Validity of the Phase-Shift Representations*

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Our previous discussion on the validity of the phase-shift representation is further developed. An example treated in a previous work based on a separable-potential model is given in an improved method to show that such a representation fails in more general cases than those considered recently by Bander, Coulter, and Shaw, based on a simplified N/D model. It is also shown that $\delta_I(s)$, the imaginary part of the elastic phase shift, is a boundary value of a function which connects the elastic S-matrix element in the physical sheet to that in other Riemann sheets. Consequently, $\delta_I(s)$ should be so chosen as to satisfy certain mathematical conditions as well as to fit experiments. The simplest example of such a function is given.

1. INTRODUCTION AND SUMMARY

THERE has been a question[1] about the validity of the phase shift representation proposed by Ball and Frazer[2] for the elastic S-matrix element:

$$S(z)=\exp\left[2i\frac{k_1(z)}{\pi}\int_{s_2}^{\infty}\frac{\delta_I(s)}{k_1(s)(s-z)}ds\right], \tag{1}$$

where s is the total energy squared, z is its complex extension, k_1 is the momentum in the elastic channel, s_2 is the inelastic threshold energy squared, and δ_I is the imaginary part of the elastic-scattering phase shift. Although similar representations have been proposed to include purely elastic effects,[3] we shall consider only the representation (1) to avoid unnecessary complications. If there is a resonance below or near the inelastic threshold, it is usually associated with a complex conjugate pair of poles of the elastic scattering amplitude in the second Riemann sheet in the squared complex-energy plane. At the positions of these poles, the elastic S-matrix element has zeros in the physical sheet due to the relation[4]

$$S=T_{11}/T_{11}^{\mathrm{II}}, \tag{2}$$

where T_{11} and T_{11}^{II} are the elastic scattering amplitudes in the physical and the second sheet, respectively. For such a resonance Eq. (1) cannot be valid because the right-hand side gives no zeros. To include such poles and zeros it has been proposed[5,6] that the form of (1) be modified to

$$S=\frac{(k_1+\gamma)(k_1-\gamma^*)}{(k_1-\gamma)(k_1+\gamma^*)}\exp\left[2i\frac{k_1}{\pi}\int_{s_2}^{\infty}\frac{\delta_I(s)}{k_1(s)(s-z)}ds\right], \tag{3}$$

with $\mathrm{Re}\gamma>0$, $\mathrm{Im}\gamma<0$. In this representation, however, we cannot predict a resonance solely on the basis of inelastic effects (δ_I).

Recently, Bander, Coulter, and Shaw[6] showed, by an exact calculation based on a soluble two-channel N/D model, that there is certainly a case in which the representation (1) is not valid. The particular case considered in detail is that of a bound-state resonance (a resonance below the inelastic threshold which goes into a bound state in the inelastic channel as the coupling between channels is switched off). For this example we can see directly that (1) is not valid for the reasons discussed above. The case of a bound-state resonance allows a simple interpretation for a weak coupling between channels. On the other hand, we can consider the opposite extreme of a strong coupling between channels, and show again that (1) is not always valid. In Sec. 2, an example in such a case is given without using the zero-range approximation as used in a previous work.[1] This suggests that the failure of (1) is more general than conjectured in Ref. 6.

This example suggests also that poles may sometimes move into another Riemann sheet, named sheet IV in the following, which is connected to the second sheet through the inelastic cut. In this case the representation (1) is certainly valid in principle. It is shown in Sec. 3, however, that $\delta_I(s)$ is a boundary value of a function which connects the S-matrix element in the physical sheet to that in sheet IV, and therefore has some logarithmic singularities in the complex s plane. Consequently, the function $\delta_I(s)$ should satisfy some mathematical conditions as well as fit the experiments. It should be noted that the same argument applies also to the case of the modified representation (3).

In Sec. 4, the simplest example of δ_I satisfying such a condition is given in terms of a variable ω which is the conformal transform of the momentum.[7] This example shows also that a strong absorption does not necessarily give a significant effect in elastic scattering.

