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Determination of the Nucleon-Nucleon Elastic Scattering Matrix. II. Phase-shift Analyses of Experiments Near 95, 142, 210, and 310 MeV†

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Phase-shift analyses have been carried out for (p,p) and (n,p) experiments at energies near 95, 142, 210, and 310 MeV. Analyses both of (p,p) data and of combined (p,p) plus (n,p) data were made at these energies. Attention was concentrated in all cases on the Stapp-type 1 solution, the objective being to obtain the best possible values for the phase shifts from the existing nucleon-nucleon data. Error matrices were calculated for all solutions, and determinations of the pion-nucleon coupling constant g^2 were carried out. It was found that reasonably consistent values for g^2 can be obtained if a suitable selection of the phenomenological phase shifts is made. The $T=1$ phase shifts determined from (p,p) data only were found to be in good agreement with the $T=1$ phase shifts obtained from combined (p,p) plus (n,p) data, thus verifying the gross features of the charge independence. A previous article, paper I of this series, gave the results of phase-shift analyses near 142 MeV. Since the publication of I, some changes in the data have occurred, and some improvements in the method of phase-shift analysis have been made. Hence new phase-shift values at 142 MeV are included here that supersede the values listed in I.

I. INTRODUCTION

THIS paper gives the results of modified phase-shift analyses[1] of nucleon-nucleon data at several energies spanning the elastic energy region. In these analyses, the low-angular-momentum phase shifts were treated as free parameters, and the higher phase shifts (through $l=18$) were calculated from the one-pion-exchange contribution (OPEC) to the nuclear potential. The (p,p) data at these energies were complete enough to give well-determined values for the isotopic spin 1 (T=1) scattering matrix. The (n,p) data are not yet complete enough to permit a separate analysis, as described in detail in paper I of this series.[2] However, when analyzed in conjunction with the (p,p) data under the hypothesis of charge independence, they yield "combined analysis" values for $T=1$ amplitudes that agree well with those calculated from the (p,p) data alone. They also yield $T=0$ amplitudes that are reasonably well determined.

The grid search procedure was used in the initial stages of these analyses. In obtaining final-phase shift values, we used a second-derivative search procedure[3] to eliminate some of the grid-search ambiguity from the phase shifts. Error matrices were obtained for all of the solution sets. Since it has now been well established (for example in paper I) that Stapp solution type 1 is the only admissible one, we studied only this type of solution.

Section II contains some comments on error-matrix calculations. Sections III–VI list the results of analyses at narrow bands of energies near 95, 142, 210, and 310 MeV, respectively. In these analyses, the data were treated at their correct energies. An energy dependence was assigned to the phase shifts, using the results of energy-dependent analyses. This is discussed in detail in Sec. IV. Section VII is a discussion of the determination of the values for the pion-nucleon coupling constant g^2, treating in particular the effect of the ordering of the phase shifts selected as free parameters. We do not agree with the ordering criterion adopted by the Pennsylvania State group in some of their phase shift analyses.[4–7] Section VIII is on the problem of charge independence. In Sec. IX, we compare the present phase-shift values with the results from other analyses.

II. VALIDITY OF ERROR-MATRIX VALUES

A comment should be made here about the significance of error limits as given by error-matrix calculations. The error-matrix calculation gives the statistical uncertainties for a set of parameters δ that are used to fit a set of data points which themselves contain quoted statistical uncertainties. However, if the data points are in some sense not complete, the error matrix may give a misleading result, corresponding to the solution ending in an incorrect local minimum on the $\chi^2(\delta)$ surface.

Another way in which the error matrix result can be misleading is if systematic errors are present. Systematic errors in the data can only be determined by a comparison with other results at the same energy

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[1] M. J. Moravcsik, University of California Lawrence Radiation Laboratory Report UCRL-5317-T (1958); A. F. Grashin, Zh. Eksperim. i Teor. Fiz. **36**, 1717 (1959) [English transl.: Soviet Phys.—JETP **9**, 1223 (1959)]. One of us (MHM) would like to comment here that the idea of calculating some of the higher partial waves from the one-pion-exchange contribution (OPEC) to the potential was communicated to him by Professor H. P. Noyes in the summer of 1957, and that this suggestion was utilized in an early publication [M. H. MacGregor, Phys. Rev. **113**, 1559 (1959)].

[2] M. H. MacGregor, R. A. Arndt, and A. A. Dubow, Phys. Rev. **135**, B628 (1964).

[3] The advantage of using this search procedure was pointed out to us by Professor Signell. We are indebted to him for several useful communications.

[4] P. Signell, N. R. Yoder, and N. M. Miskovsky, Phys. Rev. **133**, B1490 (1964).

[5] P. Signell, Phys. Rev. **135**, B1344 (1964).

[6] P. Signell and D. L. Marker, Phys. Rev. **134**, B365 (1964).

[7] P. Signell, N. R. Yoder, and J. E. Matos, Phys. Rev. **135**, B1128 (1964).

or at a neighboring energy. Systematic errors in the modified-phase-shift analysis itself are always present in the sense that some phase shifts are selected as free parameters and others are fixed at OPEC values. If the OPEC phases start at a certain angular-momentum value l_0, then the (l_0-1) phases, roughly speaking, will in general give small uncertainties in an error matrix calculation and yet will be influenced strongly by the precise choice of the phases selected as free parameters. The (l_0-2) and lower phases, on the other hand, will be relatively unaffected by the selections. For example, the 210-MeV analysis, discussed in Sec. V, shows that at least some of the H waves must be treated as free parameters. However, the data are not accurate enough to give precise values when all of the H waves are freed. As Signell[7] has noted in the case of $^3H_5$, and as we have noted in Sec. V for the case of $^3H_4$, the phase shift varies by amounts greater than predicted by the error matrix, for different combinations of free and OPEC H waves. The result is that the 210-MeV analysis requires some H-wave freedom, but it does not at present give accurate quantitative values for the H phases.

At 50 MeV [following paper, Phys. Rev. **139**, B380 (1965)], we are in a similar situation, only now at the F-wave level. The (p,p) data are complete enough to require some F-wave freedom in the analysis, but they are not yet accurate enough or complete enough to determine the F waves reliably. At 95 MeV (Sec. III), F-wave freedom is again required, but the values obtained for the F waves have little significance. At 142 MeV (Sec. IV), as at 210 MeV, H wave freedom is required, but values for the H waves are not reliably determined. The results of Sec. VI indicate that the same situation prevails at 310 MeV.

III. ANALYSIS NEAR 95 MeV

The data[8–14] used in the 95-MeV analysis are listed in Table I. The 95-MeV (p,p) $\sigma(\theta)$ data[9] were renormalized to give agreement with total cross section measurements of Goloskie and Palmieri.[15] A computer code was used to carry out the integration of the $\sigma(\theta)$ data. The 95-MeV (p,p) $P(\theta)$ data[9] were lowered by 6.7% and the 98-MeV (p,p) $P(\theta)$ data[10] were given no renormalization constraints. Similar changes were made at 142 MeV, and the reasons for these changes are discussed in Sec. IV. The 99-MeV (n,p) $\sigma(\theta)$ data[8] were renormalized to match the value $\sigma_T=71.8\pm1.7$ mb, which was interpolated from measurements by Bowen.[16] The 91-MeV (n,p) $\sigma(\theta)$ data[12] were renormalized to match the value $\sigma_T=77.7\pm1.7$ mb, as measured by Bowen.[16]

For the (p,p) analysis, 47 data were used. In preliminary searches, the least-squares sum χ^2 was about 38, 32, and 31, respectively, when 5, 7, and 9 free parameters were used in the search. Also, when $^3F_3$, or $^3F_3$ and $^3F_4$, were freed, $^3F_2$ went to a (nonphysical) negative value. Hence we selected 7 as the proper number of free (p,p) phase shifts. There were 77 (n,p) data used in the combined analysis, making a total of 124 pieces of data. For the combined analysis, a comparison of the first data column of Table II with the fifth data column illustrates the fact that the errors on all of the (p,p) phases are considerably increased if $^3F_3$ is treated phenomenologically, and there is no significant reduction in χ^2. Also, it was found that the addition of $^1F_3$ as a free parameter did not improve the analysis. Hence we selected seven $T=1$ plus six $T=0$ free phases for the combined analysis at 95 MeV.

The final phase-shift values obtained at 95 MeV are listed in Table II. The $T=1$ phase shifts are slightly different in the (p,p) and combined analyses, most notably for $^3F_2$. More complete and more accurate data will be required before we can determine F waves at 95 MeV with any accuracy. This is similar to the situation encountered at 50 MeV. The $T=1$ phase shifts from the combined analysis are probably more reliable than the $T=1$ phase shifts from the (p,p) analysis, since they reflect a wider data selection. In particular, the combined analysis value for $^3P_0$ seems to be more in line with the results from other energies (see Fig. 1). Also, the greater reliability of the combined analysis is indicated by the results of g^2 determinations, as shown in the first three rows of Table XI.

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[8] J. P. Scanlon, G. H. Stafford, J. J. Thresher, P. H. Bowen, and A. Langsford, Nucl. Phys. **41**, 401 (1963). [See note added in proof.]

[9] J. N. Palmieri, A. M. Cormack, N. F. Ramsey, and R. Wilson, Ann. Phys. (N.Y.) **5**, 299 (1958).

[10] A. E. Taylor, E. Wood, and L. Bird, Nucl. Phys. **16**, 320 (1960). O. N. Jarvis and B. Rose [Harwell report, (unpublished)] have informed us that these data should be multiplied by the factor 0.911. Since we used no normalization constraint, this change will not affect our results.

[11] E. H. Thorndike and T. R. Ophel, Phys. Rev. **119**, 362 (1960).

[12] R. H. Stahl and N. F. Ramsey, Phys. Rev. **96**, 1310 (1954).

[13] P. H. Bowen, G. C. Cox, G. B. Huxtable, A. H. Langstord, J. P. Scanlon, and J. J. Thresher, Phys. Rev. Letters **7**, 248 (1961). More recent data were taken from the listing in Wilson's book (Ref. 37, Table A-7). [See note added in proof.]

[14] G. H. Stafford, C. Whitehead, and P. Hillman, Nuovo Cimento **5**, 1589 (1957).

