

Gauge Independence and Path Independence*

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Quantum electrodynamics is formulated in a gauge-independent as well as path-independent way. The theory is manifestly Lorentz covariant. It can be obtained from the path-dependent electrodynamics of Mandelstam by an averaging process. The commutation relations and propagators in this gauge-independent formulation correspond to those of the Landau gauge in the conventional theory, giving this gauge a fundamental significance. The method is also applied to the gauge-invariant theory of neutral vector mesons with nonvanishing mass interacting with a conserved current. The resulting theory is gauge-independent and describes a pure spin-1 particle, the spin-0 contributions being identically zero.

I. INTRODUCTION

RECENTLY, considerable attention has been paid to a formulation of quantum electrodynamics in terms of fields rather than potentials.¹⁻⁴ That makes it possible to work without any reference to particular gauges and thus avoids all the complications and unphysical features associated with any of them. According to this philosophy the fundamental quantities are the fields and the quantization is carried out directly on them by specifying their commutation relations. The potentials appear as derived, strictly auxiliary quantities and their introduction into the theory is merely a mathematical device for simplifying the field equations, just as in the classical theory.

However, the formulation by Rohrlich³ is not manifestly covariant. It corresponds to the Coulomb gauge, in which surface-dependent terms occur and only the scattering matrix is invariant. The intermediate equations involve the normal vector to a space-like plane. On the other hand, the DeWitt-Mandelstam formulation is completely covariant, but involves a "path dependence" of certain quantities.

In the present paper we propose a formulation of electrodynamics in terms of field strengths which is both manifestly covariant *and* path-independent. It thus combines the advantageous features of both previous gauge-independent formulations. The basic quantity is the gauge- and path-independent operator \mathcal{A}_μ which is defined in terms of the field strengths $F_{\mu\nu}$ in Eq. (9) below.

The arbitrariness in the choice of the "gauge," which deprives the conventional potential of a direct physical meaning, no longer affects the theory. Thus, the gauge-independent operator \mathcal{A}_μ may have a direct physical interpretation as photon-field operator. The Lorentz condition is automatically satisfied by \mathcal{A}_μ as a consequence of the antisymmetry of $F_{\mu\nu}$. No spin-zero field is therefore involved in the theory, as is the case, e.g. in Stueckelberg's formulation.⁵ The commutation relations for \mathcal{A}_μ are derived by assigning those for the fields $F_{\mu\nu}$; they will be proven to be the same as in the Landau gauge. The Lorentz condition is therefore consistent with them as an operator equation, and no problem arises requiring *ad hoc* measures such as the use of an indefinite metric with the consequent appearance of unphysical states. The success of the Landau propagator ($Z_1=Z_2$, finite) may thus be justified by its connection with a gauge-independent formulation of quantum electrodynamics. It may be interesting to note that the recent attempts at a finite electrodynamics by Johnson *et al.*⁶ make use of the Landau gauge for the photon propagator in first approximation.

These results may be extended to the theory of massive vector bosons interacting with a conserved current, where analogous problems of gauge invariance arise.^{7,8} Here too, a gauge-independent field operator \mathcal{A}_μ may be introduced. The physical meaning of gauge independence in this theory is that \mathcal{A}_μ described a pure spin-1 field. The commutation relations for \mathcal{A}_μ , derived from those for $F_{\mu\nu}$, are

$$[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x')] = -i(g_{\mu\nu} - \partial_\mu \partial_\nu / \square) D(x-x').$$

In this formulation, the renormalizability of the theory is therefore explicit.⁸

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¹ S. Mandelstam, *Ann. Phys. (N. Y.)* **19**, 1 (1962). For a more general approach see M. Lévy, *Nucl. Phys.* **57**, 152 (1964).

² B. S. DeWitt, *Phys. Rev.* **125**, 2189 (1962).

³ F. Rohrlich, State University of Iowa Report 62-15, 1962 (unpublished). See also F. Rohrlich, Proceedings of the Midwest Theory Conference, Argonne National Laboratory, June 1963 (unpublished).

⁴ F. J. Belinfante, *Phys. Rev.* **128**, 2832 (1962).

⁵ E. C. G. Stueckelberg, *Helv. Phys. Acta* **11**, 225 (1938).

⁶ K. Johnson, M. Baker, and R. S. Willey, *Phys. Rev. Letters* **11**, 518 (1963); *Phys. Rev.* **136**, B1111 (1964).

⁷ V. I. Ogievetski and I. V. Polubarinov, *Zh. Eksperim. i Teor. Fiz.* **41**, 247 (1961) [English transl.: *Soviet Phys.—JETP* **14**, 179 (1962)].

⁸ A. A. Slavnov, *Zh. Eksperim. i Teor. Fiz.* **44**, 1119 (1963) [English transl.: *Soviet Phys.—JETP* **17**, 754 (1963)].