2. AN EXAMPLE IN TWO-CHANNEL MODEL

We consider a two-channel model in which each channel consists of two nonrelativistic spinless particles

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with equal masses; the channel with lower (higher) threshold is labeled by 1(2). We use the separable-potential method (or equivalently, a two-channel pair theory with different cutoff functions in each channel), which is similar to the N/D method. The particular case of a strong coupling between channels is given by assuming the following potentials:

$$\langle s|V_{ij}|s'\rangle=(s-t_i)^{-1/2}g_{ij}(s'-t_j)^{-1/2},\quad t_i,t_j<0,$$

with the particular choice,

$$g_{11}=g_{22}=0.$$

This extreme case is of theoretical interest because it gives a mechanism which is essentially the same as that in the one-pion-exchange model for the second resonance in πN scattering, taking into account the ρN channels.[2,8]

Two integrals appear which take simple forms:

$$\frac{1}{\pi}\int_{s'}^{\infty}\frac{k_i ds_i'}{(s'-t_i)(s'-s-i\epsilon)}=\frac{1}{2}\frac{1}{a_i-ik_i},$$

where $s_i(i=1,2)$ is the threshold energy squared, and

$$k_i=\tfrac{1}{2}(s-s_i)^{1/2},\quad a_i=\tfrac{1}{2}(s_i-t_i)^{1/2}>0.$$

The scattering amplitudes have a common denominator function $D(s)$ given by[1]

$$D(s)=1-G^2[(a_1-ik_1)(a_2-ik_2)]^{-1},$$

with

$$G^2=(8\pi)^{-2}s_1^{-1/2}s_2^{-1/2}g_{12}^2.$$

The poles of the amplitudes are obtained from the equation

$$f(k_1,k_2)\equiv(a_1-ik_1)(a_2-ik_2)=G^2,\tag{4}$$

or by denoting real and imaginary parts of k_i by α_i and β_i, respectively,

$$\mathrm{Re}f=(a_1+\beta_1)(a_2+\beta_2)-\alpha_1\alpha_2=G^2,\tag{5}$$

$$-\mathrm{Im}f=\alpha_1(a_2+\beta_2)+\alpha_2(a_1+\beta_1)=0.\tag{6}$$

The analytic continuation of $D(z)$, as a function of a complex squared energy z, defines four Riemann sheets.[9] Sheet I is the physical one, sheet II is connected to sheet I through the unitarity cut between s_1 and s_2, sheet III is connected to sheet I through the cut above s_2, and sheet IV is connected to sheet II (III) through the cut above s_2 (between s_1 and s_2). These sheets correspond to various choices of signs of α's and β's, as illustrated in Fig. 1 (the complex k_2 plane). It is sufficient to consider only the upper or lower half of each sheet, since the functions on the other half-plane are determined completely by the reality condition.

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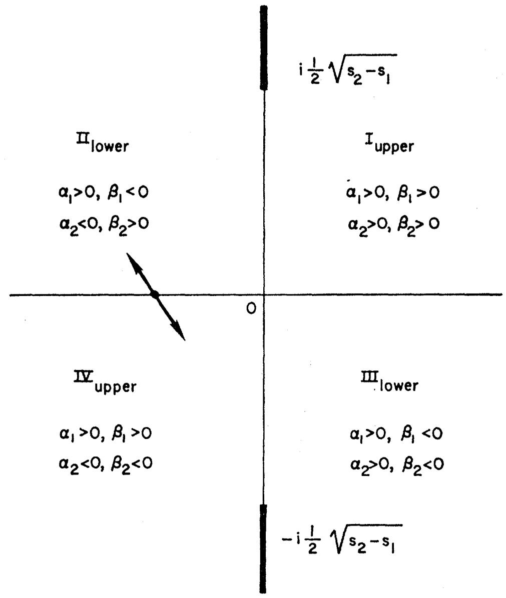

FIG. 1. Complex k_2 plane. There are two sheets connected through the cuts along the imaginary axis starting from the branch points $\pm\frac{1}{2}i(s_2-s_1)^{1/2}$. The first quadrant in the "first" sheet, for example, is mapped onto the upper half of sheet I in the complex s plane, and so on, as indicated in the figure.