[15] R. Goloskie and J. N. Palmieri, Nucl. Phys. **55**, 463 (1964). We thank these authors for communicating to us their data in advance of publication. There is a possible difficulty with these data. When the total cross sections are plotted as a function of the bombarding energy (Fig. 4 of Ref. 15), a curve drawn through the data points has a slight hump near 100 MeV. Gregory Breit commented to Richard Wilson that this hump does not seem to be consistent with an expected smoothly varying energy dependence for the cross section. The 95-MeV $\sigma(\theta)$ data (Ref. 9) we have listed in Table I were normalized to the total cross section value $\sigma_T=31.7$ mb, which is obtained by drawing a curve directly through the data points shown in Fig. 4 of Ref. 15. However, we finally adopted the value $\sigma_T=30\pm2$ mb for the cross section normalization. This leads to the normalization factor 0.946 given in Table I. The measurements of Goloskie and Palmieri have been carefully rechecked (private communication from R. Wilson) and no evidence of any error was found. Thus the way to check on the validity of the slight hump near 100 MeV would be to redo the measurements. We assigned a large normalization error to the differential-cross-section measurements (30±2 mb is a 6.7% uncertainty), but the phase-shift analysis resulted in a renormalization of only 1% (Table I), showing that in any case our results do not depend sensitively on this normalization assignment.

[16] P. H. Bowen, J. P. Scanlon, G. H. Stafford, J. J. Thresher, and P. E. Hodgson, Nucl. Phys. **22**, 640 (1961).

TABLE I. Data used in final (p,p) (set A) and combined (p,p) plus (n,p) (set D) phase-shift analyses near 95 MeV.

Energy (MeV)	Type of data	C.m. angle (deg)	Datum	Exptl. error	Normalization error	Renormalized value ($g^2=13$)		Ref.
						Set A	Set D	
95.0	$\sigma(\theta)$	20.6	4.62	0.08	0.067	0.993	0.984	9
	(p,p)	25.7	5.09	0.08	These data should be multiplied by 0.946 to match the total cross section (see text).			
		30.7	5.32	0.08				
		35.8	5.38	0.08				
		40.9	5.28	0.08				
		46.0	5.35	0.08				
		51.1	5.34	0.08				
		56.2	5.26	0.08				
		61.2	5.24	0.08				
		66.3	5.23	0.08				
		71.3	5.20	0.08				
		76.4	5.17	0.08				
		81.4	5.09	0.08				
		86.4	5.04	0.08				
95	$P(\theta)$	20.6	0.092	0.01	0.03	1.002	1.000	9
	(p,p)	25.7	0.111	0.008	These data should be multiplied by 0.933 (see text).			
		30.7	0.130	0.007				
		35.8	0.131	0.007				
		40.9	0.112	0.007				
		46.0	0.126	0.007				
		51.1	0.115	0.007				
		56.2	0.096	0.007				
		61.2	0.099	0.007				
		66.2	0.087	0.007				
		71.3	0.069	0.008				
		76.4	0.058	0.007				
		81.4	0.038	0.007				
		86.4	0.023	0.007				
98	$P(\theta)$	10.2	0.029	0.031	∞	0.980	0.978	10
	(p,p)	12.3	−0.004	0.033				
		14.3	0.024	0.039				
		16.4	0.085	0.035				
		18.5	0.123	0.035				
		20.5	0.111	0.019				
		22.6	0.093	0.018				
		25.6	0.114	0.015				
		30.7	0.125	0.013				
		40.9	0.121	0.010				
		51.1	0.105	0.010				
		61.3	0.107	0.015				
		71.4	0.073	0.012				
		81.4	0.043	0.011				
98	$D(\theta)$	21.0	0.00	0.08				11
	(p,p)	31.0	0.00	0.07				
		41.0	0.00	0.08				
		51.0	−0.12	0.10				
		61.0	−0.11	0.16				
99	$\sigma(\theta)$	7.0	11.25	0.48	0.024		1.004	8
	(n,p)	14.0	9.93	0.51	[See note added in proof.]			
		21.0	8.01	0.48				
		31.0	8.63	0.43				
		41.0	6.46	0.38				
		51.0	5.21	0.22				
		62.0	4.56	0.32				
		72.0	4.25	0.25				
		82.0	3.65	0.30				
		92.0	3.69	0.21				
		102.0	4.48	0.40				
		112.0	3.96	0.40				
		122.0	4.92	0.59				
		78.0	2.14	0.76				
		88.0	2.90	0.63				
		98.0	4.21	0.54				
		108.0	5.06	0.25				
		118.0	5.85	0.22				
		129.0	6.74	0.30				
		139.0	7.52	0.31				
		149.0	9.50	0.54				
		159.0	9.76	0.47				
		166.0	11.84	0.54				
		173.0	12.63	0.64				

TABLE I (*continued*)

Energy (MeV)	Type of data	C.m. angle (deg)	Datum	Exptl. error	Normalization error	Renormalized value ($g^2=13$)		Ref.
						Set A	Set D	
91	$\sigma(\theta)$	59.8	5.61	0.33	∞		1.036	12
	(n,p)	64.8	4.88	0.29				
		69.7	4.26	0.18				
		74.7	4.08	0.19				
		78.7	4.17	0.15				
		82.7	3.97	0.13				
		88.7	4.19	0.15				
		98.7	4.53	0.14				
		108.7	4.93	0.16				
		118.8	5.99	0.15				
		129.0	6.51	0.17				
		139.1	8.08	0.28				
		139.1	7.74	0.19				
		149.3	9.13	0.24				
		154.9	9.97	0.30				
		159.4	10.84	0.43				
		159.4	10.42	0.29				
		162.0	10.85	0.33				
		164.5	11.82	0.31				
		167.3	11.84	0.30				
		169.7	12.61	0.32				
		171.7	13.24	0.35				
		173.7	13.30	0.33				
		175.6	13.09	0.38				
		176.6	13.08	0.41				
90	$P(\theta)$	20.0	0.158	0.04	0.05		0.991	13
	(n,p)	30.0	0.298	0.047	[See note added in proof.]			
		40.0	0.299	0.043				
		50.0	0.344	0.047				
		70.0	0.544	0.123				
		120.0	0.107	0.025				
		140.0	−0.002	0.022				
100	$P(\theta)$	20.0	0.173	0.065	0.07		0.964	13
	(n,p)	30.0	0.303	0.07	[See note added in proof.]			
		40.0	0.438	0.072				
		50.0	0.452	0.086				
		120.0	0.008	0.04				
		140.0	−0.034	0.035				
95	$P(\theta)$	22.5	0.143	0.032	0.08		1.025	14
	(n,p)	29.8	0.17	0.037				
		41.0	0.32	0.06				
		52.5	0.405	0.041				
		61.5	0.56	0.064				
		76.0	0.307	0.04				
		78.5	0.386	0.034				
		88.5	0.291	0.032				
		98.5	0.256	0.048				
		108.0	0.07	0.047				
		118.5	0.049	0.055				
		128.5	−0.055	0.035				
		138.5	−0.016	0.028				
		149.0	−0.073	0.025				
		159.5	−0.037	0.024				

IV. ANALYSIS NEAR 142 MeV

Analysis of (p,p) and (n,p) data have been given in detail in paper I of this series. Since publication of this paper, changes have occurred both in our method of analysis and in the data used in the analysis. Table III of the present report contains our final values, which supersede the values given in Tables IX and X of I.

The principal changes made in the phase-shift analyses subsequent to the publishing of I was the inclusion of an energy dependence for the phase shifts. The data covered in the (p,p) analysis range from 137.5 to 155 MeV, and the (n,p) data range from 126 to 156 MeV (see Table I of I). Over this large an energy interval some of the phase shifts can change in value by as much as 30%. In I we had investigated the effect of the phase-shift energy dependence by using a linear dependence, $\delta=\delta(a+bE)$, and searching first on a and then on b (Table IV of I). It was found that the parameter b searched to small values, which were often of the wrong sign to agree with the results of energy-dependent analyses. However, this does not mean that an energy-dependent effect is not present, but only that the data included in the analysis were not of sufficient accuracy

TABLE II. Phase-shift solutions at 95 MeV.

(p,p) data	47	47	47	47	47	47	47	56[a]	Slope (°/MeV)
(n,p) data	77	...	77	...	77	...	77	77	
χ^2	129.52	29.60	131.23	29.56	129.59	29.57	133.01	134.81	
g^2	13	11	11	13	13	15	15	13	
$^1S_0$	24.31±3.71	26.85±1.93	26.76±1.90	25.67±2.03	25.61±1.94	24.34±2.11	24.29±2.0	25.08±2.41	−0.25
$^1D_2$	3.89±0.52	3.78±0.26	3.62±0.27	3.91±0.28	3.72±0.29	4.06±0.30	3.84±0.30	3.48±0.38	0.028
$^1G_4$					(0.335)[b]				
$^3P_0$	14.36±2.43	16.40±3.24	14.17±2.37	16.05±3.08	13.90±2.31	15.63±2.89	13.64±2.23	12.83±1.88	−0.1
$^3P_1$	−12.36±0.54	−11.86±0.91	−12.37±0.55	−12.09±0.81	−12.45±0.52	−12.31±0.73	−12.53±0.50	−12.95±0.50	−0.075
$^3P_2$	10.77±0.72	9.90±0.67	10.41±0.49	10.13±0.61	10.55±0.46	10.37±0.55	10.76±0.45	10.55±0.53	0.09
ϵ_2	− 3.03±0.48	− 3.03±0.30	− 2.74±0.27	− 3.16±0.30	− 2.85±0.28	− 3.29±0.29	− 2.97±0.27	− 2.75±0.33	0
$^3F_2$	0.90±0.73	− 0.068±0.41	0.35±0.28	0.25±0.38	0.62±0.27	0.58±0.35	0.92±0.26	1.29±0.74	0.007
$^3F_3$	− 1.91±0.83				(−1.54)			− 1.62±0.59	
$^3F_4$					(0.199)			0.62±0.24	
ϵ_4					(−0.478)				
$^3H_4$					(0.092)				
$^3H_5$					(−0.273)				
$^3H_6$					(0.030)				
$^1P_1$	−13.97±2.37		−13.48±2.19		−13.41±2.08		−13.55±1.93	−16.53±3.34	−0.11
$^1F_3$					(−2.13)				
$^3S_1$	43.91±1.79		42.93±1.68		43.54±1.71		44.44±1.76	44.54±1.71	−0.24
ϵ_1	0.01±1.42		− 0.13±1.16		0.16±1.30		− 0.05±1.60	− 0.61±1.74	0.035
$^3D_1$	−11.95±0.71		−12.10±0.67		−12.05±0.67		−11.98±0.73	−11.35±0.81	−0.07
$^3D_2$	15.85±2.05		17.14±1.61		16.46±1.70		15.30±1.81	13.59±2.62	0.11
$^3D_3$	1.29±0.56		1.06±0.50		1.19±0.52		1.35±0.56	1.93±0.63	0.04
ϵ_3					(3.34)				
$^3G_3$					(−0.77)				
$^3G_4$					(1.88)				
$^1H_5$					(−0.49)				

[a] Includes R and R' data (see Addendum).

