nodal region (inside the allowed configuration space $\mathfrak{C}$). In order to see this, we recall that for low enough densities and reasonable shapes of the box $\mathfrak{R}$, the configuration space $\mathfrak{C}$ is connected[16]; we shall temporarily restrict ourselves to such a case. Suppose, then, that c_P is $+1$ at some point Q_1 in $\mathfrak{C}$ and -1 at some other point Q_2, jumping from $+1$ to -1, as we follow a continuous path[16] from Q_1 to Q_2, at some point on one of the rods (26) through which this path passes. But since $\mathfrak{C}$ is connected, there exists some other continuous path from Q_1 to Q_2 which lies entirely within $\mathfrak{C}$, i.e., avoids all of the rods (26). Since c_P is $+1$ on one end of this latter path, -1 on the other end, and ± 1 at every intermediate point, it must jump from $+1$ to -1 at at least one point on the path, hence within $\mathfrak{C}$. But by the previous arguments this is not permitted since it would make $c_P\psi$ discontinuous (at

[19] We have in mind cases where the potential-energy part V of H is a function only of positions $\mathbf{r}_1\cdots\mathbf{r}_n$. However, the same conclusion follows if V is momentum-dependent but, at most, quadratic in the momenta. This is the case, e.g., for interacting particles in an external magnetic field, where the momentum-dependent terms arising from the magnetic field are linear in the momenta.

[20] It can be objected that the overlap regions are outside the configuration space $\mathfrak{C}$ owing to the requirement that in allowed configurations no two hard spheres may overlap. However, the global properties of c_p within $\mathfrak{C}$ may be related to its discontinuities outside $\mathfrak{C}$, as shown by the one-dimensional models of Sec. 2; hence the terminology "constraint nodes" seem justified.

[21] A fixed, impenetrable obstacle also gives rise to a constraint node. The proof for such a node goes through in the same way as for a constraint node due to overlap of movable hard spheres.

[22] As in the rest of this section, we are considering the simplified case of particles without internal degrees of freedom, hence, in particular, spinless. The actual case of particles with spin will be considered in the next section.

[23] Thus, by the same argument as for the Fermi case, any wave function belonging to a single Young diagram (irreducible representation of the permutation group) changes sign at a symmetry node, since it must be antisymmetric under permutations within each column of the Young diagram.

least for some ψ). We conclude, then, that c_P cannot have discontinuities anywhere, not even on the rods (26), i.e., *c_P is independent of $\mathbf{r}_1\cdots\mathbf{r}_n$, equal to either +1 or −1, throughout the configuration space* $\mathcal{C}$. This argument is illustrated by Fig. 2.

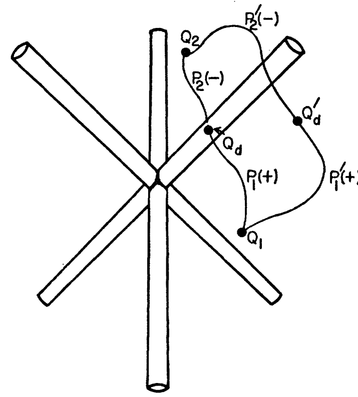

FIG. 2. c_P is +1 on the path segment P_1 and −1 on P_2, with a discontinuity at the point Q_d inside one of the rods $|\mathbf{r}_j-\mathbf{r}_l|\leqslant a$ which make up the constraint nodal region. Some other path P' can be constructed from Q_1 to Q_2 but avoiding all constraint nodes. c_P is +1 on part P_1' of this path and −1 on some other part P_2', and hence necessarily has a discontinuity at some point Q_d' in the allowed configuration space $\mathcal{C}$.

Since we have now established the constancy of c_P, the group-theoretical argument (3)–(6) then shows immediately that only the Bose case (5) or the Fermi case (6) are possible, i.e., the symmetrization postulate is proved for spinless particles in three dimensions. This proof is completely general for point particles confined to a connected region $\mathcal{R}$, but for particles with hard cores it is restricted to sufficiently reasonable box shapes $\mathcal{R}$ and sufficiently low densities.[16] For particles with hard cores, topological arguments related to the connectivity of configuration space played a crucial role[24] in the proof; *the theorem is true only for those systems whose spatial configuration space is connected.* In particular, it is not true for hard spheres in one dimension, as shown by the counterexamples of Sec. 2.