II. DEWITT-MANDELSTAM FORMULATION

Maxwell's equations may be simplified by introducing the potential A_μ , ($\mu=0, \dots, 3$), through the equation

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1}$$

Actually, Eq. (1) does not determine the potential A_μ uniquely. The field tensor $F_{\mu\nu}$ does not change if A_μ undergoes a gauge transformation (of the second kind),

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \Lambda,$$

where $\Lambda = \Lambda(x)$ is a scalar function which vanishes at infinity

$$\Lambda(x) \rightarrow 0 \text{ as } x^\mu \rightarrow \infty \quad (\mu=0, \dots, 3).$$

If the electromagnetic (e.m.) field is coupled to a matter field the gauge transformation on A_μ must be accompanied by a gauge transformation (of the first kind) of the matter field, in order that the field equations remain unchanged,

$$\psi \rightarrow \psi' = \exp[ie\Lambda]\psi.$$

The arbitrariness in the choice of A_μ may be regarded as the origin of difficulties which arise in connection with the quantization of the e.m. field.

The DeWitt-Mandelstam^{1,2} formulation of electro-

dynamics tries to overcome these difficulties by introducing *gauge-independent* field operators. The tensor $F_{\mu\nu}$ and a suitably defined gauge-independent operator Ψ are chosen as the basic fields.

We start by defining a gauge-independent potential. Let a_μ and ψ be the potential and the matter field operator in some *arbitrary* gauge. A gauge-independent potential A_μ may be defined by the following gauge transformation:

$$A_\mu = a_\mu + \partial_\mu \Lambda, \tag{2}$$

where

$$\Lambda = - \int_{-\infty}^0 a_\sigma(z) \frac{\partial z^\sigma}{d\xi} d\xi. \tag{2'}$$

Here $z^\mu(x, \xi)$ are four arbitrary single-valued differential functions of the space variables x_μ and a parameter ξ , which transform like a four-vector and satisfy the boundary conditions

$$z^\mu(x, 0) = x^\mu, \tag{3}$$

$$\lim_{\xi \rightarrow -\infty} z^\mu(x, \xi) = \text{spatial infinity.}^9 \tag{4}$$

The path integral in (2) is to follow a space-like path.

It is easy to show that A_μ is gauge-independent. One has in fact

$$A_\mu - a_\mu \equiv \frac{\partial \Lambda}{\partial x^\mu} = - \int_{-\infty}^0 \frac{\partial a_\sigma}{\partial z^\rho} \frac{\partial z^\rho}{\partial x^\mu} \frac{\partial z^\sigma}{\partial \xi} d\xi - \int_{-\infty}^0 a_\sigma(z) \frac{\partial}{\partial \xi} \frac{\partial z^\sigma}{\partial x^\mu} d\xi = - \int_{-\infty}^0 \frac{\partial a_\sigma}{\partial z^\rho} \frac{\partial z^\rho}{\partial x^\mu} \frac{\partial z^\sigma}{\partial \xi} d\xi + \int_{-\infty}^0 \frac{\partial a_\sigma}{\partial \xi} \frac{\partial z^\sigma}{\partial x^\mu} d\xi - a_\sigma(z) \frac{\partial z^\sigma}{\partial x^\mu} \Big|_{-\infty}^0 = \int_{-\infty}^0 F_{\nu\sigma}(z) \frac{\partial z^\sigma}{\partial x^\mu} \frac{\partial z^\nu}{\partial \xi} d\xi - a_\mu,$$

i.e.,

$$A_\mu(x) = \int_{-\infty}^0 F_{\nu\sigma}(z) \frac{\partial z^\sigma}{\partial x^\mu} \frac{\partial z^\nu}{\partial \xi} d\xi. \tag{5}$$

Then, if a_μ and a'_μ describe the same field, but in two different gauges, the operators A_μ and A'_μ ,

$$A_\mu \equiv a_\mu + \partial_\mu \Lambda, \quad \Lambda \equiv - \int_{-\infty}^0 \alpha_\sigma(z) \frac{\partial z^\sigma}{\partial \xi} d\xi$$

and

$$A'_\mu \equiv a'_\mu + \partial_\mu \Lambda', \quad \Lambda' \equiv - \int_{-\infty}^0 \alpha'_\sigma(z) \frac{\partial z^\sigma}{\partial \xi} d\xi,$$

coincide (gauge independence of A_μ). Thus Eqs. (2) and (2') enable us to pass from a class of gauge-dependent potentials to a field variable which is gauge-independent.

The gauge-independent matter-field operator is defined by

$$\Psi = \exp[ie\Lambda]\psi,$$

where Λ is given by Eq. (2'). The gauge independence of Ψ is easily shown:

$$\begin{aligned} \Psi &= \exp[ie\Lambda]\psi = \exp[ie\Lambda'] \\ &\quad \times \exp[ie(\Lambda - \Lambda')]\psi = \exp[ie\Lambda']\psi'. \end{aligned}$$

These operations are possible also in the quantized theory as long as the paths for Λ and Λ' are the same. It is not difficult to verify that Ψ coincides with the field operator introduced by Mandelstam.