We can prove that the amplitudes considered here have no poles in sheets I or III. For example, in the upper half of sheet I ($\alpha_i>0$, $\beta_i>0$), each factor in $-\mathrm{Im}f$ is positive, so that there is no solution of (6). In the lower half of sheet III ($\alpha_i>0$, $\beta_i<0$), Eq. (6) can be satisfied. In this case, however, it is evident that

$$(a_1+\beta_1)(a_2+\beta_2)<0,$$

so that (5) cannot be satisfied. On the other hand, poles can appear either in sheets II or IV. We shall focus our attention on a pole on the real axis above s_2 which is the boundary between sheets II and IV and corresponds to the negative real axis in the k_2 plane ($\alpha_1>0$, $\alpha_2<0$, $\beta_i=0$). Equation (6) gives

$$\alpha_1/(-\alpha_2)=a_1/a_2.$$

The left-hand side is larger than unity; therefore we must choose

$$a_1>a_2,$$

or equivalently,

$$|t_1|-|t_2|>s_2-s_1>0.$$

In the model considered in Ref. 6 with $t_1=t_2$, this relation cannot be satisfied, and this is the reason why this model gives no poles in this region. A little further investigation[10] shows that the pole moves in the upper-left direction (as indicated by the arrow in Fig. 1) into sheet II

[10] Along the real axis below s_1 [or, in Fig. 1, along the imaginary axis above $\frac{1}{2}i(s_2-s_1)^{1/2}$ and below $-\frac{1}{2}i(s_2-s_1)^{1/2}$], poles can appear in any sheet.

for an increasing g_{12}^2, and in the lower right direction into sheet IV for a decreasing g_{12}^2.

The foregoing arguments show that, in addition to the case of a bound-state resonance (characterized by a sufficiently negative g_{22} and relatively small g_{12}), there is certainly some other case in which poles appear in sheet II, and zeros of the elastic S-matrix element appear in the physical sheet due to the relation (2), thus invalidating the representation (1). We must, therefore, be careful in applying (1) when a resonance occurs below or near the inelastic threshold whatever its cause. It seems rather difficult to get a more definite criterion for the validity of (1) or (3).

3. ANALYTIC CONTINUATION OF ELASTIC S-MATRIX ELEMENT

The foregoing example shows also that poles can appear in sheet IV; the associated effect is a large cusp in the inelastic threshold. This situation can be realized in general, as emphasized by us[1,11] and more recently by Frazer and Hendry,[12] by changing some parameters slightly from the values corresponding to a resonance just below the inelastic threshold. In this case the modification as in (3) is not necessary. It is, however, expected that the presence of poles in sheet IV imposes some restrictions on the form of $\delta_I(s)$. This will be confirmed by investigating the analytic continuation of the S-matrix element to other sheets. It will be also found that the same kind of restriction on $\delta_I(s)$ is necessary even in the representation (3) in which poles appear in sheet II.

Using Eq. (2) we can prove the relation[5]

$$S^{\rm II}=(S^{\rm I})^{-1}. \qquad (7)$$

An exactly parallel calculation gives

$$S^{\rm III}=(S^{\rm IV})^{-1}, \qquad (8)$$

which shows that at the positions of a complex conjugate pair of poles of $S^{\rm IV}$, there must be a pair of zeros of $S^{\rm III}$ and vice versa. The continuation of $S^{\rm I}$ or $S^{\rm II}$ to $S^{\rm III}$ or $S^{\rm IV}$ through the inelastic cut is provided by (1). For example, along the real axis $s>s_2$, we have the relation

$$\begin{aligned}
S^{\rm III}(s\pm i\epsilon)&=S^{\rm I}(s\mp i\epsilon)\\
&=\exp\left[\mp 2i\frac{k_1}{\pi}\int_{s_2}^{\infty}\frac{\delta_I(s')ds'}{k_1'(s'-s\pm i\epsilon)}\right]\\
&=\exp\left[\mp 2i\frac{k_1}{\pi}\int_{s_2}^{\infty}\frac{\delta_I(s')ds'}{k_1'(s'-s\mp i\epsilon)}-4\delta_I(s)\right] \qquad (9)\\
&=S^{\rm I}(s\pm i\epsilon)^{-1}e^{-4\delta_I(s)}.
\end{aligned}$$