[b] OPEC values.

to give reliable values for b, using only a narrow band of energies. For the final values in I, we set $b=0$ and assumed that the phase-shift values would represent average values over the energy range. From our subsequent work, described in the next paragraph, it turns out that phase shifts averaged in this manner do in fact represent quite a good average value. However, since the results of analyses at all energies in the elastic range give reliable values for b, this information should be used in the analyses.

The phase-shift analyses described in I were redone using a linear energy dependence for the phase shifts. The energy derivatives, listed in the last column of Table VI, were obtained from the phase shift versus energy values given in Figs. 1 and 2 of the present report. These derivatives were held fixed. Hence the phase shifts still represent only a single value per phase for each energy range.

When a realistic energy dependence was added to the 142-MeV analysis, complications developed. In Table XI of I, the Saclay (p,p) differential cross-section data at 155 MeV required a renormalization of only 1.5% (footnote a) when no energy dependence was used. However, when we now included a realistic energy dependence, these same data required a renormalization of 9%, or more than two standard deviations in the quoted normalization error (Ref. 14 of I). This indicated that some of the data in Table XI of I are inconsistent with one another. Furthermore, in footnotes b and g of Table XI, it can be seen that the normalizations of the Harvard (p,p) $P(\theta)$ data at 147 MeV (Ref. 11 of I) and the Harwell (p,p) $P(\theta)$ data at 142 MeV (Ref. 12 of I) were in close agreement with each other (within 2%). When we inserted a realistic energy dependence, this was no longer true. The Harvard normalization value changed from 0.988 to 1.020 (footnote b of XI), and the Harwell value changed from 0.986 to 0.963 (footnote g of XI). Thus the two normalizations differed by somewhat over 5%, which was outside of their combined quoted errors.

Part of this difficulty was resolved by receipt of a letter from Wilson and Palmieri[17] in which they stated that all Harvard (p,p) and (p,n) polarization measurements should be lowered by 6.7%. The Harvard cyclotron proton-beam polarization after scattering off a carbon target was in error by this amount. This change affected the Harvard data listed in Refs. 11, 25, and 31 of I. The (n,p) $P(\theta)$ data of Refs. 29 and 30 were, of course, not affected. When we inserted these changes for Refs. 11 and 31 (we had already eliminated 25), the difficulty with the Saclay data vanished. The required renormalization was now less than 4% (footnote a of XI). However, this change made the discrepancy between the Harvard and Harwell (p,p) $P(\theta)$ normalizations even worse if we did not change the Harwell values. They now differed by 12%, and the least-squares sum χ^2 increased by 10 owing to the normalization mismatch. As a way out of this difficulty, we released all constraints on the Harwell (p,p) $P(\theta)$ normalization, under the assumption that the Harvard (p,p) $P(\theta)$ as revised is probably correct. A letter subsequently received from A. E. Taylor confirmed that any changes in the Harvard $P(\theta)$ normalization assignments should also be made in the Harwell $P(\theta)$ normalization assignments, since they were both based on the same (p, carbon) scattering measurements. (Jarvis and Rose have made new measurements,[10] which differ somewhat from the values previously used.) Similar changes were

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[17] R. Wilson and J. Palmieri (private communication).

TABLE III. Phase-shift solutions at 142 MeV.

(p,p) data	127	127	127	127	127	127	127	127	127	127	
(n,p) data	…	…	93	93	…	93	…	93	…	93	
χ^2	102.57	97.96	236.80	231.25	101.55	232.48	101.47	231.08	101.86	230.70	Slope
g^2	13	13	13	13	11	11	13	13	15	15	(°/MeV)
$^1S_0$	16.51±0.72	16.72±0.75	16.63±0.70	16.58±0.70	16.93±0.79	16.78±0.74	16.82±0.73	16.63±0.73	16.71±0.79	16.49±0.73	−0.17
$^1D_2$	5.04±0.21	5.37±0.24	5.17±0.20	5.16±0.20	5.19±0.23	5.23±0.22	5.14±0.23	5.17±0.22	5.09±0.23	5.10±0.22	0.028
$^1G_4$	0.66±0.19	0.61±0.12	0.57±0.12	0.58±0.12	0.61±0.12	0.55±0.12	0.64±0.12	0.58±0.12	0.66±0.12	0.61±0.12	0.0055
$^3P_0$	6.25±0.61	6.21±0.65	6.31±0.55	6.33±0.55	6.54±0.67	6.35±0.57	6.53±0.68	6.32±0.57	6.53±0.69	6.27±0.58	−0.11
$^3P_1$	−16.97±0.41	−16.66±0.44	−17.18±0.39	−17.13±0.38	−16.68±0.46	−17.11±0.41	−16.68±0.46	−17.13±0.41	−16.69±0.47	−17.16±0.41	−0.07
$^3P_2$	13.72±0.22	13.75±0.21	13.73±0.21	13.73±0.22	13.77±0.22	13.75±0.22	13.76±0.22	13.74±0.22	13.75±0.22	13.73±0.22	0.055
ϵ_2	−2.81±0.16	−2.72±0.15	−2.90±0.15	−2.89±0.15	−2.69±0.17	−2.86±0.15	−2.72±0.18	−2.89±0.15	−2.74±0.18	−2.92±0.15	0
$^3F_2$	0.77±0.34	0.58±0.45	0.76±0.28	0.76±0.27	0.39±0.47	0.71±0.33	0.41±0.47	0.75±0.32	0.42±0.46	0.80±0.32	0.0065
$^3F_3$	−2.02±0.23	−1.75±0.24	−2.08+0.21	−2.08±0.20	−1.83±0.27	−2.03±0.21	−1.87±0.27	−2.08±0.21	−1.89±0.27	−2.14±0.20	−0.013
$^3F_4$	0.95±0.17	0.87±0.23	0.93±0.14	0.93±0.14	0.73±0.26	0.88±0.18	0.75±0.25	0.92±0.17	0.77±0.25	0.96±0.17	0.01
ϵ_4	−0.65±0.07	−0.57±0.08	−0.64±0.06	−0.64±0.06	−0.60±0.75	−0.63±0.07	−0.62±0.07	−0.64±0.07	−0.64±0.07	−0.66±0.07	−0.006
$^3H_4$	0.14±0.07	−0.30±0.22	(0.18)[a]		0.36±0.84	0.12±0.08	0.08±0.08	0.16±0.08	0.12±0.08	0.20±0.08	0.002
$^3H_5$		−0.06±0.20	(−0.50)		−0.30±0.16	−0.45±0.14	−0.33±0.16	−0.48±0.14	−0.35±0.16	−0.50±0.14	−0.0047
$^3H_6$		−0.22±0.16	(0.07)								0.0008
$^1P_1$			−16.83±1.43	−13.14±2.39		−14.79±2.42		−13.45±2.43		−12.54±2.44	−0.11
$^1F_3$			−0.69±0.61	−1.01±0.68		−0.97±0.68		−0.93±0.70		−0.88±0.70	−0.036
$^3S_1$			29.27±0.87	29.67±0.90		29.59±0.90		29.65±0.90		29.65±0.89	−0.2
ϵ_1			1.70±0.74	1.39±0.73		1.52±0.78		1.40±0.74		1.39±0.70	0.035
$^3D_1$			−14.48±0.61	−14.97±0.70		−14.81±0.70		−14.94±0.70		−15.00±0.70	−0.07
$^3D_2$			23.60±0.90	24.50±1.01		24.14±1.07		24.42±1.02		24.57±0.99	0.035
$^3D_3$			2.54±0.56	0.81±0.88		1.39±0.95		0.91±0.91		0.60±0.88	0.04
ϵ_3			3.31±0.44	2.55±0.56		2.73±0.60		2.56±0.57		2.44±0.54	0
$^3D_3$			(−1.36)	−2.60±0.53		−2.22±0.58		−2.54±0.54		−2.78±0.51	−0.012
$^3G_4$			(3.10)	4.73±0.92		4.16±0.92		4.64±0.93		4.97±0.94	0.024
$^1H_5$			(−0.81)								

[a] OPEC values.