The reader may wonder why the analysis has been based on ordinary Schrödinger wave functions, rather than more general abstract Hilbert vectors in an arbitrary representation. The essential point is that experiments are carried out in a real space-time world, not an arbitrary one; hence boundary and regularity conditions on wave functions are expressed in a spatial representation. It has thus been shown that SP is a necessary property of wave functions of systems of identical particles whose *spatial* configuration space is *connected*. Once this has been shown, it is then trivial to prove that *for such systems*, SP also holds for the wave functions expressed in *any* representation in terms of single-particle dynamical variables. The argument is found in most textbooks, and thus need not be repeated here.

4. PROOF OF THE SYMMETRIZATION POSTULATE FOR PARTICLES WITH INTERNAL DEGREES OF FREEDOM

We now consider the more realistic case of particles which possess not only positions $\mathbf{r}_1\cdots\mathbf{r}_n$, but also internal degrees of freedom labeled by variables $\sigma_1\cdots\sigma_n$. Although our notation is motivated by the case that the σ_j are spin variables, it is not necessary to make this restriction; σ_j may be any set of internal variables of the jth identical particle. A wave function will then be denoted by $\psi(\mathbf{r}_1\sigma_1\cdots\mathbf{r}_n\sigma_n)$, and our starting point will again be the assumed permutation invariance of probability densities, i.e.,

$$|P\psi(\mathbf{r}_1\sigma_1\cdots\mathbf{r}_n\sigma_n)|^2=|\psi(\mathbf{r}_1\sigma_1\cdots\mathbf{r}_n\sigma_n)|^2 \quad (27)$$

in analogy with (15); here P simultaneously permutes the $\mathbf{r}_j$ and σ_j.

An examination of the previous proof shows that the continuity and connectedness of the configuration space were brought in only in the last few steps; the argument from (15) through (25) goes through unchanged except for notation. In particular, (18) and (24) are replaced by

$$P\psi(\mathbf{r}_1\sigma_1\cdots\mathbf{r}_n\sigma_n)=c_P(\mathbf{r}_1\sigma_1\cdots\mathbf{r}_n\sigma_n)\psi(\mathbf{r}_1\sigma_1\cdots\mathbf{r}_n\sigma_n) \quad (28)$$

and

$$c_P(\mathbf{r}_1\sigma_1\cdots\mathbf{r}_n\sigma_n)=\pm 1. \quad (29)$$

But for *fixed* $\sigma_1\cdots\sigma_n$, the previous arguments based on the connectedness of configuration space and nature of the nodes still go through, with the conclusions that c_P is independent of $\mathbf{r}_1\cdots\mathbf{r}_n$ for fixed $\sigma_1\cdots\sigma_n$, i.e., c_P can depend at most on the internal variables:

$$c_P=c_P(\sigma_1\cdots\sigma_n). \quad (30)$$

However, for discrete internal variables the total configuration space $\mathcal{C}$ [space of all points $(\mathbf{r}_1\sigma_1\cdots\mathbf{r}_n\sigma_n)$] is always disconnected (the parts with different $\sigma_1\cdots\sigma_n$ are disconnected from each other with any reasonable definition of connectivity) so the proof must be completed by a method which does not assume connectedness of $\mathcal{C}$.

The method we shall use is based on the fact that any observable O of a system of *identical* particles must be *symmetric* in single-particle variables, so that[25]

$$[O,P]=0 \quad (31)$$

for every permutation P. Consider, in particular, operators which affect only the internal variables, of the form[26]

$$Of(\sigma_1\cdots\sigma_n)=\sum_{\sigma_1'\cdots\sigma_n'}K(\sigma_1\cdots\sigma_n|\sigma_1'\cdots\sigma_n')f(\sigma_1'\cdots\sigma_n') \quad (32)$$

[24] For hard spheres in one dimension, the configuration space is disconnected even for arbitrarily low densities.

