Variational Principle in S-Matrix Dynamics

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By extending previous results, it is shown that a variational principle similar to that of Lippmann and Schwinger also exists in the N/D formalism of multichannel S-matrix theory.

In dynamical theories of particles and fields, variational principles have played very important roles in formulating theories and in solving problems approximately. In the recently developed theory of the S matrix, especially in its N/D formalism, the dynamical problem is replaced by the determination of singularities of the S matrix instead of pursuing the time development of the system. In spite of this difference in the basic idea, we have shown that there also exists a kind of variational principle in S-matrix theory, which is similar to that of Lippmann and Schwinger[1] in quantum-mechanical scattering theory. The purpose of the present note is to extend the previous result[2] to multichannel scattering.

The S-matrix elements of a specified partial wave are written as

$$S_{ab}=\delta_{ab}+2i(\rho_a)^{1/2}f_{ab}(\rho_b)^{1/2}, \tag{1}$$

where f_{ab} is the scattering amplitude, and ρ_a is the density function in channel a multiplied by a suitable step function θ. In the N/D formalism, f_{ab} is written as

$$f_{ab}=N_{ac}(D^{-1})_{cb} \tag{2}$$

and the $\mathbf{N}$ and $\mathbf{D}$ matrix functions are subjected to the following dispersion relations:

$$\begin{aligned}\mathbf{D}(s)&=\mathbf{D}(s_0)-\frac{s-s_0}{\pi}\int_R ds'\frac{\boldsymbol{\varrho}(s')\mathbf{N}(s')}{(s'-s_0)(s'-s)},\\ \mathbf{N}(s)&=\frac{1}{\pi}\int_L ds'\frac{\mathbf{I}(s')\mathbf{D}(s')}{s'-s},\end{aligned} \tag{3}$$

where

$$I_{ab}(s)=\mathrm{Im}f_{ab}(s) \tag{4}$$

and L and R denote integrals along the left and right halves of the real axis.

Now we shall show that the integral equation for the $\mathbf{D}$ function can be derived from the following variational principle:

$$\delta_D\mathbf{J}_{ab}(s_0)=0, \tag{5}$$

where

$$\begin{aligned}\mathbf{J}(s_0)=&\left[\frac{1}{\pi}\int_L ds'\frac{\mathbf{D}^T(s)\mathbf{I}(s)}{s-s_0}\right]^{-1}\left[\frac{1}{\pi}\int_L ds\frac{\mathbf{D}^T(s)\mathbf{I}(s)\mathbf{D}(s)}{s-s_0}\right.\\ &\left.+\frac{1}{\pi^3}\int_L ds\int_R ds'\int_L ds''\frac{\mathbf{D}^T(s)\mathbf{I}(s)\boldsymbol{\varrho}(s')\mathbf{I}(s'')\mathbf{D}(s'')}{(s-s')(s'-s_0)(s'-s'')}\right]\\ &\times\left[\frac{1}{\pi}\int_L ds\frac{\mathbf{I}(s)\mathbf{D}(s)}{s-s_0}\right]^{-1}\end{aligned} \tag{6}$$

$$\equiv\mathbf{N}^T(s_0)^{-1}\mathbf{A}(s_0)\mathbf{N}(s_0)^{-1}, \tag{6'}$$

δ_D denotes the variation due to a small change of the $\mathbf{D}$ function, and $\mathbf{D}^T$ means the transposed matrix of $\mathbf{D}$. It is easily verified that

$$\mathbf{J}(s_0)^T=\mathbf{J}(s_0), \tag{7}$$

if $\mathbf{I}(s)^T=\mathbf{I}(s)$.[3]

Differentiating (6′) functionally, and using the identity[4] $\delta(\mathbf{N}^{-1})=-\mathbf{N}^{-1}\delta\mathbf{N}\mathbf{N}^{-1}$, one finds

$$\delta_D\mathbf{A}(s_0)-\delta_D\mathbf{N}^T(s_0)\mathbf{J}(s_0)\mathbf{N}(s_0)-\mathbf{N}^T(s_0)\mathbf{J}(s_0)\delta_D\mathbf{N}(s_0)=0, \tag{8}$$