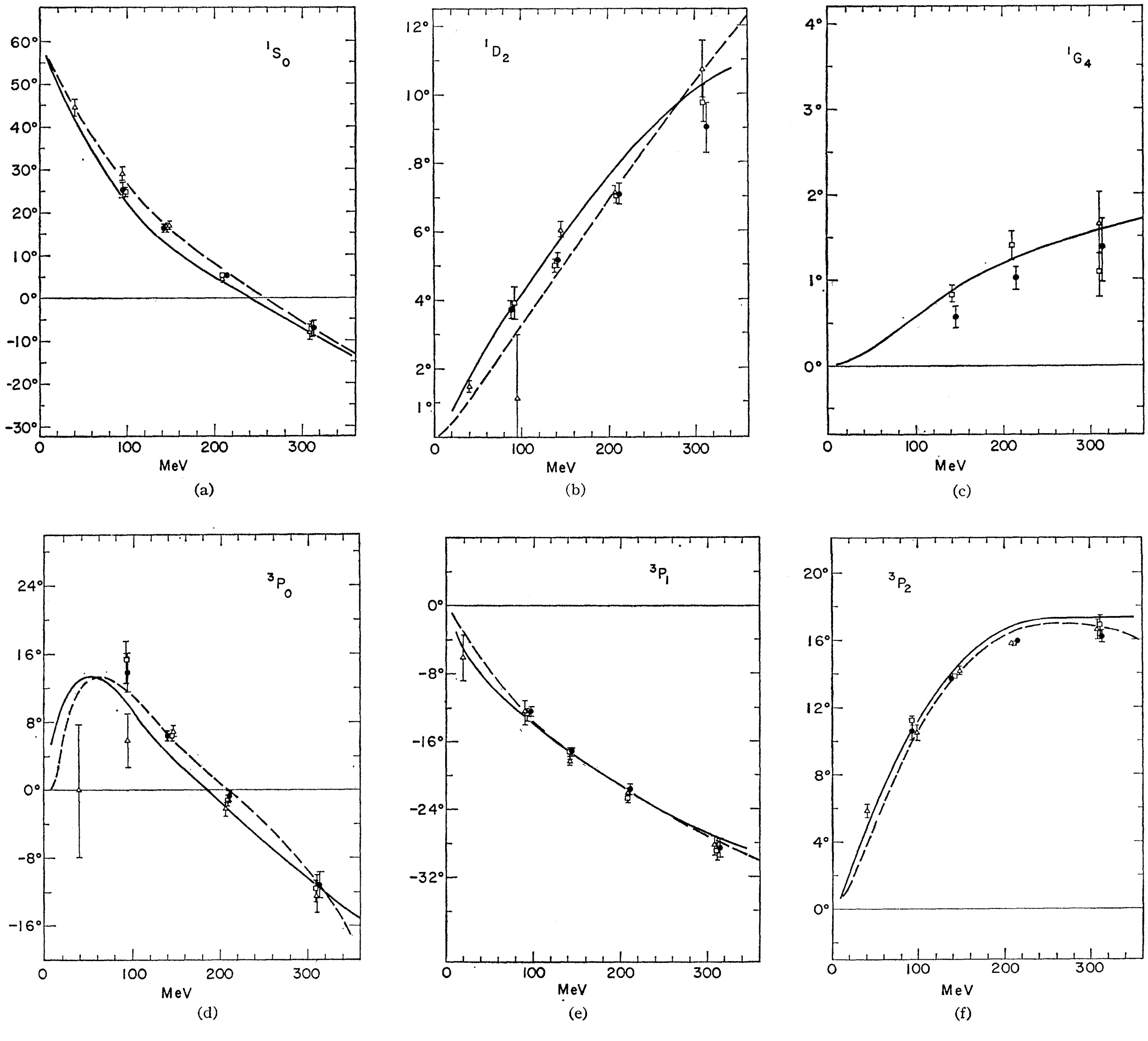
also made in (p,p) $P(\theta)$ data at 95 MeV, as noted in Sec. III.

One additional change was made in the search procedure that differs from I. Our derivative search routine[3] did not include the normalization constants in I. These were searched separately. This has the effect that the phase-shift errors given by the error matrix are slightly too small, since they don't reflect the normalization uncertainties. Hence for the present paper we included the normalization parameters also in the derivative search routine. The only parameter not included in this search routine was g^2, the pion-nucleon coupling constant.

The changes we have made in the 142-MeV data selection and the normalization constants as given in Table XI of I can be summarized by making the following changes in the footnotes to that table (we quote results for the $g^2=13$ solution): a: A' 1.006, D' 1.032; b: data multiplied by 0.933, NE 2.4%, A' 1.004, D' 0.998; e: A' 1.013, D' 1.007; f: A' 1.025, D' 1.033; g: NE ∞, A' 0.880, D' 0.875; *1* D' 0.992; m: D' 0.958; o: D' 1.040; q: D' 1.029; r: D' 1.010; s: data multiplied by 0.933, NE ∞, D' 1.155; t: D' 0.989. In addition, the (n,p) $R(\theta)$ at 137 MeV should have the following footnote: Re D', 1.

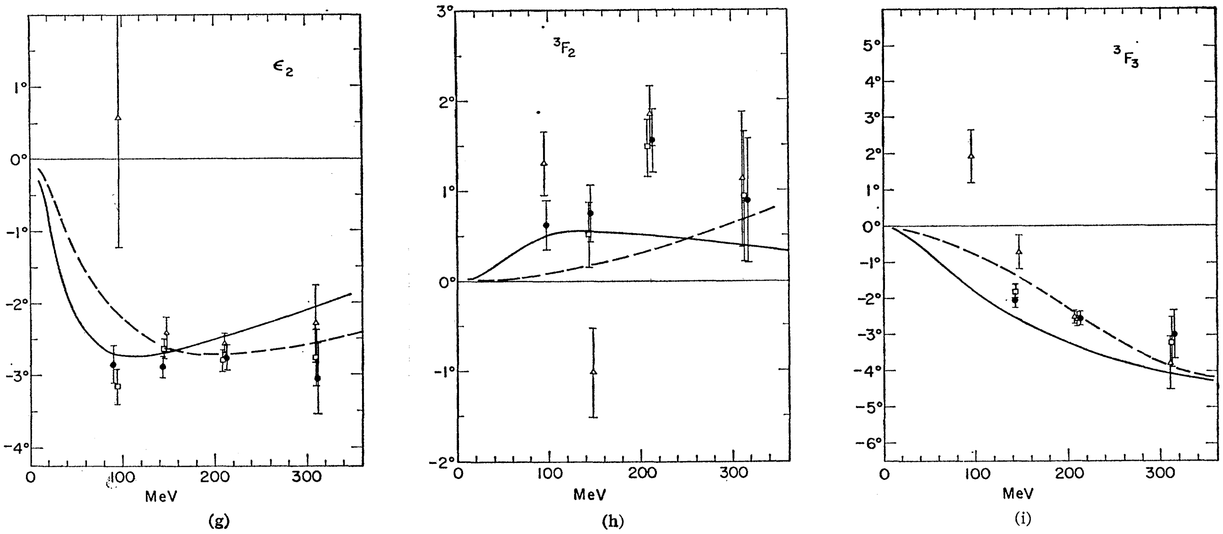
Table III lists the final 142-MeV phase shift values from the present analysis. These values do not differ radically from those quoted in I, although the differences are substantial in many cases when measured in terms of the quoted errors. The first and second data columns of Table III illustrate the fact that 14 free (p,p) phases give a χ^2 value almost 5 lower than for 12 free (p,p) phases, although the H waves for the former seem to show anomalously large departures from OPEC. The 142-MeV (p,p) data are not yet accurate enough to give any reliable information about H waves. For our final solution we selected 13 free (p,p) phases.

The third and fourth data columns of Table III



(a) (b) (c)

(d) (e) (f)



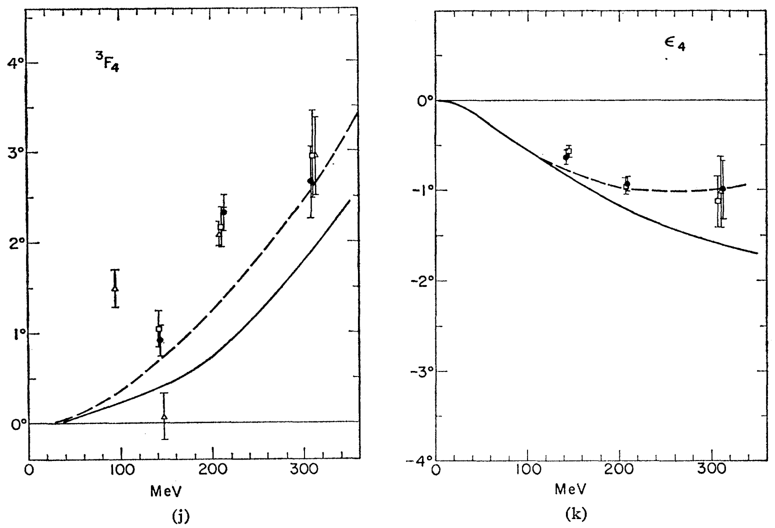


FIG. 1. Comparison of $T=1$ phase-shift values. ● Livermore—MacGregor and Arndt (present paper). □ Penn State—Signell (Ref. 40). △ Dubna—Kazarinov and Silin (Ref. 41). —— Yale (YLAM)—Breit *et al.* (Ref. 42). --- Livermore (MIDPOP)—Moravcsik, Noyes, and Stapp (Ref. 43).

show that the (n,p) data appear to be accurate and complete enough to warrant adding $^3G_3$ and $^3G_4$ as free parameters in the search. Hence our final combined analysis includes thirteen $T=1$ and ten $T=0$ free phase shifts. The values obtained for $^3G_3$ and $^3G_4$ are probably not too meaningful, but the $T=0$ phase shifts up through ϵ_3 should be reliable. Also, the $T=1$ phase shifts up through ϵ_4 should be reliable.

The value for $^3F_2$ as given by the 13-phase (p,p) solution is not fully consistent with the value from combined analysis and it differs also from the 12- and 14-phase (p,p) values. This phase shift at 142 MeV seems to be quite sensitive to small changes in the normalization parameters. A comparison with values for $^3F_2$ at 210 MeV indicates that the value as given by the combined analysis is more nearly correct.