As the Eq. (5) uniquely determines the gauge-independent potential once the field tensor $F_{\mu\nu}$ is given, electrodynamics may be formulated either in terms of A_μ or in terms of $F_{\mu\nu}$. The latter alternative has been followed by Mandelstam.

The above definition of A_μ needs two remarks. First, on starting from Eq. (5) and using $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$, the relation (1) may be obtained only if explicit use is made of condition (4). Secondly, according to the definition (5), A_μ depends not only on $F_{\mu\nu}$ but also on the chosen path $z_\mu(x, \xi)$ along which the integration is performed. Thus the theory, while no longer gauge-dependent, now becomes path-dependent, and this may be regarded as just another kind of "gauge."

The above-mentioned path dependence may be re-

⁹ Spatial infinity here means any limit which is sufficiently remote in a space-like direction from x_μ , so that the field vanishes there; at this limit A_μ may without loss of generality be set equal to zero. We shall return to this assumption later.

moved by performing a sort of average over all possible paths. We shall see that such a procedure leads to results related to the usual gauges.

III. NONCOVARIANT PATH AVERAGE

Following Belinfante,⁴ we put

$$z^\mu = x^\mu + \epsilon^\mu \xi$$

and choose the Lorentz frame in which

$$\epsilon^0 = 0, \quad \epsilon^2 = 1, \quad \xi = -r \equiv -|\mathbf{z} - \mathbf{x}|.$$

The lines $z^\mu(x, \xi)$ are then confined to straight lines at $t = \text{const}$, converging to the field points \mathbf{x} in arbitrary directions given by unit vectors $\boldsymbol{\epsilon}$. Equation (5) reduces to

$$A_\mu(x) = \int_{-\infty}^0 F_{i\mu}(z) \epsilon^i d\xi \quad (6)$$

(sum over dummy indices). The potential A_μ now depends on the direction of $\boldsymbol{\epsilon}$. This dependence is removed by averaging Eq. (6) over all possible directions $\boldsymbol{\epsilon}$. By straightforward manipulation one obtains

$$\begin{aligned} \bar{A}_\mu(x) &= \int A_\mu \frac{d\Omega}{4\pi} = \int d^3z F_{i\mu}(z) \frac{\partial}{\partial z_i} \left(\frac{1}{4\pi r} \right) \\ &= - \int d^3z [\nabla^i F_{i\mu}(z)] / (4\pi r). \end{aligned} \quad (7)$$

Here $d\Omega$ is the infinitesimal solid angle in the direction $\boldsymbol{\epsilon}$. Equation (7) gives explicitly

$$\bar{A}^0(x) = \int d^3z \frac{\text{div} \mathbf{E}(\mathbf{z}, t)}{4\pi r}, \quad \bar{\mathbf{A}} = \int d^3z \frac{\text{curl} \mathbf{E}(\mathbf{z}, t)}{4\pi r}. \quad (7')$$

Thus the potential \bar{A}_μ coincides with the potential in the Coulomb gauge.

As it should be expected, the average performed above has led to noncovariant results due to the choice of the plane $t = \text{const}$ as a privileged plane in a given Lorentz frame. This breaks the manifest covariance just like the definition of the Coulomb gauge, $\text{div} \mathbf{A} = 0$.

IV. COVARIANT PATH AVERAGE

In order to obtain a Lorentz-invariant result we must perform a covariant average, i.e., an average over paths covering a Lorentz-invariant region. Now, the Lorentz-invariant subregions of Minkowski space are the space-like region outside the light cone, the time-like regions inside it, the light cone itself, and the whole space. Averaging over straight lines covering any of the first two regions leads to divergent integrals unless special prescriptions are given.¹ Therefore, we shall average over *all* straight lines converging to the point where the potential has to be calculated.

Putting

$$z_\nu = x_\nu + \epsilon_\nu \xi, \quad \epsilon_\nu \epsilon^\nu = \pm 1 \quad (8)$$

(with the plus sign if the line is space-like and the minus if it is time-like), and using Eq. (5), we have:

$$\begin{aligned} A_\mu(x) &= \int_{-\infty}^0 F_{\nu\mu}(x + \epsilon\xi) \frac{z^\nu - x^\nu}{\xi} d\xi \\ &= \int_0^\infty F_{\mu\nu}(x + y) \frac{y^\nu}{4\xi} d\xi, \end{aligned}$$

where the new variables $y^\nu \equiv z^\nu - x^\nu$ and $\zeta \equiv (y^\nu y_\nu)^2$, have been introduced.