[25] In fact, (31) is best taken as the *definition* of precisely what is meant by the statement that O is symmetric in single-particle variables.

[26] E.g., if the σ_j are spin variables (+ or −) of a system of spin-$\frac{1}{2}$ particles, then the x component S_x of spin angular momentum corresponds to

$$K(\sigma_1\cdots\sigma_n|\sigma_1'\cdots\sigma_n')=\tfrac{1}{2}\hbar\sum_{j=1}^{n}\delta_{\sigma_1\sigma_1'}\cdots\delta_{\sigma_{j-1}\sigma_{j-1}'}\delta_{\sigma_j,-\sigma_j'}\delta_{\sigma_{j+1}\sigma_{j+1}'}\cdots\delta_{\sigma_n\sigma_n'}$$

if one chooses the usual representation for the Pauli spin matrices.

where the $\mathbf{r}_1\cdots\mathbf{r}_n$ dependence of f is not indicated explicitly since it will not enter the argument. Then[27] O is symmetric if and only if the kernel K is invariant under application of the *same* arbitrary permutation to both $\sigma_1\cdots\sigma_n$ and $\sigma_1'\cdots\sigma_n'$, i.e.,

$$K(P\sigma_1\cdots P\sigma_n|P\sigma_1'\cdots P\sigma_n') = K(\sigma_1\cdots\sigma_n|\sigma_1'\cdots\sigma_n'). \quad (33)$$

A rather trivial class of kernels satisfying (33) is obtained by choosing K to be independent of $\sigma_1\cdots\sigma_n$; then (33) is satisfied if and only if K is symmetric in $\sigma_1'\cdots\sigma_n'$. As a particular case of such a K, we choose

$$K(\sigma_1\cdots\sigma_n|\sigma_1'\cdots\sigma_n') = (n!)^{-1}\sum_P P(\delta_{\sigma_1'\sigma_1^0}\cdots\delta_{\sigma_n'\sigma_n^0}) \quad (34)$$

where P permutes $\sigma_1'\cdots\sigma_n'$, whereas $\sigma_1^0\cdots\sigma_n^0$ is any *fixed* set of values of the internal variables. The corresponding operator O then has the following effect on any f[28]:

$$Of(\sigma_1\cdots\sigma_n) = (n!)^{-1}\sum_P f(P\sigma_1^0\cdots P\sigma_n^0); \quad (35)$$

note in particular that O changes f into a constant (independent of $\sigma_1\cdots\sigma_n$). Next we note from (31), (28), and (30) that

$$O[c_P(\sigma_1\cdots\sigma_n)\psi(\mathbf{r}_1\sigma_1\cdots\mathbf{r}_n\sigma_n)] = c_P(\sigma_1\cdots\sigma_n)O\psi(\mathbf{r}_1\sigma_1\cdots\mathbf{r}_n\sigma_n) \quad (36)$$

provided that O is symmetric and is such that $O\psi$ is an allowable wave function if ψ is. These requirements are satisfied for the operator O defined by (35); hence

$$\sum_Q c_P(Q\sigma_1^0\cdots Q\sigma_n^0)\psi(\mathbf{r}_1Q\sigma_1^0\cdots\mathbf{r}_nQ\sigma_n^0) = c_P(\sigma_1\cdots\sigma_n)\sum_Q\psi(\mathbf{r}_1Q\sigma_1^0\cdots\mathbf{r}_nQ\sigma_n^0). \quad (37)$$

But the left side of (37) is independent of $\sigma_1\cdots\sigma_n$, as is the $\sum_Q$ on the right side. Hence $c_P(\sigma_1\cdots\sigma_n)$ *must in fact be independent of* $\sigma_1\cdots\sigma_n$ provided that the $\sum_Q$ on the right side does not vanish. Although this sum vanishes for some ψ and some choices of $\sigma_1^0\cdots\sigma_n^0$, it cannot vanish for all ψ and all choices of $\sigma_1^0\cdots\sigma_n^0$.[29] Thus, c_P is indeed independent of $\sigma_1\cdots\sigma_n$. But then, as in Sec. 3, the proof of the symmetrization postulate is completed immediately by (3)–(6).