V. ANALYSIS NEAR 210 MeV

The data used in this analysis[18–25] are listed in Table IV. The (p,p) $\sigma(\theta)$ data[18] at 210 MeV were re-

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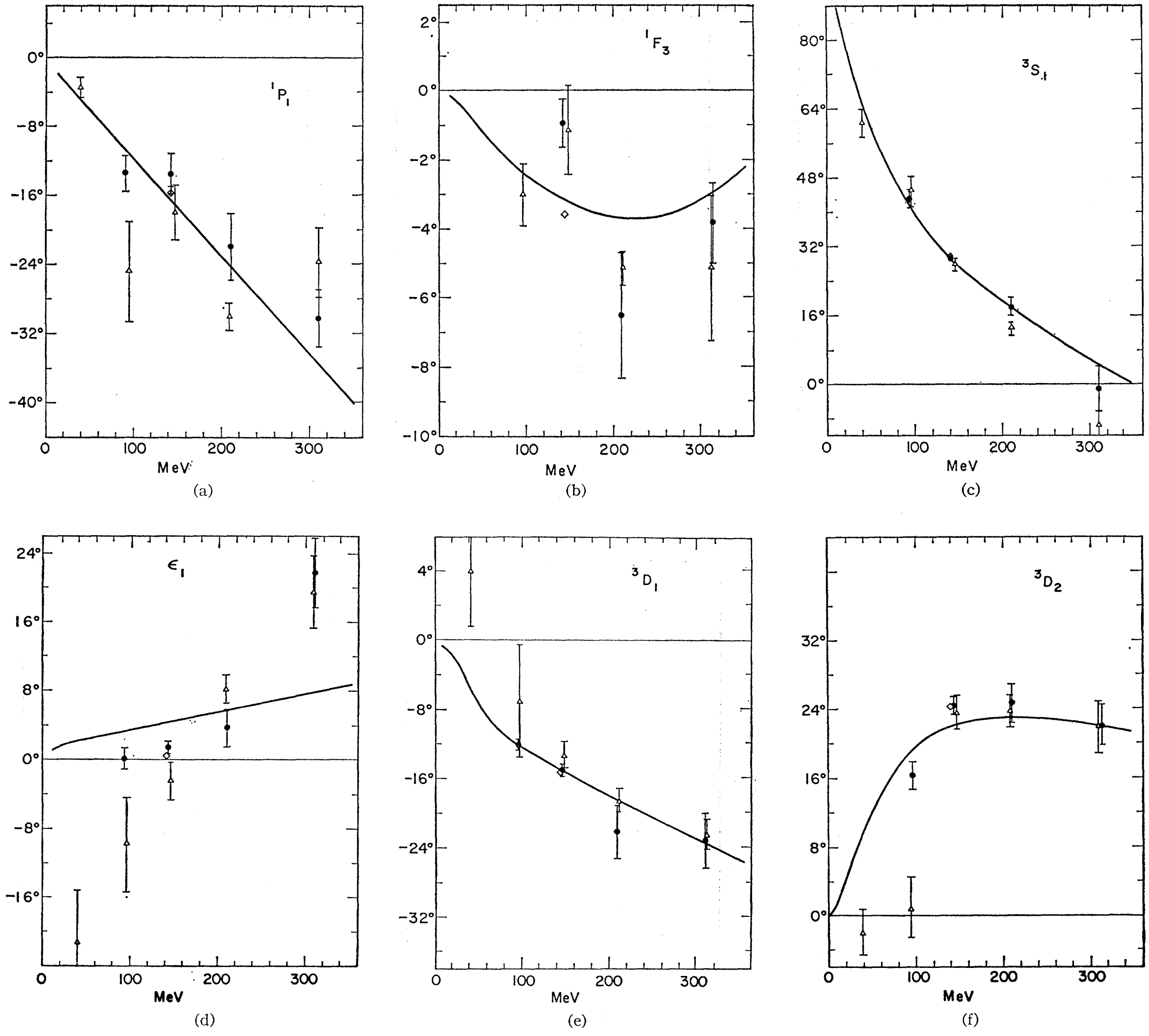
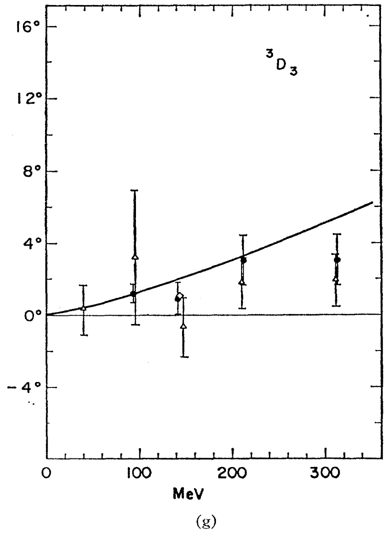


(a) (b) (c)

(d) (e) (f)



(g)

FIG. 2. Comparison of $T=0$ phase shift values. ● Livermore—MacGregor and Arndt (present paper). △ Dubna—Kazarinov and Silin (Ref. 41). —— Yale (YLAN3M)—Hull *et al.* (Ref. 44). ◇ Harwell—Perring (Ref. 45).

TABLE IV. Data used in final (p,p) (set *A*) and combined (p,p) plus (n,p) (set *D*) phase shift analyses near 210 MeV.

Energy (MeV)	Type of data	C.m. angle (deg)	Datum	Exptl. error	Normalization error	Renormalized value ($g^2=13$)		Ref.
						Set *A*	Set *D*	
213	$\sigma(\theta)$ (p,p)	30	3.71	0.06	0.042	0.995	0.988	18
		40	3.74	0.05				
		50	3.64	0.04				
		60	3.56	0.04				
		70	3.57	0.04				
		80	3.57	0.03				
		90	3.52	0.04				
210	$P(\theta)$ (p,p)	30	0.312	0.006	0.022	0.976	0.977	19
		40	0.319	0.0085				
		50	0.303	0.0075				
		60	0.240	0.006				
		70	0.163	0.007				
		80	0.084	0.007				
		90	−0.002	0.007				
217	$P(\theta)$ (p,p)	60	0.246	0.007	0.022	1.010	1.012	19
		70	0.153	0.008				
		80	0.079	0.008				
		90	0.014	0.011				
		100	−0.09	0.009				
		110	−0.153	0.009				
		120	−0.218	0.009				
213	$D(\theta)$ (p,p)	30	0.2	0.016				20
		40	0.232	0.026				
		50	0.240	0.018				
		60	0.319	0.021				
		70	0.297	0.03				
		80	0.36	0.07				
		90	0.5	0.18				
213	$R(\theta)$ (p,p)	30	−0.203	0.012				20
		40	−0.133	0.017				
		50	−0.041	0.018				
		60	+0.071	0.026				
		70	0.147	0.029				
		80	0.248	0.042				
		90	0.223	0.055				
213	$E(\theta)$ (p,p)	30	−0.449	0.016				20
		40	−0.343	0.015				
		50	−0.202	0.017				
		60	−0.059	0.018				
		70	+0.053	0.029				
213	$R'(\theta)$ (p,p)	30	0.538	0.028				21
		40	0.390	0.028				
		50	0.193	0.026				
		60	−0.055	0.066				
200	$\sigma(\theta)$ (n,p)	6.25	9.5	2.50	0.021		1.003	22
		10.5	8.3	0.80				
		21.3	4.7	0.70				
		31.5	4.1	0.50				
		41.7	3.0	0.40				
		62.7	2.4	0.40				
		67.3	2.16	0.16				
		77.3	1.91	0.07				
		87	1.87	0.08				
		97	2.2	0.08				
		109.3	2.79	0.16				
		117.5	3.51	0.24				
		129.6	3.85	0.16				
		139.3	4.63	0.16				
		148.5	5.79	0.12				
		159	7.02	0.13				
		163	7.78	0.24				
		165	9.22	0.26				
		169.5	10.33	0.23				
		173.75	11.29	0.24				
215	$P(\theta)$ (n,p)	40	0.501	0.035	0.12		1.095	23 25
		50	0.466	0.038				
		60	0.362	0.044				
		70	0.24	0.035				
		80	0.012	0.038				
		90	−0.087	0.034				
212	$D(\theta)$ (n,p)	40	0.79	0.1				24 25
		50	0.90	0.1				
		60	0.82	0.09				
		70	1.01	0.14				
		80	1.06	0.45				

normalized downward by 2.5% to give agreement with the measured total cross section at 225 MeV, which is $\sigma_T(\theta>20°)=21.3\pm0.7$ mb.[26] Numerical integration of $\sigma(\theta)$ was carried out using a computer calculation. The $E(\theta)$ data at 213 MeV[20] are linear combinations of $A(\theta)$ and $R(\theta)$, namely $E=A\ \sin\chi+R\ \cos\chi$, where $\chi=63.3°$. Values for $E(\theta)$ at 80° and at 90° contributed 16 to the χ^2 sum and were eliminated from the data selection. Knowing R, we can extract values for A from the E data. Problems run using first A data and then E data showed very small differences in χ^2 or in the phase shift values. From an experimental point of view, the E data are to be preferred. The Penn State group, in their 213-MeV analyses,[7] have used a recent set of data for $R'R$ (in Signell's notation), where $R'R=R'\sin\chi+R\cos\chi$, with $\chi=61°$. They found it necessary to reject data points at 60 and 70°, and they found the $R'R$ value at 80° to be two standard deviations off the calculated curve corresponding to the final phase-shift solution. Using the $R'R$ values as listed in Ref. 21, we extracted values for R' to use in our existing computer code. Analysis indicated that we should remove the data at 60, 70, and 80°, in agreement with Signell's result.[7] As a practical matter, the statistical errors for R' are large enough that the data have little effect on the phase shift analysis. Hence, using $R'R$ or R' for the analysis will produce no observable difference in the results. In principle, of course, the $R'R$ data are preferred from an experimental point of view.

A computer integration of the (n,p) differential cross section[22] at 200 MeV gave a total cross-section value of 42.48 mb. The experimental measurement of σ_T at 200 MeV is 42.7 ± 0.9 mb.[22] Hence the published cross section values are correct in absolute value. The 180° point was omitted because of its large contribution to χ^2. The (n,p) $P(\theta)$ [23] and $D(\theta)$ [24] data were used as corrected for deuteron binding effects by Cromer and Thorndike.[25] In treating these data, one should use a normalization error that varies with angle, due to the nature of the binding correction.[25] For the $D(\theta)$ data, this effect makes little difference, due to the large statistical uncertainties. For the $P(\theta)$ data, the effect is more pronounced. It was not convenient in the computer code to use normalization errors in the form suggested by Thorndike. Instead we tried two approximations. In the first, the normalization errors were combined quadratically with the statistical errors, and the over-all normalization constant was eliminated. In the second approximation, an "average" normalization error of 12% was used together with the published statistical errors. This is the form listed in Table IV. The two approximations gave essentially identical $T=1$ phase shifts. They gave $T=0$ phase shifts which were not identical, but which in every case differed by less than half of the phase shift uncertainty as given by the error matrix. The phase shift values listed in Table V are based on the second approximation.

For the (p,p) analysis, 44 data were used. With 10 free phase shifts ($^1S_0$, $^1D_2$, and $^3P_0-\epsilon_4$), χ^2 was about 57. Adding $^1G_4$ and $^3H_4$ dropped χ^2 to 28.2 for 12 free phase shifts. Adding $^3H_5$ and $^3H_6$ dropped χ^2 to 27.6, and the H waves deviated only moderately from the OPEC values. Hence we chose 14 free phase shifts as the proper number to represent the $T=1$ scattering matrix. The phase shift solutions are shown in Table V. The combined analysis was based on 75 data. The (n,p) data at 210 MeV are not as complete as at 142 MeV. From the results of Table V we selected 9 as the proper number of free $T=0$ phases. As can be seen in Table V, the $T=0$ phase shifts are fairly accurately determined, in spite of the incompleteness of the (n,p) data selection. The $T=1$ phase shifts are accurately determined up through $l=5$. The $T=1$ H waves are not well determined by the present data near 210 MeV. In particular, if $^3H_5$ and $^3H_6$ phases are added to the search, the $^3H_4$ phase changes from the value 0.16 ± 0.12 shown in Table V ($g^2=13$) to the value 0.35 ± 0.36. Signell[7] has remarked on a similar instability for $^3H_5$. Thus neither the 142-MeV nor the 210-MeV data are sufficient at the present time to give accurate values for H waves.