We now wish to average over all directions of the unit vector ϵ^ν . To this end we note first that in 4-dimensional *Euclidean* space

$$\begin{aligned} d^4y &= d^3y dy_4 = \frac{1}{4} d\zeta d^2\Omega (d\mu_E / (1 + \mu_E^2)^2), \\ \zeta &= (y_\mu y^\mu)^2, \quad \mu_E = y_4 / |\mathbf{y}|, \end{aligned}$$

where $0 \leq \zeta < \infty$ and $-\infty < \mu < +\infty$. The transition to Minkowski space is governed by the relation $y_4 = iy^0$ but permits two alternative distortions of the path along the real y_4 axis to one along the imaginary y_4 axis, viz., by a rotation by $\frac{1}{2}\pi - \epsilon$ or $-(\frac{1}{2}\pi - \epsilon)$. Those alternatives give contours in the complex y^0 plane which avoid the poles according to $y^0 = \pm(|\mathbf{y}| + i\epsilon)$ or $y^0 = \pm(|\mathbf{y}| - i\epsilon)$. They correspond to the D_{1R} and D_{1A} functions, respectively. Time symmetry is preserved if we take the contour for $\frac{1}{2}(D_{1R} - D_{1A}) = \frac{1}{2}(D_R - D_A) = D_P$ as indicated in the Appendix. The choice of any of these contours is completely determined by the asymptotic conditions.

The integrals in Minkowski space are thus evaluated over $\mu = y^0 / |\mathbf{y}| = -i\mu_E$ with the above contour choices in the μ plane,

$$d^4y = d^3y dy^0 = \frac{1}{4} d\zeta d^2\Omega (d\mu / (1 - \mu^2)^2).$$

The surface of the pseudosphere in M_4 is

$$\int d^3\Omega = \int d^2\Omega \int_{-\infty}^{\infty} \frac{d\mu}{(1 - \mu^2)^2} = 2\pi^2 i.$$

The direction average of $A_\mu(x)$ now becomes, choosing one of the above contours, C_R , C_A , C_{1R} , or C_{1A} ,

$$\begin{aligned} \bar{\alpha}_\mu(x) &= \frac{1}{2\pi^2 i} \int A_\mu(x, \epsilon) d^3\Omega \\ &= \frac{1}{2\pi^2 i} \int F_{\mu\nu}(x + y) \frac{y^\nu}{4\xi} d\zeta d^2\Omega \frac{d\mu}{(1 - \mu^2)^2} \\ &= \frac{1}{2\pi^2 i} \int F_{\mu\nu}(x + y) \frac{y^\nu}{(y_\alpha y^\alpha)^2} d^4y \\ &= \frac{i}{4\pi^2} \int F_{\mu\nu}(x + y) \frac{\partial}{\partial y_\nu} \left(\frac{1}{y^2} \right) d^4y. \end{aligned} \quad (9)$$

Here $y^2 \equiv y_\alpha y^\alpha$ and the integral extends over all Minkowski space, restricted only by the path C_α . The resultant \mathcal{Q}_μ are obviously Lorentz-covariant.

The result (9) could also have been obtained as follows. One notes that $1/x^2$ is the Green function for the inhomogeneous d'Alembert equation,

$$\square(1/x^2) = -4\pi^2 i \delta(x), \quad (10)$$

provided that both sides of the equation are regarded as distributions, the left side being defined by one of the paths C_R , C_A , C_{1R} , or C_{1A} . Consequently, for any of these paths

$$\begin{aligned} F_{\mu\nu}(x) &= \frac{i}{4\pi^2} \int F_{\mu\nu}(x+y) \partial_\alpha \partial^\alpha \left(\frac{1}{y^2} \right) d^4y \\ &= \frac{-i}{4\pi^2} \int \partial_\alpha F_{\mu\nu}(x+y) \partial^\alpha \left(\frac{1}{y^2} \right) d^4y, \end{aligned}$$

since the surface integral vanishes. If the homogeneous Maxwell equations hold, this becomes

$$F_{\mu\nu}(x) = \frac{i}{4\pi^2} \int (\partial_\mu F_{\nu\alpha} - \partial_\nu F_{\mu\alpha}) \partial^\alpha \left(\frac{1}{y^2} \right) d^4y.$$

Hence, if we define $\mathcal{Q}_\mu(x)$ as in (9), this equation yields

$$\partial_\mu \mathcal{Q}_\nu - \partial_\nu \mathcal{Q}_\mu = F_{\mu\nu}. \quad (11)$$

Equation (9) may be put into the form:

$$\begin{aligned} \mathcal{Q}_\mu(x) &= \frac{i}{4\pi^2} \int \partial^\nu F_{\nu\mu}(x+y) \frac{d^4y}{y^2} + \frac{i}{4\pi^2} \int \partial^\nu \left(\frac{F_{\nu\mu}(x+y)}{y^2} \right) d^4y \\ &= -\frac{i}{4\pi^2} \int \frac{j_\mu(x+y)}{y^2} d^4y + \frac{i}{4\pi^2} \int \partial^\nu \left(\frac{F_{\nu\mu}(x+y)}{y^2} \right) d^4y, \end{aligned} \quad (12)$$

where an integration by parts has been performed and use has been made of the inhomogeneous Maxwell equations,

$$\partial^\nu F_{\nu\mu} = -j_\mu.$$

Equation (12) provides a decomposition of \mathcal{Q}_μ into a "bound" and a free field

$$\mathcal{Q}_\mu = \mathcal{Q}_\mu^{(b)} + \mathcal{Q}_\mu^{(f)}, \quad (13)$$

because

$$\square \mathcal{Q}_\mu^{(b)} = -j_\mu \quad \text{and} \quad \square \mathcal{Q}_\mu^{(f)} = 0. \quad (14)$$