where,

$$\begin{aligned}\delta_D\mathbf{A}(s_0)=&\frac{1}{\pi}\int_L ds\frac{\delta\mathbf{D}^T(s)\mathbf{I}(s)}{s-s_0}\left[\mathbf{D}(s)+\frac{s-s_0}{\pi^2}\right.\\ &\left.\times\int_R ds'\int_L ds''\frac{\rho(s')\mathbf{I}(s'')\mathbf{D}(s'')}{(s-s')(s'-s_0)(s'-s'')}\right]\\ &+\frac{1}{\pi}\int_L ds\left[\mathbf{D}^T(s)+\frac{s-s_0}{\pi}\int_R ds'\int_L ds''\right.\\ &\left.\times\frac{\mathbf{D}^T(s'')\mathbf{I}(s'')\boldsymbol{\varrho}(s')}{(s''-s')(s'-s_0)(s'-s)}\right]\frac{\mathbf{I}(s)\delta\mathbf{D}(s)}{s-s_0},\end{aligned} \tag{9a}$$

$$\delta_D\mathbf{N}^T(s_0)=\frac{1}{\pi}\int_L ds\frac{\delta\mathbf{D}^T(s)\mathbf{I}(s)}{s-s_0}, \tag{9b}$$

$$\delta_D\mathbf{N}(s_0)=\frac{1}{\pi}\int_L ds\frac{\mathbf{I}(s)\delta\mathbf{D}(s)}{s-s_0}. \tag{9c}$$

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[1] B. A. Lippman and J. Schwinger, Phys. Rev. **79**, 469 (1950).

[2] D. Itô, Progr. Theoret. Phys. (Kyoto) **32**, 171 (1964); **32**, 172 (1964).

[3] This assumption comes from the time-reversal invariance of the S matrix.

[4] This follows from $\delta(\mathbf{N}^{-1}\mathbf{N})=\delta\mathbf{N}^{-1}\cdot\mathbf{N}+\mathbf{N}^{-1}\cdot\delta\mathbf{N}=0$.

Inserting (9a)–(9c) into (8), we have

$$\frac{1}{\pi}\int_L \frac{\delta\mathbf{D}^T(s)\mathbf{I}(s)}{s-s_0}ds\left[\mathbf{D}(s)+\frac{s-s_0}{\pi^2}\int_R ds'\int_L ds'' \times\frac{\varrho(s')\mathbf{I}(s'')\mathbf{D}(s'')}{(s-s')(s'-s_0)(s'-s'')}-\mathbf{J}(s_0)\mathbf{N}(s_0)\right]+\frac{1}{\pi}\int_L ds \times\left[\mathbf{D}^T(s)+\frac{s-s_0}{\pi^2}\int_R ds'\int_L ds''\frac{\mathbf{D}^T(s'')\mathbf{I}(s'')\varrho(s')}{(s''-s')(s'-s_0)(s'-s)} -\mathbf{N}^T(s_0)\mathbf{J}(s_0)\right]\frac{\mathbf{I}(s)\delta\mathbf{D}(s)}{s-s_0}=0. \quad (10)$$

From (10), we can derive[5]

$$\mathbf{D}(s)+\frac{s-s_0}{\pi^2}\int_R ds'\int_L ds''\frac{\varrho(s')\mathbf{I}(s'')\mathbf{D}(s'')}{(s-s')(s'-s_0)(s'-s'')} -\mathbf{J}(s_0)\mathbf{N}(s_0)=0, \quad (11a)$$

$$\mathbf{D}^T(s)+\frac{s-s_0}{\pi^2}\int_R ds'\int_L ds''\frac{\mathbf{D}^T(s'')\mathbf{I}(s'')\varrho(s')}{(s''-s')(s'-s_0)(s'-s)} -\mathbf{N}^T(s_0)\mathbf{J}(s_0)=0, \quad (11b)$$

and by putting $s=s_0$ in (11a) and (11b), we see that

$$\mathbf{J}(s_0)\mathbf{N}(s_0)=\mathbf{D}(s_0) \quad \text{and} \quad \mathbf{N}^T(s_0)\mathbf{J}(s_0)=\mathbf{D}^T(s_0). \quad (12)$$

Thus, we get

$$\mathbf{D}(s)=\mathbf{D}(s_0)-\frac{s-s_0}{\pi^2}\int_R ds' \times\int_L ds''\frac{\varrho(s')\mathbf{I}(s'')\mathbf{D}(s'')}{(s-s')(s'-s_0)(s'-s'')}, \quad (13a)$$

$$\mathbf{D}^T(s)=\mathbf{D}^T(s_0)-\frac{s-s_0}{\pi^2}\int_R ds' \times\int_L ds''\frac{\mathbf{D}^T(s'')\mathbf{I}(s'')\varrho(s')}{(s''-s')(s'-s_0)(s'-s)}. \quad (13b)$$

These are nothing but the integral equations for the **D** functions, which are obtained from Eq. (3) or its transpose by eliminating the **N** functions.