In the third column of Table V we have listed the favored solution from the Penn State analysis.[7] For phases through $l=5$, the results of the Penn State analysis and the present (p,p) analysis are essentially identical. The order of selection of phase shifts in the two analyses differs somewhat. This is discussed in Sec. VII.

VI. ANALYSIS NEAR 310 MeV

The data used in this analysis[27–37] are listed in Table VI. In selecting (p,p) $\sigma(\theta)$ data, we rejected points more than two standard deviations away from the theoretical values. The (p,p) $\sigma(\theta)$ data at 330 MeV[28] are essentially relative data. They were adjusted in normalization during the search to match the (p,p)

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TABLE V. Phase-shift solutions at 210 MeV.

(p,p) data	44	44	Penn	44	44	44	44	44	44	
(n,p) data	...	31	State	...	31	...	31	...	31	
χ^2	28.16	53.65	solution[7]	28.77	53.07	27.60	52.70	26.85	54.70	Slope
g^2	13	13	14.4	11	11	13	13	15	15	(°/MeV)
$^1S_0$	5.45±0.51	5.01±0.55	5.19±0.44	5.24±0.57	5.11±0.56	5.27±0.57	5.14±0.56	5.29±0.58	5.15±0.57	−0.13
$^1D_2$	7.09±0.26	6.97±0.30	7.13±0.29	7.02±0.33	6.99±0.32	7.06±0.33	7.05±0.32	7.10±0.33	7.11±0.32	0.028
$^1G_4$	1.07±0.15	1.01±0.15	1.12±0.14	1.04±0.16	1.02±0.15	1.06±0.16	1.03±0.15	1.07±0.16	1.04±0.15	0.0045
$^3P_0$	− 0.98±0.59	− 0.62±0.60	− 1.37±0.58	− 0.79±0.59	− 0.603+0.59	− 0.89±0.60	− 0.68±0.60	− 0.99±0.60	− 0.77±0.60	−0.13
$^3P_1$	−22.01±0.55	−21.55±0.61	−22.09±0.71	−21.81±0.62	−21.59±0.60	−21.84±0.62	−21.59±0.61	−21.86±0.62	−21.60±0.61	−0.06
$^3P_2$	15.75±0.24	15.90±0.27	15.99±0.28	15.86±0.27	15.90±0.27	15.83±0.27	15.88±0.27	15.81±0.27	15.87±0.27	0.02
ϵ_2	− 2.77±0.16	− 2.78±0.19	− 2.85±0.19	− 2.71±0.19	− 2.69±0.19	− 2.79±0.20	− 2.75±0.19	− 2.87±0.20	− 2.83±0.19	0
$^3F_2$	1.49±0.33	1.55±0.36	1.49±0.37	1.65±0.36	1.64±0.35	1.55±0.36	1.55±0.35	1.46±0.36	1.47±0.36	0.006
$^3F_3$	− 2.61±0.21	− 2.57±0.22	− 2.62±0.21	− 2.59±0.22	− 2.57±0.22	− 2.60±0.22	− 2.57±0.22	− 2.61±0.22	− 2.57±0.22	−0.01
$^3F_4$	2.21±0.18	2.32±0.21	2.25±0.20	2.34±0.21	2.37±0.20	2.28±0.21	2.32±0.20	2.22±0.21	2.27±0.21	0.013
ϵ_4	− 0.94±0.10	− 0.95±0.09	− 1.03±0.09	− 0.91±0.10	− 0.89±0.09	− 0.97±0.10	− 0.94±0.09	− 1.03±0.10	− 0.99±0.09	−0.001
$^3H_4$	0.16±0.12	0.59±0.29		0.39±0.35	0.48±0.33	0.35±0.36	0.43±0.34	0.31±0.36	0.40±0.35	0
$^3H_5$	(−0.81)[a]	− 0.65±0.21	− 0.64±0.18	− 0.72±0.22	− 0.64±0.21	− 0.73±0.22	− 0.62±0.21	− 0.74±0.22	− 0.60±0.21	−0.005
$^3H_6$	(0.13)	0.49±0.22	0.35±0.09	0.36±0.26	0.44±0.24	0.30±0.27	0.38±0.25	0.25±0.27	0.33±0.26	0.001
$^1P_1$		−20.18±3.75			−24.46±3.79		−21.83±4.00		−18.93±3.94	−0.11
$^1F_3$		− 7.79±1.40			− 6.39±1.63		− 6.51±1.85		− 6.93±1.97	−0.036
$^3S_1$		18.10±2.33			17.71±2.45		17.60±2.44		17.54±2.45	−0.14
ϵ_1		4.30±2.11			3.87±2.32		3.72±2.27		3.78±2.24	0.035
$^3D_1$		−23.77±2.67			−21.66±2.96		−22.13±3.10		−22.86±3.06	−0.07
$^3D_2$		24.17±2.31			23.53±2.44		24.63±2.32		25.39±2.27	0
$^3D_3$		2.64±1.41			3.08±1.48		3.07±1.41		2.85±1.38	0.04
ϵ_3		6.50±0.70			7.22±0.70		6.90±0.71		6.46±0.77	0
$^3G_3$		(−2.19)			− 0.25±1.29		− 0.68±1.38		− 1.34±1.51	−0.01
$^3G_4$		(4.67)								
$^1H_5$		(1.19)								

[a] OPEC values.

TABLE VI. Data used in final (p,p) (set A) and combined (p,p) plus (n,p) (set D) phase-shift analyses near 310 MeV.

Energy (MeV)	Type of data	C.m. angle (deg)	Datum	Exptl. error	Normalization error	Renormalized value ($g^2=13$) Set A	Renormalized value ($g^2=13$) Set D	Ref.
345	$\sigma(\theta)$	36.4	3.93	0.15	0.05	0.962	0.961	27
	(p,p)	43.4	3.79	0.15				
		45.8	3.64	0.07				
		52.4	3.77	0.10				
		60.8	3.83	0.13				
		64.0	3.55	0.11				
		64.0	3.74	0.14				
		70.6	3.67	0.16				
		72.2	3.67	0.11				
		80.2	3.95	0.12				
		87.6	3.86	0.10				
		88.2	3.91	0.08				
		88.2	3.70	0.08				
		88.6	3.85	0.06				
		88.6	3.54	0.09				
		89.2	4.15	0.36				
		15.2	3.71	0.22				
		15.2	3.21	0.17				
		21.1	3.51	0.10				
		32.5	3.52	0.09				
		33.1	3.51	0.11				
		42.8	3.48	0.10				
330	$\sigma(\theta)$	6.52	8.59	0.82	0.2	0.926	0.926	28
	(p,p)	7.28	6.34	0.61				
		11.43	3.14	0.36				
		12.93	3.45	0.31				
		14.80	3.49	0.29				
		16.77	3.58	0.23				
		18.63	3.44	0.27				
		20.87	4.02	0.24				
		22.80	3.62	0.29				
		24.27	3.75	0.31				
		26.03	3.66	0.31				
		27.57	3.63	0.35				
		29.70	3.81	0.35				
315	$P(\theta)$	21.6	0.305	0.023	0.04	1.007	1.017	29
	(p,p)	32.3	0.378	0.027				
		42.9	0.379	0.02				
		53.4	0.303	0.025				
		63.9	0.251	0.027				
		76.2	0.142	0.025				
		89.4	−0.005	0.016				
310	$P(\theta)$	6.5	−0.21	0.27	0.04	0.989	0.991	29
	(p,p)	7.6	0.11	0.28				
		8.7	0.02	0.13				
		11.0	0.19	0.07				
		13.0	0.25	0.05				
		17.3	0.25	0.04				
		21.7	0.37	0.04				
315	$C_{NN}(\theta)$ (p,p)	90	0.84	0.16				30
320	$C_{NN}(\theta)$ (p,p)	90	0.77	0.11				31
310	$D(\theta)$	23.1	0.245	0.079				29
	(p,p)	25.8	0.299	0.055				
		36.5	0.456	0.081				
		52.0	0.533	0.06				
		65.0	0.503	0.048				
		80.5	0.472	0.063				
310	$R(\theta)$	22.4	−0.324	0.139				29
	(p,p)	34.4	−0.167	0.08				
		41.8	0.104	0.071				
		54.1	0.287	0.052				
		70.9	0.310	0.072				
		80.1	0.576	0.087				
316	$A(\theta)$	25.4	−0.339	0.064				32
	(p,p)	51.4	0.007	0.045				
		76.3	0.236	0.05				
300.0	$\sigma(\theta)$	35.0	3.81	0.41	0.10		1.059	33
	(n,p)	45.0	3.5	0.35				
		55.0	2.96	0.28				
		65.0	2.31	0.31				

TABLE VI (*continued*)