Consider, for example, the bound field produced by the contour C_A ; according to the Appendix,

$$\begin{aligned} \frac{1}{4\pi^2 i} \int_{C_A} \frac{j_\mu(x+y)}{y^2} d^4y &= \int j_\mu(x+y) D_A(y) d^4y \\ &= \int j_\mu(z) D_A(z-x) d^4z = \int D_R(x-z) j_\mu(z) d^4z. \end{aligned}$$

The well-known retarded and advanced functions are here denoted by D_R and D_A . Thus,

$$\begin{aligned} \mathcal{Q}_\mu^{\text{ret}}(x) &= \int D_R(x-z) j_\mu(z) d^4z \\ &= \frac{1}{4\pi^2 i} \int_{C_A} \frac{j_\mu(x+y)}{y^2} d^4y. \end{aligned} \quad (15)$$

The asymptotic behavior of this function leads to the identification

$$\mathcal{Q}_\mu^{\text{in}}(x) = \frac{1}{4\pi^2 i} \int_{C_A} \partial^\nu \left(\frac{F_{\nu\mu}(x+y)}{y^2} \right) d^4y. \quad (16)$$

The expression (15) is easily seen to be equivalent to the Sommerfeld representation of the retarded potential.¹⁰

The decomposition (13) can thus be written as

$$\mathcal{Q}_\mu = \mathcal{Q}_\mu^{\text{ret}} + \mathcal{Q}_\mu^{\text{in}} = \mathcal{Q}_\mu^{\text{adv}} + \mathcal{Q}_\mu^{\text{out}} \quad (13')$$

depending on the choice of the contour, C_A or C_R , respectively. Of special interest is the decomposition

$$A_\mu(x) = A_\mu^{(+)}(x) + \bar{A}_\mu(x), \quad (17)$$

which is obtained as the arithmetic mean of Eqs. (13') and which is time-symmetric. It plays an important role in the classical theory of point charges.¹¹ Here

$$\mathcal{Q}_\mu^{(+)}(x) = \frac{1}{2} (\mathcal{Q}_\mu^{\text{ret}} + \mathcal{Q}_\mu^{\text{adv}}) = \frac{1}{8\pi^2 i} \int_{C_P} \frac{j_\mu(x+y)}{y^2} d^4y \quad (18)$$

and

$$\begin{aligned} \bar{\mathcal{Q}}_\mu(x) &= \frac{1}{2} (\mathcal{Q}_\mu^{\text{in}} + \mathcal{Q}_\mu^{\text{out}}) \\ &= \frac{1}{8\pi^2 i} \int_{C_P} \partial^\nu \left(\frac{F_{\nu\mu}(x+y)}{y^2} \right) d^4y. \end{aligned} \quad (19)$$

They also satisfy (14).

We note parenthetically that a decomposition of \mathcal{Q}_μ according to the contours C_{1R} and C_{1A} also leads to (13) and (14) and the arithmetic mean of these two leads also to (17).

The antisymmetry of $F_{\mu\nu}$ assures that the Lorentz condition is satisfied as an identity:

$$\begin{aligned} \partial^\mu \mathcal{Q}_\mu(x) &= \frac{i}{4\pi^2} \int \partial^\mu F_{\mu\nu}(x+y) \partial^\nu \frac{1}{y^2} d^4y \\ &= \frac{1}{4\pi^2 i} \int F_{\mu\nu}(x+y) \partial^\mu \partial^\nu \frac{1}{y^2} d^4y \equiv 0. \end{aligned} \quad (20)$$

The absence of a spin-zero field is thus proven also for the case of interactions. This result combined with (14) yields charge conservation.

¹⁰ A. Sommerfeld, *Ann. Physik* **33**, 649 (1910). See also C. Møller, *The Theory of Relativity* (Oxford University Press, Oxford, 1960), p. 144 ff.

¹¹ F. Rohrlich, *Phys. Rev. Letters* **12**, 375 (1964). See also F. Rohrlich, *Classical Charged Particles* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1965).

The path average performed on the potential can also be applied to the electron field. The path-dependent, though gauge-independent field

$$\Psi(x, \epsilon) = L(x, \epsilon) \psi(x), \quad L(x, \epsilon) = \exp \left[- \int_{-\infty}^0 a_\sigma(z) \epsilon^\sigma d\xi \right]$$

involves the unit vector ϵ_ν of Eq. (8). The path independent Ψ is now related to the path-dependent one by

$$\Psi(x) = \frac{1}{2\pi^2 i} \int \Psi(x, \epsilon) d^3\Omega = L(x) \psi(x)$$

in complete analogy to (9).