[27] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover Publications, Inc., New York, 1950), p. 282, Eq. (1.2).

[28] It is readily verified directly from (35) that (31) is satisfied.

[29] If one were to try to apply a similar argument to the function $c_P(\mathbf{r}_1\cdots\mathbf{r}_n)$ of Sec. 3 in order to shorten the proof there, one would encounter the difficulty that $\sum_Q\psi(Q\mathbf{r}_1^0\cdots Q\mathbf{r}_n^0)$ vanishes identically if ψ has any irreducible permutation symmetry other than Bose (completely symmetric). Thus the proof would go through only if one assumed Bose symmetry from the beginning, clearly a circular argument. On the other hand, for any choice of irreducible permutation symmetry (or any mixture of such symmetries) with respect to identical simultaneous permutations of both the $\mathbf{r}_j$ and σ_j, there are some ψ which possess a nonvanishing completely symmetric part with respect to permutations of $\sigma_1\cdots\sigma_n$ alone.

5. DISCUSSION

It has been shown for identical particles without internal degrees of freedom that the symmetrization postulate can be proved starting with the following assumed properties: (a) probability densities are permutation-invariant, (b) allowable wave functions are continuous with continuous gradient, (c) the $3n$-dimensional configuration space is connected, (d) the Hamiltonian is symmetric (commutes with all permutations), and (e) the nodes of allowed wave functions have certain properties described in detail in Sec. 3. Requirement (c) played a crucial role in the proof; it was shown by means of counterexamples that the symmetrization postulate is not true for hard spheres in one dimension, for which (c) fails. A similar counterexample in *three* dimensions is provided by hard spheres of diameter a moving in a channel of diameter $<2a$. Actually, however, such a system is basically one-dimensional, the configuration space being disconnected for the same reason as in the counterexamples of Sec. 2. Similarly, a system confined in a region $\mathfrak{R}$ consisting of two or more disjoint subregions has a configuration space $\mathfrak{C}$ which is disconnected in an entirely trivial fashion. It is well known that in such a case the wave functions need not have any particular symmetry under exchanges between disjoint pieces of $\mathfrak{R}$. For normal three-dimensional systems with a connected $\mathfrak{R}$, the space $\mathfrak{C}$ is connected; this connectivity played a crucial role in the proof. The proof was extended to particles with internal degrees of freedom by using the additional properties (f) that any symmetric observable (operator) commutes with every permutation and that the particular symmetric operator O defined by (34) is such that $O\psi$ is an allowable wave function if ψ is.

Our basic conclusion, then, is that for systems of identical particles with a *connected spatial* configuration space, SP is a *necessary* property of allowable wave functions; if the spatial configuration space is not connected, then SP is *not* necessary, although, as is well known, it may, for purposes of convenience, be *consistently imposed* without contradicting observable properties of the system. It will be noted that the assumptions (a)–(f) on which the proof is based are all generally accepted and physically understandable; extraneous mathematical assumptions, e.g., the existence of a "complete" set of commuting observables, were not employed. It has, however, been pointed out by Messiah and Greenberg[2] that assumption (a) is not an inescapable consequence of the indistinguishability of identical particles. However, we have adopted the more conservative viewpoint that probability densities should be invariant under permutation of identical particles.

It is important to realize, however, that the decision as to whether two particles are identical or not is partly a matter of convention. The elaboration of this viewpoint occupies the following section.

An entirely separate question, with which we have not concerned ourselves, is that of how one decides whether a given species of identical particles should have Bose or Fermi symmetry, having established the fact that these two are the only possibilities. The well-known answer, i.e., the spin-statistics theorem which asserts that particles with integral spin are bosons whereas those with half-odd-integral spin are fermions, has been proved within the framework of relativistic field theory; no proof that operates in the realm of nonrelativistic wave mechanics is known at present.

