Dynamics of Cylindrical Electromagnetic Universes

M. A. MELVIN*
Florida State University, Tallahassee, Florida

(Received 7 December 1964; revised manuscript received 1 March 1965)

In a previous publication the writer presented a rigorous solution of the Einstein-Maxwell equations which corresponds to a configuration of parallel magnetic lines of force in equilibrium under their mutual gravitational attraction. This "magnetic geon"—which it is appropriate to rename a "magnetic universe" because the magnetic-field energy falls off so slowly with distance—is unstable according to an elementary Newtonian analysis but is *stable* according to the analysis reported here. In this connection the general time-dependent equations for electromagnetic-gravitational fields in the case of cylindrical symmetry are discussed in detail and certain conclusions drawn. Then the solutions as functions of radius and time are found for the two gravitational potential fields and the electromagnetic field which appear when a magnetic universe is subjected to a radial perturbation. It is shown that the magnetic universe is stable (oscillatory) under such perturbations, i.e., all admissible frequencies are real. The converse is also true: All real frequencies are admissible. Every solution is a superposition of proper modes of which there are two types, a "g-type" and an "h-type" for each and every real frequency, and for each of the physically interesting fields, i.e., for the electromagnetic field and for the two gravitational fields ("Newtonian potential" and "C energy," respectively). The g-type modes may be identified roughly as "dominantly gravitational" since in them gravitational perturbations dominate electromagnetic both at small and large distances; the h-type may be identified as "locally electromagnetic" since in these modes, near the axis, the electric and magnetic perturbations dominate the gravitational perturbations. The radial dependence of each of the mode types for the various fields is expressed linearly, with radius-dependent coefficients, in terms of Bessel coefficients of order zero and one, respectively. The manner in which an initial perturbation is shaken off is analyzed, and causality is verified. The relevance of the analysis to the understanding of gravitational collapse is discussed. Pure magnetic (or electric) fields are the only presently known systems which resist gravitational collapse. It is suggested that extended magnetic fields may play a role in retarding and finally halting gravitational collapse of material systems.

1. INTRODUCTION AND SUMMARY

A. Motivation and Outcome

GRAVITATIONAL collapse is a physical phenomenon of the very greatest interest at the present time.[1] The dynamics of collapse in the case of a star presents so many issues that one asks for a simpler model on which to begin the physical analysis. Such a model is supplied by a geon,[2] a configuration which contains nothing but electromagnetic field energy (no particles) and which is held together by its own gravitational attraction. Recently[3] it occurred to the present writer to try the simpler case of a static pure magnetic (or electric) field, specifically a collection of parallel magnetic lines of force held together by mutual gravitation. In addition to its greater simplicity, this configuration has the merit over previously studied geons, that it is a rigorous equilibrium solution of the Einstein-Maxwell equations and not merely a statistical equilibrium. Is this equilibrium unstable? If so, then a small perturbation away from equilibrium in the sense of a more concentrated configuration would provide a particularly simple starting point for the analysis of gravitational collapse. The present report analyzes the character of the equilibrium. It turns out that this particular type of magnetic field configuration is *stable*, contrary to the first expectation, and contrary to the behavior of simple spherical geons, which Wheeler has reasoned[1(c)]—and Brill and Hartle have proved[4]—to be unstable against collapse.

The results of the somewhat difficult perturbation analysis can themselves be summarized rather simply (Tables I and II, and Figs. 1 and 2) and interpreted rather generally: Any system which is described by a set of linear differential equations with coefficients which, though variable in position, do not contain the time may be called "a system of constant structure." One can expect quite generally that such a system on account of its time-translation symmetry exhibits normal modes. Our magnetic universe when perturbed to the first order away from equilibrium is of this type. The

* Dedicated to Oskar Klein on his seventieth birthday. This work was performed at Oak Ridge National Laboratory and in part at Los Alamos Scientific Laboratory and Argonne National Laboratory.