In order to see the meaning of our "action function" $\mathbf{J}(s_0)$, let us insert (13) into

$$\mathbf{J}(s_0)=\mathbf{N}^T(s_0)^{-1}\frac{1}{\pi}\int_L\frac{\mathbf{D}^T(s)\mathbf{I}(s)}{s-s_0}ds\left[\mathbf{D}(s)+\frac{s-s_0}{\pi^3} \times\int_R ds'\int_L ds''\frac{\varrho(s')\mathbf{I}(s'')\mathbf{D}(s'')}{(s-s')(s'-s_0)(s'-s'')}\right]\mathbf{N}^{-1}(s_0). \quad (14)$$

Then, (14) reduces to

$$\mathbf{J}(s_0)\rightarrow\mathbf{D}(s_0)\mathbf{N}(s_0)^{-1}\equiv\mathbf{f}^{-1}(s_0). \quad (15)$$

This shows that our $\mathbf{J}(s_0)$ will reduce to $\mathbf{f}^{-1}(s_0)$, if the "correct" **D** function is inserted. As is well known, $\mathbf{f}^{-1}(s_0)$ can be expressed by a **K** matrix defined by

$$\mathbf{S}(s_0)=\frac{1+i\mathbf{K}(s_0)}{1-i\mathbf{K}(s_0)}=\mathbf{1}+2i(\sqrt{\varrho})\mathbf{f}(\sqrt{\varrho}), \quad (16)$$

i.e.,

$$\varrho^{-1/2}\mathbf{f}^{-1}\varrho^{-1/2}=\mathbf{k}^{-1}-i\mathbf{1}. \quad (17)$$

On the other hand, (6) can be written as

$$\varrho^{-1/2}\mathbf{J}\varrho^{-1/2}=\varrho^{-1/2}\mathbf{N}^T(s_0)^{-1}\left[\frac{1}{\pi}\int_L ds\frac{\mathbf{D}^T(s)\mathbf{I}(s)\mathbf{D}(s)}{s-s_0} +\frac{P}{\pi^3}\int_L ds\int_R ds'\int_L ds''\frac{\mathbf{D}^T(s)\mathbf{I}(s)\varrho(s')\mathbf{I}(s'')\mathbf{D}(s'')}{(s-s')(s'-s_0)(s'-s'')}\right] \times\mathbf{N}(s_0)^{-1}\varrho^{-1/2}-i\mathbf{1}. \quad (18)$$

Therefore, the second term of (18) is of no importance in our variational principle, and we see from (15), (17), and (18) that the first term of (18) reduces to $\mathbf{K}^{-1}(s_0)$, if the "correct" **D** function is inserted. Thus our "action function" can be written as

$$\varrho^{-1/2}\mathbf{J}\varrho^{-1/2}='\mathbf{K}^{-1}'-i\mathbf{1} \quad (19)$$

and our variational principle states essentially that

$$\delta_D'\mathbf{K}^{-1}'=\mathbf{0} \quad \text{or} \quad \delta_D'\cot\boldsymbol{\delta}'=\mathbf{0}, \quad (20)$$

just as in the case of Lippmann and Schwinger.[1]

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[5] The left-hand side of (10) can be written as

$$\int_L F_{ac}{}^T(s)G_{cb}(s)ds+\int_L G_{ac}{}^T(s)F_{cb}(s)ds =\int_L ds[G_{cb}(s)F_{ca}(s)+G_{ca}(s)F_{cb}(s)] =\delta_{bi}\delta_{aj}\int_L ds[G_{ci}(s)F_{cj}(s)+G_{cj}(s)F_{ci}(s)] =(\delta_{bi}\delta_{aj}+\delta_{bj}\delta_{ai})\int_L dsG_{ci}(s)F_{cj}(s)=0,$$

from which we can get (11a). In a similar way, we can also derive (11b).