6. SIMILARITY VERSUS IDENTITY

It is well known that similar[30] particles can be viewed and treated in two distinct but equivalent ways, either as separate species of particles or as different states of a single species. Thus, although in the old viewpoint neutrons and protons were regarded as distinct species, it is now customary to regard them as merely different states of a single species of particle, the nucleon; one then differentiates between the neutron and the proton by asserting that they have different values of an internal variable called the isotopic spin. In spite of the fact that the neutron and proton differ not only in charge but also (slightly) in mass and magnetic moment, there nevertheless exists a one-one correspondence[31] between the two viewpoints, such that one obtains the same answers in all calculations regardless of whether one uses the old formalism in which wave functions are antisymmetric under permutations of neutrons among themselves and protons among themselves but have no particular symmetry under exchange of a neutron with a proton, or the isotopic-spin formalism in which the wave function is completely antisymmetric under all permutations of nucleons provided that the isotopic spin variable, which distinguishes between neutrons and protons, is exchanged along with the position and ordinary spin. Although this correspondence is well understood, it seems not out of place to restate it here, in view of the close connection with the symmetrization postulate.

Before stating the correspondence we shall define precisely what is meant by the term "similar." We shall say that two particles are similar if and only if they both have integral or both half-odd-integral spin and their internal variables can be enumerated in the same way, i.e., put into one-one correspondence.[32]

As an example, protons and electrons are similar. Consider a system of one proton and one electron (hydrogen atom) described by wave functions of the form

$$\Psi(\mathbf{r}_p\sigma_p,\mathbf{r}_e\sigma_e)\,, \tag{38}$$

where $\mathbf{r}$ and σ denote position and spin z component, whereas the subscripts p and e refer to the proton and electron. These wave functions Ψ have no symmetry with respect to interchange of the proton and electron. The Hamiltonian is of the structure

$$\begin{aligned}\mathcal{H}=-(\hbar^2/2m_p)\nabla_p^2-(\hbar^2/2m_e)\nabla_e^2\\ +V_p(\mathbf{r}_p\sigma_p)+V_e(\mathbf{r}_e\sigma_e)+V_{pe}(\mathbf{r}_p\sigma_p,\mathbf{r}_e\sigma_e)\end{aligned} \tag{39}$$

in the general case where effects of external fields and spin-spin and spin-orbit interactions are included. If we now identify the proton and electron, i.e., regard them as different states of the same particle, we may describe the system by wave functions of the form

$$\psi(\mathbf{r}_1\sigma_1s_1,\mathbf{r}_2\sigma_2s_2)\,, \tag{40}$$

where $\mathbf{r}_j$ and σ_j are the position and spin,[33] whereas s_j is the *species label* (analogous to isotopic spin z-component), which may take on the two values $+$(proton) and $-$(electron). We can now set up a one-one correspondence between the two descriptions by the prescription

$$\begin{aligned}\Psi(\mathbf{r}_p\sigma_p,\mathbf{r}_e\sigma_e)&=2^{1/2}\psi(\mathbf{r}_p\sigma_p+,\mathbf{r}_e\sigma_e-)\,,\\ \psi(\mathbf{r}_1\sigma_1+,\mathbf{r}_2\sigma_2-)&=2^{-1/2}\Psi(\mathbf{r}_1\sigma_1,\mathbf{r}_2\sigma_2)\,,\\ \psi(\mathbf{r}_1\sigma_1+,\mathbf{r}_2\sigma_2+)&=\psi(\mathbf{r}_1\sigma_1-,\mathbf{r}_2\sigma_2-)=0\end{aligned} \tag{41}$$

together with the requirement[34] that ψ be antisymmetric under exchange of $(\mathbf{r}_1\sigma_1s_1)$ with $(\mathbf{r}_2\sigma_2s_2)$. The factors $2^{\pm1/2}$ are necessary because of the additional summation over s_1 and s_2 involved in computing the norm of ψ; it is trivial to verify that with inclusion of these factors, the correspondence (41) preserves all norms and scalar products. Furthermore, if the Hamiltonian H acting on ψ is related to that $\mathcal{H}$ acting on Ψ by