[1] For a summary see: (a) *Proceedings of the December 1963 Dallas International Conference on Gravitational Collapse and Relativistic Astrophysics*, edited by I. Robinson and E. Schücking (University of Chicago Press, Chicago, 1964), Vol. I; (b) *Gravitation Theory and Gravitational Collapse*, B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler (University of Chicago Press, Chicago, 1965), Vol. II; (c) J. A. Wheeler, *Relativity, Groups and Topology*, edited by C. and B. DeWitt (Gordon and Breach, Publishers, New York, 1964).

[2] J. A. Wheeler, Phys. Rev. **97**, 511 (1955); see also F. J. Ernst, *ibid.* **105**, 1665 (1957).

[3] M. A. Melvin, Phys. Letters **8**, 65 (1964). I am indebted to B. K. Harrison for discovering, subsequent to this publication, that the solution had been obtained earlier though not explicitly. It is contained implicitly as a special case among the solutions given by M. Misra and L. Radhakrishna, Proc. Natl. Inst. Sci. India **28A**, 632 (1962). [*Note added in proof*. In the meantime, Professor W. B. Bonnor has kindly informed me that the static solution is contained in his earlier work. It is indicated on p. 230 of his paper Proc. Phys. Soc. **67A**, 225 (1953).]

[4] D. R. Brill and J. B. Hartle, Phys. Rev. **135**, B271 (1964); Bull. Am. Phys. Soc. **9**, 425 (1964).

TABLE I. Normal modes of electromagnetic waves appearing when a static magnetic universe is perturbed away from equilibrium (first-order theory). The electric vector e_{physical} is in the ϕ direction, azimuthal to the axis. The magnetic flux vector b_{physical} is in the ζ direction, parallel to the axis. The electric vector is a quarter period out of phase in time with respect to the magnetic vector. The amplitudes are expressed in units of $\frac{1}{2}B_0$ where B_0 is the flux on the axis in the static magnetic universe. The h-type modes may be identified as "locally electromagnetic" since in these, at small values of ρ, the electric and magnetic perturbations e_{physical} and b_{physical} dominate the gravitational potential perturbations $\delta\psi$ and $\delta\gamma$.

	e_{physical}	b_{physical}
g-type normal mode	$\omega^{3/2}\dfrac{\rho}{(1+\rho^2)^2}J_0(\omega\rho)\sin\omega\tau$	$-\dfrac{\omega^{1/2}}{(1+\rho^2)^2}\left\{\dfrac{4\rho^2}{1+\rho^2}J_0(\omega\rho)+\omega\rho J_1(\omega\rho)\right\}\cos\omega\tau$
h-type normal mode	$\omega^{3/2}\dfrac{1-\rho^2}{(1+\rho^2)^2}J_1(\omega\rho)\sin\omega\tau$	$\dfrac{\omega^{1/2}}{(1+\rho^2)^2}\left\{\omega(1-\rho^2)J_0(\omega\rho)-\dfrac{2\rho(3-\rho^2)}{1+\rho^2}J_1(\omega\rho)\right\}\cos\omega\tau$

dependent variables which describe the space-time structure are two in number: the "Newtonian gravitational potential" ψ, and the "Weyl-Einstein-Rosen gravitational potential" γ, which has recently received an added interpretation by Thorne[5] as an energy-like quantity ("C energy") characteristic of cylindrically symmetric systems. The remaining two dependent variables are the longitudinal magnetic field B_{physical} and the azimuthal electric field E_{physical}; though these two are derivable from a single-component vector potential, it is physically much more instructive to study their development individually.