V. GAUGE- AND PATH-INDEPENDENT ELECTRODYNAMICS

In the classical case we can start with the *nonlocal* modification of the usual Lagrangian density

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j_\mu \mathcal{A}^\mu + L_f[\Psi], \quad (21)$$

where $\mathcal{A}_\mu(x)$ is defined by (9). The basic fields are therefore $F_{\mu\nu}$ and Ψ . The homogeneous Maxwell equations then imply (11) and the antisymmetry of $F_{\mu\nu}$ implies (20). The inhomogeneous equations follow in the usual fashion.

We note that of the various gauge-dependent choices which would follow from (11) and which lead to different

representations of the commutation relations,¹²⁻¹⁴ Eq. (9) fixes the one which is gauge-independent.

Consider now the quantization of the free fields,

$$\begin{aligned} [F_{\mu\nu}(x), F_{\rho\sigma}(x')] &= i d_{\mu\rho, \nu\sigma} D(x-x'), \\ d_{\mu\rho, \nu\sigma} &\equiv g_{\mu\rho} \partial_\nu \partial_\sigma + g_{\nu\sigma} \partial_\mu \partial_\rho \\ &\quad - g_{\mu\sigma} \partial_\nu \partial_\rho - g_{\nu\rho} \partial_\mu \partial_\sigma. \end{aligned} \quad (22)$$

In order to find the corresponding commutation relations for \mathcal{A}_μ , we note first that (10) implies, for any function $f(x)$ with appropriate asymptotic behavior and any one of the indicated contours, that

$$\begin{aligned} f(x) &= \frac{1}{4\pi^2 i} \int \partial^\mu \frac{1}{y^2} \partial_\mu^{(y)} f(x \pm y) d^4 y \\ &= \pm \frac{1}{4\pi^2 i} \partial_\mu \int \partial^\mu \left(\frac{1}{y^2} \right) f(x \pm y) d^4 y. \end{aligned} \quad (23)$$

The vector operator

$$(\partial^{-1})_\mu \equiv \partial^{-1} \frac{\partial_\mu}{\square}$$

can thus be defined by¹⁵

$$\partial^{-1} f(x) \equiv \pm \frac{1}{4\pi^2 i} \int \partial_\mu \left(\frac{1}{y^2} \right) f(x \pm y) d^4 y. \quad (24)$$

A simple calculation now yields

$$\begin{aligned} [\mathcal{A}_\mu(x), \mathcal{A}_\nu(x')] &= \left(\frac{i}{4\pi^2} \right)^2 \int \partial^\rho \left(\frac{1}{y^2} \right) \partial'^\sigma \left(\frac{1}{y'^2} \right) d^4 y d^4 y' [F_{\mu\rho}(x+y), F_{\nu\sigma}(x'+y')] \\ &= i \left(\frac{i}{4\pi^2} \right)^2 \int \partial^\rho \left(\frac{1}{y^2} \right) \partial'^\sigma \left(\frac{1}{y'^2} \right) d^4 y d^4 y' (-g_{\mu\nu} \partial_\rho^{(y)} \partial_\sigma'^{(y)} + g_{\rho\sigma} \partial_\mu^{(x)} \partial_\nu^{(x)} \\ &\quad - g_{\mu\sigma} \partial_\rho^{(y)} \partial_\nu^{(x)} + g_{\rho\nu} \partial_\mu^{(x)} \partial_\sigma'^{(y)}) D(x-x'+y-y') \\ &= -i (g_{\mu\nu} - \partial^{-1} \frac{\partial_\nu}{\square}) D(x-x'). \end{aligned} \quad (25)$$

The gauge- and path-independent potentials therefore have nonlocal commutation relations which are formally just those employed in the Landau gauge.

However, the essentially nonlocal nature of the potentials plays an important role: Depending on the asymptotic conditions, a different contour is to be taken in the definition of ∂^{-1} , Eq. (24), which is implicit in (25). It follows that the commutation relations of the interacting fields depend on their asymptotic behavior.

It is obvious that the Lorentz condition (20) holds as an *operator* equation and that it is consistent with the commutation relations (25). The usual difficulties arising from the inconsistency of (20) with the commutation relations in various gauges therefore do not occur here.

In exactly the same way as in (25) one finds

$$\langle \mathcal{A}_\mu(x) \mathcal{A}_\nu(x') \rangle_0 = -i \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\square} \right) D_+(x-x')$$

¹² A. Peres, Nuovo Cimento **34**, 346 (1964).

¹³ B. Zumino, J. Math. Phys. **1**, 1 (1961).

¹⁴ S. Kamefuchi and H. Umezawa, Nuovo Cimento **31**, 429 (1964); **32**, 448 (1964).