$$\begin{aligned}H&=H_1+H_2+H_{12}\,,\\ H_1&=\delta_{s_1+}[-(\hbar^2/2m_p)\nabla_1^2+V_p(\mathbf{r}_1\sigma_1)]\\ &\quad+\delta_{s_1-}[-(\hbar^2/2m_e)\nabla_1^2+V_e(\mathbf{r}_1\sigma_1)]\,,\\ H_2&=\delta_{s_2+}[-(\hbar^2/2m_p)\nabla_2^2+V_p(\mathbf{r}_2\sigma_2)]\\ &\quad+\delta_{s_2-}[-(\hbar^2/2m_e)\nabla_2^2+V_e(\mathbf{r}_2\sigma_2)]\,,\\ H_{12}&=\delta_{s_1+}\delta_{s_2-}V_{pe}(\mathbf{r}_1\sigma_1,\mathbf{r}_2\sigma_2)+\delta_{s_1-}\delta_{s_2+}V_{pe}(\mathbf{r}_2\sigma_2,\mathbf{r}_1\sigma_1)\,,\end{aligned} \tag{42}$$

then it is readily verified that

$$(\psi_b,H\psi_a)=(\Psi_b,\mathcal{H}\Psi_a) \tag{43}$$

for wave functions related by (41); furthermore, H is symmetric under exchange of $(\mathbf{r}_1\sigma_1s_1)$ with $(\mathbf{r}_2\sigma_2s_2)$, as is required of any observable of a system of *identical* particles. The same correspondence holds for matrix elements of any observable provided that the correspondence between operators is set up in the same way as that between (39) and (42). Thus the results of all calculations are the same regardless of whether one takes the usual viewpoint that p and e are distinct species or instead (admittedly artificially) views them

[30] We shall presently give a precise definition of the term "similar."

[31] See, e.g., A. Messiah, Ref. 1, pp. 619 ff.

[32] Cf. Messiah and Greenberg, Ref. 2, Sec. 4 and Messiah, Ref. 1, p. 626.

[33] At this point we make use of the fact that the proton and electron spin may be enumerated in the same way.

[34] This requirement determines $\psi(\mathbf{r}_1\sigma_1-,\mathbf{r}_2\sigma_2+)$, given Ψ.

as identical particles in different internal states (differentiated by s_j), provided that one calculates consistently in both cases.

In the general case of ν similar species of particles with n_j identical particles in the jth species, described in the conventional formulation by wave functions of the form

$$\Psi(\mathbf{r}_1{}^{(1)}\sigma_1{}^{(1)}\cdots\mathbf{r}_{n_1}{}^{(1)}\sigma_{n_1}{}^{(1)}|\mathbf{r}_1{}^{(2)}\sigma_1{}^{(2)}\cdots\mathbf{r}_{n_2}{}^{(2)}\sigma_{n_2}{}^{(2)}|\cdots \times|\mathbf{r}_1{}^{(\nu)}\sigma_1{}^{(\nu)}\cdots\mathbf{r}_{n_\nu}{}^{(\nu)}\sigma_{n_\nu}{}^{(\nu)}), \quad (44)$$

where the superscript enumerates species, one can identify all species and hence describe the system by wave functions of the form

$$\psi(\mathbf{r}_1\sigma_1 s_1\cdots\mathbf{r}_n\sigma_n s_n), \quad (45)$$

where

$$\sum_{j=1}^{\nu} n_j = n \quad (46)$$

and the species labels s_j take on all values from 1 to ν. The one-one correspondence between wave functions is given by the following generalization of (41):

$$\Psi(\mathbf{r}_1{}^{(1)}\sigma_1{}^{(1)}\cdots\mathbf{r}_{n_\nu}{}^{(\nu)}\sigma_{n_\nu}{}^{(\nu)}) = (n!/n_1!\cdots n_\nu!)^{1/2}\psi(\mathbf{r}_1{}^{(1)}\sigma_1{}^{(1)}1\cdots r_{n_1}{}^{(1)}\sigma_{n_1}{}^{(1)}1\cdots\mathbf{r}_1{}^{(\nu)}\sigma_1{}^{(\nu)}\nu\cdots\mathbf{r}_{n_\nu}{}^{(\nu)}\sigma_{n_\nu}{}^{(\nu)}\nu),$$