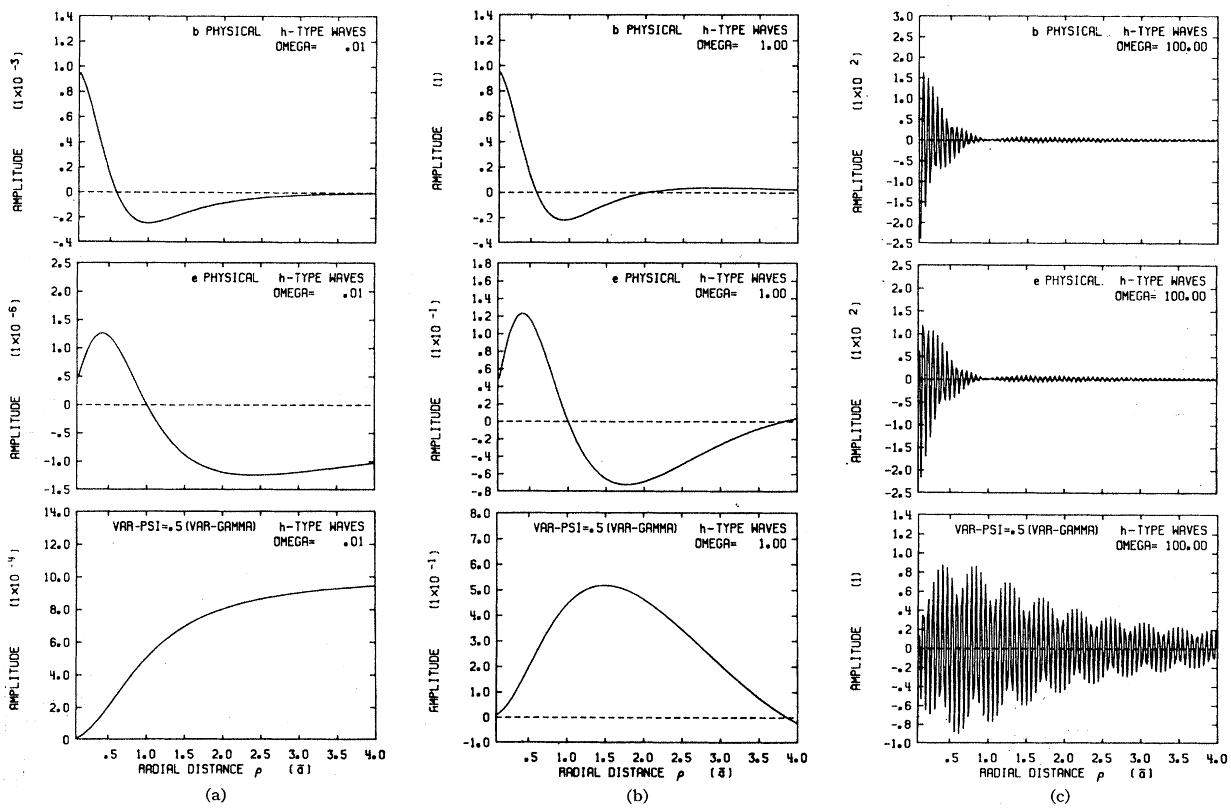

FIG. 1. Normal modes of h type in a cylindrical electromagnetic universe slightly perturbed away from the equilibrium configuration of a purely magnetic universe. The elementary standing wave of h type has a radial dependence given by a Bessel coefficient of order one. Represented here and in Fig. 2 are the proper modes of the electromagnetic field perturbations b_{phys} and e_{phys}, and the gravitational potential perturbations $\delta\gamma$ and $\delta\psi$ all expressed in terms of linear combinations (with coefficients depending on the radial variable) of the elementary g and h waves (see Tables I and II). At very low frequencies, as one would expect, the electric field has a very low amplitude relative to the magnetic field, whereas at high frequencies, as Table I also shows, they approach exact equality. It will be noted that the range radius $\rho=1$ is a universal node for e_{phys} in h-type modes (and for $\delta\psi$ in g-type modes). This "clamped membrane" type of behavior at $\rho=1$, in the two cases, has not as yet received any intuitive explanation.

[5] K. S. Thorne, Phys. Rev. **138**, B251 (1965).