¹⁵ When f is a generalized function, Eq. (24) is a convolution of such functions. These are more conveniently discussed in momentum space. In a future paper it will be shown that a consistent and unambiguous formulation of the operator ∂^{-1} can be given in that space for all the generalized functions which occur in quantum field theory. The present paper is therefore restricted entirely to x space.

and the propagator

$$\langle T(\mathcal{Q}_\mu(x)\mathcal{Q}_\nu(x')) \rangle_0 = -i \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\square} \right) D_c(x-x'). \quad (26)$$

Again, we have formally the same expression as in the Landau gauge. The success of this gauge in the recent formulation of quantum electrodynamics by Johnson *et al.*⁶ may thus find here an independent justification.

VI. MASSIVE NEUTRAL VECTOR-FIELD INTERACTING WITH A CONSERVED CURRENT

The theory of a massive neutral vector-field interacting with a conserved current may be given a form similar to electrodynamics.⁷ Gauge invariance plays a similar role in the formulation to the one it plays in electrodynamics—the Lagrangian, the field equations, and the equal-time commutation relations being invariant under gauge transformations.

The simultaneous equations

$$\begin{aligned} (\square - m^2)\phi_\mu &= -j_\mu \equiv -ig\bar{\psi}\gamma_\mu\psi, \\ (\gamma \cdot \partial + M)\psi &= i\phi^\mu\gamma_\mu\psi, \end{aligned} \quad (27)$$

are invariant under the simultaneous transformations

$$\begin{aligned} \phi_\mu &\rightarrow \phi_\mu + \partial_\mu\Lambda, \\ \psi &\rightarrow \exp[ig\Lambda]\psi, \end{aligned} \quad (28)$$

but the condition

$$\partial^\mu\phi_\mu = 0, \quad (29)$$

usually imposed to eliminate the spin-0 field, is not gauge-invariant unless Λ is restricted to solutions of the wave equation. Of course ϕ_μ itself is also gauge-dependent, as is evident from (28).

A gauge-independent formulation is, however, possible by using the fields $\phi_{\mu\nu}$ as basic operators; they are assumed to satisfy

$$\partial_\lambda\phi_{\mu\nu} + \partial_\mu\phi_{\nu\lambda} + \partial_\nu\phi_{\lambda\mu} = 0. \quad (30)$$

A gauge-independent potential Φ is now defined in terms of $\phi_{\mu\nu}$ by

$$\Phi_\mu(x) = \frac{i}{4\pi^2} \int_{C_I} \phi_{\mu\nu}(x+y) \frac{\partial^\nu}{y^2} d^4y, \quad (31)$$

where C_I is the contour of any inhomogeneous D -function.

The first Eq. (27) now reads

$$(\square - m^2)\Phi_\mu = -j_\mu, \quad (27')$$

where j_μ is conserved on account of the second Eq. (27).

Equation (30) and (31) combine to yield

$$\partial_\mu\Phi_\nu - \partial_\nu\Phi_\mu = \phi_{\mu\nu} \quad (32)$$

and

$$\partial^\mu\Phi_\mu = 0 \quad (33)$$

exactly as in (11) and (20). Equation (33) is now gauge-independent, in contradistinction to (29). It requires a conserved current as source in Eqs. (27) and (27').

This result was to be expected, because only the spin-0 part of the field ϕ_μ depends on the gauge.⁷ A gauge-independent formulation therefore must have the absence of this part as a consequence,

In the free case one quantizes the basic fields $\phi_{\mu\nu}$ according to

$$[\phi_{\mu\nu}(x), \phi_{\rho\sigma}(x')] = id_{\mu\rho, \nu\sigma} \Delta(x-x') \quad (34)$$

in analogy to (22). Exactly as before, the commutation relations for the Φ_μ are found to be

$$[\Phi_\mu(x), \Phi_\nu(x')] = -i \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\square} \right) \Delta(x-x') \quad (35)$$

and its propagator is

$$\langle T(\Phi_\mu(x), \Phi_\nu(x')) \rangle_0 = -i \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\square} \right) \Delta_c(x-x'), \quad (36)$$

where the definition (24) is used again.

This result makes the renormalizability of this theory manifest, no Dyson transformation being required.^{16,17}

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APPENDIX 1

The invariant functions $D_\alpha(x)$ will be assumed known and defined as before.¹⁸ It is then easy to verify that for any function $f(x)$ for which the integrals converge

$$\frac{1}{4\pi^2 i} \int_{C_\alpha} \frac{1}{x^2} f(x) d^4x = \int D_\alpha(x) f(x) d^4x \quad (A1)$$

or, symbolically,

$$\frac{1}{4\pi^2 i} \frac{1}{x^2} \Big|_{C_\alpha} = D_\alpha(x). \quad (A1')$$

The contours C_α are depicted in Fig. 1. The above equations hold for D_R, D_A, D_{1R} , and D_{1A} , as well as for D_+ ,

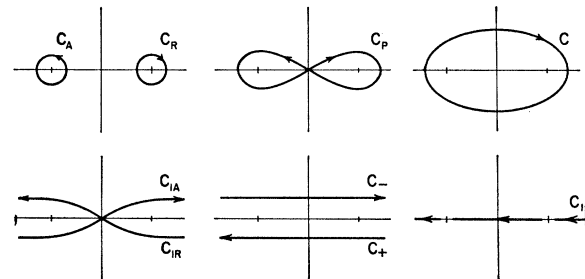


FIG. 1. The contours in the x^0 plane.