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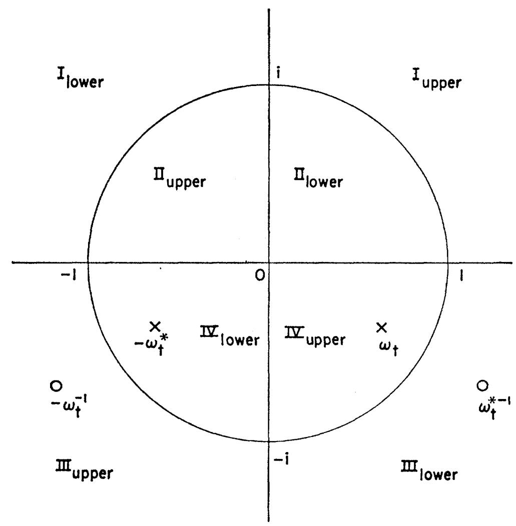

FIG. 2. Complex ω plane. Each portion is mapped onto the corresponding sheets in the complex s plane, as indicated in the figure. The poles (×) in sheet IV and zeros (○) in sheet III are shown as an example.

It should be noted that this relation can also be verified by using (3).

If we define the quantity $\bar{\delta}_I(z)$ by the equation

$$S^{\rm III}(z)=S^{\rm I}(z)^{-1}\exp[-4\bar{\delta}_I(z)], \qquad (10)$$

then Eq. (9) shows that $\delta_I(s)$ is a boundary value of $\bar{\delta}_I(z)$ along the real axis $s>s_2$. Similar relations between other sets of sheets can be obtained by using (7) and (8). For the case where poles appear in sheet IV or sheet II and no poles in any other sheets, $\bar{\delta}_I(z)$ should have corresponding logarithmic singularities according to (10).[13] Therefore $\delta_I(s)$ must be so determined as to give such analytic properties when it is continued to the complex s plane. An arbitrary function $\delta_I(s)$, even if it appears to fit the experimental data in a limited range of energy, does not always satisfy such a requirement, and may lead to a wrong result when it is substituted into (1) or (3).

In the example in Ref. 6, $\bar{\delta}_I(z)$ actually has such singularities, because it was calculated from the two-channel amplitudes which have poles in appropriate sheets. (The zeros of S that appeared in the physical sheet for $g_{22}<\bar{g}_{22}$, the critical value, should go into sheet III for $g_{22}>\bar{g}_{22}$.) This is the reason why an agreement was obtained between two methods of calculation for $g_{22}>\bar{g}_{22}$.

4. SIMPLEST EXAMPLE OF δ_I

We shall give the simplest example of the form of $\delta_I(s)$ which satisfies the requirement stated above.

[13] The same conclusion is obtained also when two pairs of poles appear in sheets II and III, or III and IV, unless they exactly coincide with each other. Such a coincidence may be exceptional as long as the poles appear near a threshold. See, for example, the paper by Y. Fujii, M. Ichimura and K. Yazaki, Progr. Theoret. Phys. (Kyoto) **32**, 320 (1964).

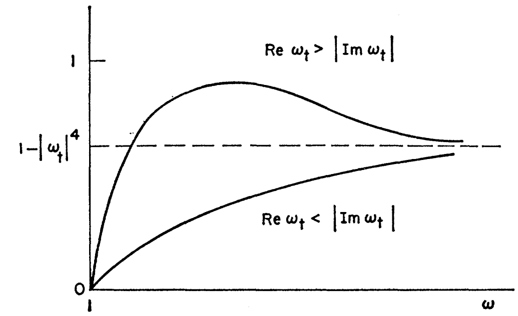

FIG. 3. Schematic plots of $\xi(\omega)=(k_1^2/\pi)\sigma_{\mathrm{inel}}=1-\exp[-4\delta_I(\omega)]$, versus ω.

A convenient representation of the S-matrix is obtained in terms of the variable ω which, introduced by Kato,[7] is a conformal transform of the momentum k_1, given by

$$k_1=\tfrac{1}{4}(s_2-s_1)^{1/2}(\omega+1/\omega)\,.$$

The four Riemann sheets in the complex s plane are mapped onto a single sheet in the complex ω plane, as indicated in Fig. 2. The branch points s_1 and s_2 correspond to $\omega=\pm i$ and ± 1, respectively. (The *physical* thresholds correspond to $\omega=+i$ and $+1$). The portions along the real axis $s<s_1$, $s_1<s<s_2$, $s>s_2$ are mapped onto the imaginary axis, the circle of unit radius centered at the origin, and the real axis, respectively.