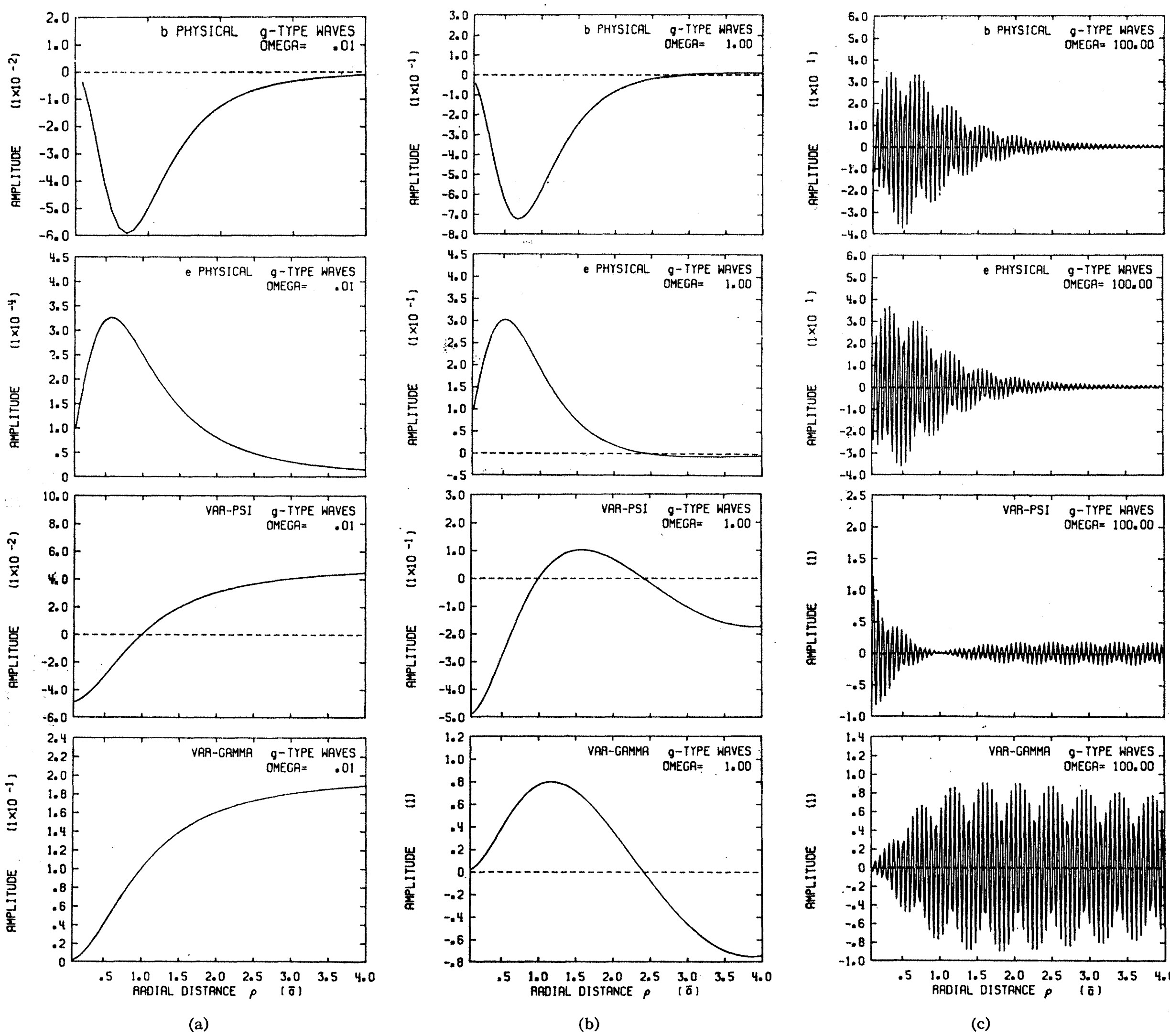

FIG. 2. Normal modes of g type in a cylindrical electromagnetic universe slightly perturbed away from the equilibrium configuration of a purely magnetic universe. The elementary standing wave of g type has a radial dependence given by a Bessel coefficient of order zero. Represented here and in Fig. 1 are the proper modes of the electromagnetic-field perturbations b_{phys} and e_{phys}, and of the gravitational potential perturbations $\delta\gamma$ and $\delta\psi$ all expressed in terms of linear combinations (with coefficients depending on the radial variable) of the elementary g and h waves (see Tables I and II). At very low frequencies, as one would expect, the electric field has a very low amplitude relative to the magnetic field, whereas at high frequencies, as Table I also shows, they approach exact equality. It will be noted that the range radius $\rho=1$ is a universal node for $\delta\psi$ in g-type modes and e_{phys} in h-type modes. This "clamped membrane" type of behavior at $\rho=1$, in the two cases, has not as yet received any intuitive explanation.

The general expectation that there exist normal modes of all these fields is verified. The small perturbation analysis shows: (1) that every first-order cylindrically symmetric departure from the equilibrium configuration is representable as a superposition of proper modes; (2) that every mode is associated with its own characteristic real circular frequency ω (stability) rather than with a complex frequency with real part representing an exponential growth or decay constant (instability). All real values of the frequency are admissible and for every value of the frequency there are two types of modes, "g type" and "h type." These are named after the two new gravitational potentials to which it was necessary to transform from the canonical potentials γ and ψ in order to obtain the solutions; g and h are linear combinations (with radius-dependent coefficients) of the gravitational