¹⁶ F. Dyson, Phys. Rev. **73**, 929 (1948).
¹⁷ R. J. Glauber, Progr. Theoret. Phys. (Kyoto) **9**, 295 (1953).
¹⁸ M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons*, (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955) (second printing, 1959). The second printing contains corrections of various misprints.

D_- , and D . Slight modifications are necessary for D_P and D_1 ,

$$\frac{1}{4\pi^2 i} \frac{1}{x^2} \Big|_{C_P} = 2D_P(x) \tag{A2}$$

and

$$\frac{1}{4\pi^2 i} \frac{1}{x^2} \Big|_{C_1} = \frac{1}{2} i D_1(x). \tag{A3}$$

It is to be noted that the above contours $C_\alpha(x)$ for integration in the complex x^0 plane are different from the contour $C_\alpha(p)$ for the corresponding Fourier transform.¹⁸ In fact, one has $C_R(x) = C_+(p)$, $C_+(x) = -C_A(p)$, etc. Only the contour for D is unchanged, $C(x) = C(p)$.

APPENDIX 2

In this Appendix we sketch the proof that $\mathcal{Q}_\mu(x)$ as defined by (9) is independent of the particular contour in the x^0 plane, as long as this contour corresponds to an invariant function which satisfies the inhomogeneous equation (10). To this end it suffices to show that

$$\int F_{\mu\nu}(x+y) \partial^\nu D_H(y) d^4y = 0, \tag{A4}$$

where $D_H(y)$ is an invariant function satisfying the homogeneous d'Alembert equation. Substitution of (11) and use of (20) yields

$$\int \partial^\nu (\partial_\mu \mathcal{Q}_\nu D_H - \mathcal{Q}_\mu \partial_\nu D_H) d^4y = - \left(\int_{y^0 \rightarrow +\infty} - \int_{y^0 \rightarrow -\infty} \right) (\partial_\mu \mathcal{Q}_0 D_H - \mathcal{Q}_\mu \partial_0 D_H) d^3\sigma. \tag{A5}$$

Since D_H depends only y^0 and $|\mathbf{y}|$ and is independent of the direction of \mathbf{y} , the integration over the directions can be carried out. Using the Fourier transform we have for any function $f(x+y)$ and with $\omega = |\mathbf{k}|$,

$$\begin{aligned} \int f(x+y) D_H(y) d^3y &= \frac{1}{(2\pi)^2} \int \tilde{f}(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{y}) - ik^0(x^0+y^0)} d^4k D_H(y) |\mathbf{y}|^2 d|\mathbf{y}| d^2\Omega \\ &= \frac{1}{\pi} \int \tilde{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x} - ik^0x^0} e^{-ik^0y^0} d^4y \int D_H(|\mathbf{y}|, y^0) \frac{\sin\omega|\mathbf{y}|}{\omega} |\mathbf{y}| d|\mathbf{y}|. \end{aligned}$$

Now all homogeneous functions D_H can be written as

$$D_H = \alpha D_1 + \beta D,$$

where α and β are constants. Since D_1 involves the principal part of $1/y^2$, the above integral will vanish in the limit $y^0 \rightarrow \pm\infty$ by the Riemann-Lebesgue lemma. But $D(y) \sim \epsilon(y) \delta(y^2)$. The $|\mathbf{y}|$ integration above then yields a term proportional to

$$2i e^{-ik^0y^0} \sin\omega y^0 = e^{i(\omega-k^0)y^0} - e^{-i(\omega+k^0)y^0}. \tag{A6}$$

Again, the limits $y^0 \rightarrow \pm\infty$ would give a vanishing result were it not for the possibility of a factor $\delta(\omega-k^0)$ or $\delta(\omega+k^0)$ in $f(k)$.

Consider the first such factor. In the limit $y^0 \rightarrow \pm\infty$ only the first term in (A6) will survive. But the result is independent of whether the limit $y^0 \rightarrow +\infty$ or $y^0 \rightarrow -\infty$ is taken. Thus, the two integrals over the spacelike planes in (A5) will exactly cancel. The same is true for a factor $\delta(\omega+k^0)$. This proves Eq. (A4).

Thus, the definition (9) of the total potential \mathcal{Q}_μ is indeed independent of the choice of the inhomogeneous contour C_1 . The two terms in the decomposition (13) are each of course not independent of these contours. And, in particular, the equal-time commutation relations for the retarded fields $\mathcal{Q}_\mu^{\text{ret}}$ which follow from (25) are also contour-dependent.