Energy (MeV)	Type of data	C.m. angle (deg)	Datum	Exptl. error	Normalization error	Renormalized value ($g^2=13$)		Ref.
						Set *A*	Set *D*	
		75.0	2.09	0.2				
		85.0	1.89	0.18				
		95.0	1.51	0.14				
		105.0	2.07	0.16				
		115.0	2.17	0.17				
		125.0	2.51	0.19				
		135.0	3.06	0.23				
		145.0	4.06	0.29				
		155.0	4.71	0.37				
		165.0	6.48	0.55				
		175.0	9.14	1.12				
290	$\sigma(\theta)$	10.7	5.6	1.1	0.10		1.060	34
	(n,p)	21.7	4.3	0.9				
		37.8	3.6	0.7				
350	$\sigma(\theta)$	114.2	1.94	0.03	0.03		0.977	35
	(n,p)	125.5	2.57	0.03				
		137.1	3.5	0.05				
		142.3	4.0	0.05				
		147.9	4.55	0.06				
		152.1	5.02	0.06				
		156.3	5.38	0.07				
		160.7	5.95	0.07				
		162.9	6.35	0.08				
		165.1	7.0	0.08				
		160.7	5.95	0.07				
		162.9	6.42	0.09				
		165.1	6.95	0.10				
		167.2	7.44	0.10				
		170.5	8.83	0.12				
		173.6	10.19	0.14				
		173.8	10.5	0.14				
310	$P(\theta)$	21.6	0.52	0.08	0.04		1.003	36
	(n,p)	32.3	0.44	0.05				
		42.9	0.39	0.041				
		53.4	0.23	0.034				
		63.9	0.158	0.036				
		74.2	−0.012	0.036				
		82.3	−0.09	0.034				
		82.3	−0.126	0.039				
		90.6	−0.097	0.038				
		100.7	−0.238	0.036				
		109.9	−0.249	0.072				
		110.2	−0.261	0.036				
		116.1	−0.228	0.038				
		121.3	−0.255	0.060				
		130.8	−0.222	0.072				
		137.7	−0.197	0.065				
		147.7	−0.202	0.066				
		158.4	−0.074	0.064				
		164.9	−0.023	0.070				

data at 345 MeV, which are absolute differential cross sections.[27] The (n,p) $\sigma(\theta)$ data[33–35] were treated essentially as relative data. One run in which the 300-MeV (n,p) $\sigma(\theta)$ data[33] were matched to an interpolated total cross section with a 3% uncertainty showed a slight increase in χ^2 but little change in the phase shifts. The (n,p) $P(\theta)$ data were approximately corrected for deuteron binding effects.[36]

The results of the 310-MeV analysis are shown in Table VII. For the (p,p) analysis, 14 free phases give a statistically better fit than do 12. However, the H phases are only qualitatively defined by the data. For the combined analysis, 9 free $T=0$ phases should be included, as shown in data columns 2 and 7 of Table VII. Historically, the 300-MeV region was the first energy region where double and triple scattering experiments were carried out for nucleon-nucleon scattering. The measurements that we have used in the phase-shift analysis at 310 MeV predate the measurements at 142 and 210 MeV. Hence they are, in general, neither as complete nor as statistically accurate as the newer measurements. This fact is reflected in the error matrices, as can be seen by comparing Tables III, V, and VII. However, from the point of view of systematic errors, we have no reason to question the 310-MeV data. If one ponders over the involved history of nucleon-nucleon measurements near 140 MeV, it becomes evident that completeness and statistical accuracy

TABLE VII. Phase-shift solutions at 310 MeV.

(p,p) data	66	66	66	66	66	66	66	66	
(n,p) data	...	54	...	54	...	54	...	54	
χ^2	63.94	93.57	59.57	102.48	58.92	92.78	59.20	94.67	Slope (°/MeV)
g^2	13	13	11	11	13	13	15	15	
$^1S_0$	-6.38 ± 1.54	-7.04 ± 1.63	-7.06 ± 1.69	-6.98 ± 1.66	-7.13 ± 1.69	-6.94 ± 1.63	-7.19 ± 1.69	-6.87 ± 1.63	-0.1
$^1D_2$	9.44 ± 0.58	8.93 ± 0.73	8.61 ± 0.77	8.73 ± 0.72	8.54 ± 0.77	8.98 ± 0.72	8.48 ± 0.77	8.98 ± 0.74	0.013
$^1G_4$	1.06 ± 0.28	1.39 ± 0.33	1.40 ± 0.36	1.48 ± 0.35	1.42 ± 0.36	1.41 ± 0.33	1.43 ± 0.36	1.45 ± 0.33	-0.0015
$^3P_0$	-11.13 ± 1.55	-11.47 ± 1.67	-10.95 ± 1.62	-10.89 ± 1.58	-11.06 ± 1.62	-11.34 ± 1.66	-11.18 ± 1.62	-11.86 ± 1.68	-0.095
$^3P_1$	-28.43 ± 1.25	-28.53 ± 1.27	-28.38 ± 1.30	-28.58 ± 1.27	-28.30 ± 1.30	-28.48 ± 1.27	-28.23 ± 1.30	-28.14 ± 1.27	-0.04
$^3P_2$	16.60 ± 0.67	16.23 ± 0.66	16.50 ± 0.69	16.50 ± 0.68	16.53 ± 0.68	16.37 ± 0.67	16.56 ± 0.68	16.43 ± 0.67	0
ϵ_2	-2.70 ± 0.43	-3.03 ± 0.49	-3.19 ± 0.53	-3.31 ± 0.48	-3.27 ± 0.54	-3.04 ± 0.50	-3.35 ± 0.54	-3.10 ± 0.53	0
$^3F_2$	1.11 ± 0.69	0.87 ± 0.68	1.05 ± 0.73	0.80 ± 0.70	1.06 ± 0.72	0.88 ± 0.69	1.06 ± 0.72	1.35 ± 0.65	0.001
$^3F_3$	-3.04 ± 0.61	-3.07 ± 0.67	-3.10 ± 0.68	-2.87 ± 0.65	-3.09 ± 0.69	-3.00 ± 0.67	$-3.06+0.69$	-3.21 ± 0.69	-0.006
$^3F_4$	2.86 ± 0.37	2.70 ± 0.40	2.72 ± 0.45	2.60 ± 0.42	2.73 ± 0.44	2.67 ± 0.40	2.74 ± 0.44	2.90 ± 0.37	0.0045
ϵ_4	-1.09 ± 0.30	-0.90 ± 0.31	-0.79 ± 0.34	-0.93 ± 0.32	-0.84 ± 0.33	-0.99 ± 0.32	-0.89 ± 0.33	-1.02 ± 0.32	0.001
$^3H_4$	0.79 ± 0.24	1.31 ± 0.39	1.59 ± 0.47	1.60 ± 0.39	1.59 ± 0.46	1.28 ± 0.39	1.57 ± 0.46	1.05 ± 0.41	0
$^3H_5$	(-1.22)[a]	-1.37 ± 0.47	-2.00 ± 0.62	-1.99 ± 0.47	-1.99 ± 0.62	01.46 ± 0.50	-1.96 ± 0.62	-1.33 ± 0.55	-0.004
$^3H_6$	(0.22)[a]	0.68 ± 0.28	0.83 ± 0.31	0.80 ± 0.28	0.78 ± 0.31	0.65 ± 0.28	0.73 ± 0.30	0.47 ± 0.28	0.001
$^1P_1$		-32.00 ± 2.25		-30.01 ± 3.19		-30.07 ± 3.19		-26.63 ± 3.93	-0.11
$^1F_3$		-3.33 ± 1.03		-3.86 ± 1.22		-3.82 ± 1.18		-4.28 ± 1.41	-0.036
$^3S_1$		2.99 ± 3.93		2.16 ± 5.39		-1.01 ± 5.19		-7.60 ± 5.50	-0.1
ϵ_1		19.39 ± 3.74		22.54 ± 4.32		21.67 ± 4.29		26.29 ± 4.27	0.035
$^3D_1$		-25.72 ± 2.79		-23.85 ± 3.72		-23.19 ± 3.16		-20.45 ± 2.66	-0.07
$^3D_2$		22.15 ± 2.34		19.20 ± 2.49		22.01 ± 2.32		20.62 ± 3.28	-0.02
$^3D_3$		2.72 ± 1.41		3.50 ± 1.54		3.08 ± 1.41		1.78 ± 1.43	0.04
ϵ_3		8.60 ± 0.88		9.71 ± 1.07		8.12 ± 0.95		5.80 ± 1.01	0
$^3G_3$		(-3.30)[a]		-4.73 ± 1.74		-5.28 ± 1.86		-7.45 ± 1.83	-0.01
$^3G_4$		(6.60)[a]							
$^1H_5$		(-1.59)[a]							

[a] OPEC values.

per se are a necessary but not a sufficient criterion for an accurate determination of the scattering amplitudes.

VII. SELECTION OF FREE PHASES AND DETERMINATION OF g^2

In the phase-shift analysis publications by the Penn State group[4–7] it was pointed out that the value of the pion-nucleon coupling constant g^2 as determined from the modified analysis depends strongly on the particular selection of phenomenological and of OPEC phases used to represent the high-angular-momentum partial waves. This led them to somewhat pessimistic conclusions about the possibility of extracting a reliable value for g^2 from the nucleon-nucleon data by a modified phase-shift analysis.[6] However, there is a way in which the high phase shifts can be selected that does give quite consistent values for g^2. The considerations which we use to draw this conclusion also show that the particular ordering of phase shifts used by the Penn State group[4–7] can give misleading results.

The difficulties in determining g^2 from the modified phase-shift analysis can be ascertained from Table VIII, which lists the phase-shift deviations from OPEC, defined as $(\delta-\delta_{\text{OPEC}})/\delta_{\text{OPEC}}$. The phase-shift values were obtained mostly from the present combined analysis results. The H waves are not reliably given from the existing data, but are included to illustrate the qualitative features of the deviations from OPEC. From Table VIII we see that the deviations from OPEC, which we may denote as two-pion and three-pion effects (TPEC), persist even through H waves in sizable percentages.[38] If we want to eliminate TPEC contributions, we would have to treat H waves and even higher l values as free parameters. However, the accuracy and completeness of the existing nucleon-nucleon data do not permit this many degrees of freedom. Table IX lists the minimum number of phase shifts required at each energy to minimize χ^2. If additional phases are added beyond the numbers shown in Table IX, the high phases become indeterminate, and the value for g^2 acquires a large statistical uncertainty.

From Tables VIII and IX, it is apparent that a "modified analysis" performed on the existing nucleon-nucleon data will necessarily have sizable TPEC effects present in the higher-phase shifts that are represented by OPEC. Hence to minimize the TPEC effects, the free phase shifts should be selected so that TPEC effects in the higher waves will tend to cancel out. If we consider (p,p) phase shift analyses, Table VIII shows that this cancellation will tend to occur if we select the triplet phase shifts in the order listed. This ordering according to the angular momentum l is what we would naturally choose from range arguments and the dominance of OPEC at large impact parameters. The Penn State group, however, have used a different criterion in selecting phase shifts for some of their analyses.[4–7] They select first the phase shifts that have the greatest effect in minimizing χ^2. These are the phase shifts that are removed from OPEC values by the largest number of standard deviations. However, as shown in Table VIII, the nature of the TPEC contribution to triplet (p,p) phases is to have a very large constructive TPEC–OPEC effect for the $l=J-1$ phases, and much smaller destructive TPEC–OPEC effects for the $l=J$ and $l=J+1$ phases and for the coupled phase. The Penn State selection procedure chooses these large constructive amplitudes first, as illustrated in Table X. Thus the remaining phases, which are to be represented by OPEC, have predominantly a desctructive TPEC contribution. Hence a $\chi^2(g^2)$ parabola will have a minimum at a low value for g^2, reflecting the destructive TPEC interference.