potential perturbations $\delta\gamma$ and $\delta\psi$. The elementary g and h waves have a radial dependence given by Bessel coefficients of order zero and one, respectively. The proper modes of the electromagnetic field perturbations b_{physical} and e_{physical} and of $\delta\gamma$ and $\delta\psi$ are expressed linearly, with coefficients depending on the radial variable ρ in terms of the elementary g and h waves (Tables I and II). The

TABLE II. Normal modes for gravitational waves appearing when a static magnetic universe is perturbed away from equilibrium (first-order theory). The Newtonian potential "waves" $\delta\psi$ and the "C-energy" waves $\delta\gamma$ are in phase in time. In the h-type modes $\delta\gamma$ everywhere bears the same ratio to $\delta\psi$ that γ bears to ψ in the static magnetic universe (2 to 1). In the g-type modes, though it starts with zero magnitude at $\rho=0$, $\delta\gamma$ soon approaches a steady ratio of 4 to 1 to $\delta\psi$. The g-type modes or standing waves may be identified as "dominantly gravitational" since in these modes, both at small and large values of the radial coordinate ρ, the "Newtonian gravitational potential" perturbation $\delta\psi$ dominates the electric and magnetic field perturbations e_{physical} and b_{physical}.

	$\delta\psi$ or VAR–PSI (Newtonian potential)	$\delta\gamma$ or VAR–GAMMA (C energy)
g-type normal mode	$-\frac{1}{2}\omega^{1/2}\dfrac{1-\rho^2}{1+\rho^2}J_0(\omega\rho)\cos\omega\tau$	$\omega^{1/2}\dfrac{2\rho^2}{1+\rho^2}J_0(\omega\rho)\cos\omega\tau$
h-type normal mode	$\omega^{1/2}\dfrac{2\rho}{1+\rho^2}J_1(\omega\rho)\cos\omega\tau$	$2\omega^{1/2}\dfrac{2\rho}{1+\rho^2}J_1(\omega\rho)\cos\omega\tau$

graphical representation of these proper modes (Figs. 1 and 2) shows a number of interesting features: At very low frequencies, as one would expect, the electric field has a very low amplitude relative to the magnetic field whereas at high frequencies, as Table I also shows, they approach exact equality.