$$\psi(\mathbf{r}_1\sigma_1 1\cdots\mathbf{r}_{n_1}\sigma_{n_1}1\mathbf{r}_{n_1+1}\sigma_{n_1+1}2\cdots\mathbf{r}_{n_1+n_2}\sigma_{n_1+n_2}2\cdots\mathbf{r}_n\sigma_n\nu) = (n!/n_1!\cdots n_\nu!)^{-1/2}\Psi(\mathbf{r}_1\sigma_1\cdots\mathbf{r}_{n_1}\sigma_{n_1}|\mathbf{r}_{n_1+1}\sigma_{n_1+1}\cdots\mathbf{r}_{n_1+n_2}\sigma_{n_1+n_2}|\cdots\mathbf{r}_n\sigma_n), \quad (47)$$

$$\psi(\mathbf{r}_1\sigma_1 s_1\cdots\mathbf{r}_n\sigma_n s_n)=0 \quad \text{unless} \quad \sum_{j=1}^{n}\delta_{s_j k}=n_k \quad \text{for} \quad 1\leqslant k\leqslant\nu.$$

If one requires that configurational probabilities $|\psi|^2$ be permutation-invariant (including species labels as part of the specification of configuration), then the truth of the symmetrization postulate for systems with a connected spatial configuration space follows by the proof of Sec. 4, the choice between Bose or Fermi being determined by the spin-statistics theorem. Thus ψ is determined for all orders of the species labels, in spite of the fact that (47) only directly specifies it for the special order $1\cdots1\ 2\cdots2\cdots\nu\cdots\nu$. Defining the operator correspondence by the obvious generalization of (42), one finds that the two treatments are again completely equivalent.

If one is willing to remove the restriction that each species contains a fixed number of particles, then the correspondence may be operated in the other direction: Given a system of identical particles with internal degrees of freedom, one may choose to regard particles differing in the value of some discrete internal variable as distinct species. Since, however, there will in general be dynamical processes which change the value of that internal variable and hence, in the latter viewpoint, change a particle of one species into that of another, the wave function must then in general be represented in Fock space. For example, a system of electrons can consistently be regarded as a mixture of two distinct species, a "spin-up particle" species and a "spin-down particle" species. Then an external magnetic field or spin-spin or spin-orbit interactions will lead to terms in the new Hamiltonian which change spin-up particles into spin-down ones and vice versa. The familiar prescription, according to which spin-independent properties of a system of electrons in the absence of magnetic fields, spin-spin, and spin-orbit interactions may be calculated by ignoring the electron spin but doubling the density of states in final expressions, is readily interpretable in terms of the spin-up–spin-down formalism, since for a spin-independent Hamiltonian H in the usual formulation the spin-up–spin-down Hamiltonian $\mathcal{H}$ differs from H only in notation. Nevertheless, in a second-quantized formalism annihilation and creation operators for electrons of opposite spin *anti*-commute, whereas annihilation and creation operators for spin-up particles *commute* with those for spin-down ones.

This property of the annihilation and creation operators clearly extends to the case of a mixture of an arbitrary number of similar species. If one has a system of several similar species, then it is clear that if the species are regarded as distinct, then field operators referring to different species commute regardless of whether the particles are bosons of fermions. On the other hand, if different species are identified, then the corresponding field operators satisfy commutation (Bose) or *anti*commutation (Fermi) relations. Hence the familiar fiat of relativistic field theory, "kinematically *independent* fermion fields *anti*commute[35]" would seem to imply that all fermions are being viewed as various states of a single universal fermion, although this is contradicted by the terminology "kinematically independent." From our nonrelativistic point of view, commutation relations between distinct fermion species would seem much more natural.

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[35] See, e.g., R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That* (W. A. Benjamin, Inc., New York, 1964), pp. 146 ff.