TABLE VIII. Phase-shift deviations from OPEC values. The deviation is defined as $(\delta-\delta_{\text{OPEC}})/\delta_{\text{OPEC}}$ in percent.

	Deviation, %, at energy of:			
	95 MeV	142 MeV	210 MeV	310 MeV
$^1D_2$	+140	+181	+242	+312
$^1G_4$		+16	+52	+70
$^3P_1$	−26	−18	−15	−5
$^3P_2$	+430	+384	+307	+224
ϵ_2	−1	−24	−43	−48
$^3F_2$	−23	−41	−18	−66
$^3F_3$		−10	−21	−31
$^3F_4$		+163	+300	+214
ϵ_4		−16	−16	−34
$^3H_4$		−11	+30	+146
$^3H_5$		−4	−24	+20
$^3H_6$			+192	+195
$^1P_1$	+25	+17	+90	+167
$^1F_3$		−67	+87	−3
$^3D_1$	+220	+310	+638	+1000
$^3D_2$	+28	+41	+10	+20
$^3D_3$	−169	−133	−175	−154
ϵ_3		−47	+6	−2

There is one other way that the TPEC effects can be averaged out—by using a combined (p,p) plus (n,p) analysis. This doubles the number of partial-wave states and should produce better TPEC cancellations.

In Table XI are summarized the results of g^2 determinations. The combined (p,p) plus (n,p) analyses give more consistent results than do the (p,p) analyses. The fact that both the (p,p) analyses and combined analyses give "low" values for g^2 (we expect a value of about 15) probably indicates that some destructive (on the average) TPEC effects are present in the high phases that we represent by OPEC. These TPEC effects

TABLE IX. Approximate number of free phase shifts required to minimize the least-squares sum χ^2.

	95 MeV	142 MeV	210 MeV	310 MeV
$T=1$	7	13	14	14
$T=0$	6	10	9	9

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[38] As an example of a calculation of one-pion and two-pion effects, see G. Breit, Rev. Mod. Phys. 34, 766 (1962).

TABLE X. Selection of phase shifts in the Penn State (p,p) analysis[7] at 213 MeV.

	Standard deviations from OPEC	Order of selection	Sign of TPEC relative to OPEC
$^1D_2$	16.5[a]	5	+
$^1G_4$	2.4	9	+
$^3P_1$	11.0		−
$^3P_2$	41.0		+
ϵ_2	14.5	7	−
$^3F_2$	1.9	12	−
$^3F_3$	5.6	8	−
$^3F_4$	7.9	6	+
ϵ_4	3.0	11	−
$^3H_4$	0.8	17	−
$^3H_5$	1.8	13	−
$^3H_6$	2.1	10	+

[a] These values are from the SYM 13 solution[7] except for $^3H_4$, which is from SYM 19.

cannot be eliminated from modified-analysis g^2 determinations until the experimental data are improved in both completeness and accuracy. At the bottom of Table XI, we have listed preliminary values from recent Yale energy-dependent analyses.[39] The agreement between these values and a suitable "average" of our values is good.

It is of some interest to note that the experimental phase-shift deviations from OPEC can be qualitatively accounted for by the addition of ABC and ρ Born-term resonance contributions, as shown in Table XII. In particular, the large deviations for $^3P_2$, $^3F_4$, and $^3H_6$ can be accounted for by the ABC resonance.

TABLE XI. Values for the π-n coupling constant.

Energy (MeV)	Free phases $T=1$	$T=0$	(p,p) analysis	$(p,p)+(n,p)$ analysis
95	7		13.6±12.7	
95	7	6		12.7±1.3
95	8	6		12.2±1.5
142	6		15.6±0.7	
142	9		10.0±1.3	
142	11		12.8±2.5	
142	13		12.4±4.1	
142	14		11.3±4.1	
142	11	8		12.8±2.2
142	11	10		14.0±2.3
142	13	10		14.8±2.8
210	12		14.5±2.7	
210	14		17.5±4.3	
210	12	8		12.9±1.5
210	14	8		12.7±1.9
210	14	9		12.3±1.8
310	12		19.0±3.8	
310	14		12.6±4.4	
310	12	8		13.3±0.6
310	14	8		13.5±0.8
310	14	9		13.7±0.8
10–345[a]			14.6±0.4	
14–350[a]				13.9±0.2

[a] These values are preliminary results from recent Yale energy-dependent analyses and were sent to us by Professor Breit (private communication).

VIII. CHARGE INDEPENDENCE

The gross features of charge independence were substantiated in two ways. First, comparisons of $T=1$ phases as determined from (p,p) data with the $T=1$ phases as determined from combined $[(p,p)$ plus $(n,p)]$ data (see Tables II, III, V, VII) show differences which are generally well within the statistical error on those phases. Second, determinations of the pion-nucleon coupling constant g^2 (see Sec. VII) were essentially the same whether they were achieved with (p,p) or combined $[(p,p)$ plus $(n,p)]$ data (see Table XI). The Yale group have studied this problem in considerable detail. An excellent review of the subject is given in the article listed in Ref. 38.

TABLE XII. Phase-shift deviations from OPEC at 213 MeV.

	Experimental deviation, %[a]	Deviation, %, produced by ABC[b]	ABC+ρ[b]
$^1D_2$	+242	+240	+290
$^1G_4$	+52	+41	+45
$^3P_1$	−15	−75	−48
$^3P_2$	+307	+590	+400
ϵ_2	−43	0	−31
$^3F_2$	−18	+52	−1
$^3F_3$	−21	−36	−31
$^3F_4$	+300	+210	+185
ϵ_4	−16	0	−3
$^3H_4$	+30	+18	+11
$^3H_5$	−24	−8	−8
$^3H_6$	+192	+60	+57
$^1P_1$	+90	−186	+390
$^1F_3$	+87	−33	−37
$^3D_1$	+638	−265	+780
$^3D_2$	+10	+21	+34
$^3D_3$	−175	+130	−104
ϵ_3	+6	0	+3

[a] Values from Table VIII.
[b] The ABC Born term has mass 437 MeV and coupling constant 3.05. The ρ has mass 591 MeV and coupling constants 1.13 and 3.38. These are the Scott-Wong values.

IX. COMPARISON WITH OTHER WORKERS

Figures 1 and 2 show, graphically, a comparison of our results with those of other recent phase-shift analyses. The energy-independent (p,p) analyses at Livermore (present paper) and the Pennsylvania State analyses[4–7,40] give essentially identical results. The Dubna results[41] are similar. These results are also in general agreement with the Yale[42] and Livermore[43]

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[39] We would like to thank Professor Breit for sending us these values and for several useful comments. These results were quoted in a paper by the Yale group that was presented at the Dubna High Energy Conference in the summer of 1964.

[40] P. Signell, Phys. Rev. **133**, B982 (1964). The phase shift values shown here were taken from the graphs presented in this reference.

[41] Y. M. Kazarinov and I. N. Silin, Zh. Eksperim. i Teor. Fiz. **43**, 1385 (1962) [English transl.: Soviet Phys.—JETP **16**, 983 (1963).

[42] G. Breit, M. H. Hull, Jr., K. E. Lassila, K. D. Pyatt, Jr., and H. M. Ruppel, Phys. Rev. **128**, 826 (1962).

[43] M. J. Moravcsik, H. P. Noyes, and H. P. Stapp (to be published).

energy-dependent (p,p) analyses. Hence the $T=1$ scattering matrix has been reliably determined in the elastic scattering region. Some difficulties still remain, but resolution of these will require more complete and more accurate nucleon-nucleon data, not merely more phase-shift analyses.

The $T=0$ phase shifts from the present analysis are in reasonable agreement with the Yale energy-dependent $T=0$ phase shifts[44] and with the 140-MeV values of Perring.[45] The agreement with the Dubna values[41] is not quite as good. We consider that the $T=0$ elastic scattering matrix has been qualitatively determined, although not as accurately as the $T=1$ matrix. Again, better values for the phase shifts will require better (n,p) nucleon-nucleon data before the analyses can be pushed much farther than shown here.

The Yale group have obtained results that are more recent than those shown here.[46] The differences from the solutions shown in Figs. 1 and 2 are in general not large.

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ADDENDUM

After the completion of this paper, we obtained new (p,p) R and R' measurements at 98 MeV.[47] To investigate the effect of these data on our solutions, we ran a problem with these data added to the data shown in Table I, and using nine free $T=1$ and six free $T=0$ phases. The solution is listed in the last column of Table II. It presumably represents our best solution at 95 MeV. The main effect of these data on the $T=1$ phases is to lower the value for $^3P_0$ and bring it more into line with the values obtained at neighboring energies. The $T=0$ phases are all altered by amounts that are comparable to the errors quoted for the phase shifts. This points up the desirability of having complete data selections at each energy. The coupling constant value obtained with this expanded data selection near 95 MeV was $g^2=12.2\pm1.5$.

Note added in proof. The data of Refs. 8 and 13 were taken from Wilson's book.[37] The errors from Ref. 8 should be increased somewhat, and newer data are available to replace Ref. 13. These changes, along with others, have been incorporated in paper IV of this series. These changes have only a slight effect on the results of the analyses. Paper IV (to be submitted to Phys. Rev.) includes energy-dependent and energy-independent analyses of the data included in I–III.

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[44] M. H. Hull, Jr., K. E. Lassila, H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. **128**, 830 (1962).

[45] J. K. Perring, Nucl. Phys. **42**, 306 (1963).

[46] Some of these results were included in a rapporteur's talk by D. Amati at the International Congress of Nuclear Physics in Paris, July 1964, and also at the Dubna Conference (Ref. 39).

[47] O. N. Jarvis, B. Rose, G. F. Cox, and G. H. Eaton, Harwell Report No. AERE–NP/GEN 37, 1964 (unpublished).