The elastic S-matrix element in the "one-pole approximation" is given by[7]

$$S(\omega)=|\omega_t|^2\frac{(\omega-\omega_t^{*-1})(\omega+\omega_t^{-1})}{(\omega-\omega_t)(\omega+\omega_t^*)}\,. \qquad (11)$$

If ω_t is chosen as

$$\mathrm{Re}\omega_t>0,\quad \mathrm{Im}\omega_t(\gtrless)0,\quad |\omega_t|<1,$$

then $S(z)$ has a pair of poles in sheet II (IV) (corresponding to ω_t and $-\omega_t^*$), and a pair of zeros in sheet I (III) (corresponding to ω_t^{*-1} and $-\omega_t^{-1}$). The function $\delta_I(\omega)$, defined by

$$|S(\omega)|^2=e^{-4\delta_I(\omega)}\,,$$

takes the form

$$\delta_I(\omega)=-\tfrac{1}{4}\ln\left(|\omega_t|^4\left|\frac{(\omega-\omega_t^{*-1})(\omega+\omega_t^{-1})}{(\omega-\omega_t)(\omega+\omega_t^*)}\right|^2\right), \qquad (12)$$

which certainly vanishes along the unit circle (corresponding to the real axis $s_1<s<s_2$). It can also be noted that the right-hand side of (12) does not depend on the sign of $\mathrm{Im}\omega_t$ for real ω. This means that the poles on sheet II and those in sheet IV give the same $\delta_I(s)$ if they are in the same position in the complex s plane; therefore we cannot determine which resonance and cusp phenomena occur only by knowing δ_I, as was already pointed out in Ref. 7.

On the other hand, the function $\bar{\delta}_I(\omega)$ is defined by

$$S(\omega)S(-\omega)=\exp[-4\bar{\delta}_I(\omega)]\,,$$

according to (10), since by changing the sign of ω, $\omega\to-\omega$, S^{I} and S^{II} go into S^{III} and S^{IV}, respectively, and vice versa. By using the form (11) we have

$$\bar{\delta}_I(\omega)=-\tfrac{1}{4}\ln\left(|\omega_t|^4\frac{(\omega^2-\omega_t^{*-2})(\omega^2-\omega_t^{-2})}{(\omega^2-\omega_t^2)(\omega^2-\omega_t^{*2})}\right), \qquad (13)$$

which coincides with (12) for real ω (corresponding to $s>s_2$).

In Fig. 3, we show schematic plots of the quantity $\xi(\omega)=(k_1^2/\pi)\sigma_{\mathrm{inel}}=1-e^{-4\delta_I(\omega)}$ for $\omega>1$ (physical region $s>s_2$). There are two distinguished forms according to the location (actually the argument) of the pole ω_t. A simple measure of the magnitude of $\xi(\omega)$ may be given by its asymptotic value $1-|\omega_t|^4$ for $\omega\to\infty$, which is smaller for the pole lying closer to the unit circle (or closer to the real axis $s_1<s<s_2$ in the complex s plane). This suggests that a strong absorption does not necessarily give a significant effect in elastic scattering (a sharp resonance below, or a big cusp at the inelastic threshold).

On the other hand, the ratio

$$\frac{\sigma_{\mathrm{inel}}}{\sigma_{\mathrm{el}}}=\frac{1-|S|^2}{|1-S|^2}=\frac{1+|\omega_t|^2}{1-|\omega_t|^2}\frac{\omega^2-1}{\omega^2+1}\,, \qquad (14)$$

is a monotonically increasing function of ω for $\omega>1$, and also increases as $|\omega_t|$ goes to unity. This is consistent with the fact that this ratio appears in the dispersion integral of the inverse partial-wave amplitude, whose real part is essentially k_1 Re $\cot\delta$.[1] It is also noted that the ratio in (14) depends only on the magnitude of ω_t. The argument of ω_t is evidently related to the value of the real phase shift at a particular energy, say s_2, and, therefore, corresponds to the subtraction constant which is important in the representation for k_1 Re $\cot\delta$.[1]

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