As it became clear that the static magnetic universe was stable against radial perturbations the following discussion was contributed by Professor John A. Wheeler:

"It made sense to analyze the stability of the system when it appeared to be a geon. Amongst geons, it would have been hard to think of a simpler case where one might have tested in all mathematical detail that general instability of all geons against collective gravitational collapse which is argued by Wheeler on physical grounds.[1(c)] But the present system turns out not to be a magnetic geon; it is a magnetic universe. Moreover, it is a universe of a most implausible geometry. Then what can one possibly learn about live physical issues by analyzing—as here—the small perturbations of this system?

"Thorne has pointed out[1(a)] that the magnetic universe is directly relevant to geon physics and to the theory of gravitational collapse. Envisage in asymptotically flat (Schwarzschildean) space a toroidal region of minor radius a very small in comparison with its major radius b. Let magnetic lines of force go around in the direction of the major circumference so that one really *does* have a magnetic geon. Moreover, let the bundle of magnetic lines of force be so dense that this toroidal region curves itself up (geometry of the cross section $\sim\pi a^2$) into almost complete closure; that is, into almost complete isolation from the surrounding asymptotically flat space. In approaching this limit and the limit of very large b/a it is reasonable—Thorne points out—*to compare the active region of the magnetic geon with an almost closed universe*. Thus the present analysis, which refers to the limiting case itself, with complete closure (at least in the sense that $b/a\to\infty$), can be expected to serve as one natural *starting point for analyzing gravitational collapse of a toroidal magnetic geon*."

TABLE III. Asymptotic expressions for the magnetic and electric perturbation fields b_{physical} and e_{physical}. It is assumed that the time τ is very great compared with: (1) the radial distance ρ to the field point; (2) a bounding radial distance ρ'_{bound} which may be estimated as follows: Let $\tilde{G}$, $\tilde{G}_\tau$ be the first moments of the initial distributions $g(\rho',0)$, $g_\tau(\rho',0)$, and let $\tilde{H}$, $\tilde{H}_\tau$ be the second moments of the initial distributions $h(\rho',0)$, $h_\tau(\rho',0)$. Then ρ'_{bound} is the greatest lower bound for distances beyond which the contributions of the initial distributions to $\tilde{G}$, $\tilde{G}_\tau$, $\tilde{H}$, $\tilde{H}_\tau$ are negligible.

	$(2/B_0)b_{\text{physical}}$	$(2/B_0)e_{\text{physical}}$
g-type running waves	$\dfrac{4\rho^2}{(1+\rho^2)^3}\left\{\dfrac{\tilde{G}}{\tau^2}-\dfrac{\tilde{G}_\tau}{\tau}\right\}$	$-\dfrac{\rho}{(1+\rho^2)^2}\left\{\dfrac{2}{\tau^3}\tilde{G}-\dfrac{1}{\tau^2}\tilde{G}_\tau\right\}$
h-type running waves	$\dfrac{1-3\rho^2}{(1+\rho^2)^3}\left\{\dfrac{3}{\tau^4}\tilde{H}-\dfrac{1}{\tau^3}\tilde{H}_\tau\right\}$	$-\dfrac{1-\rho^2}{(1+\rho^2)^2}\dfrac{3\rho}{2}\left\{-\dfrac{4\tilde{H}}{\tau^5}+\dfrac{\tilde{H}_\tau}{\tau^4}\right\}$

Aside from these considerations which refer to the value of the cylindrical-toroidal model as an idealized system whereby one might come to a better understanding of the *theory* of gravitational collapse, there are also the following considerations to give the present investigation considerable physical interest for the *phenomenon* of gravitational collapse as it occurs in the quasistellar sources.

Many strong radio sources take the form of two emitting regions situated on opposite sides of a galaxy. The most popular theory to account for this is that the magnetic fields and high-energy particles responsible for the synchrotron radiation were blown out of the galactic nucleus in a giant explosion. Such an explosion would have to be highly directional in order to explain the observations. After the present investigation had been carried out Dicke remarked that the explosions in actual quasistellar sources could result from the gravitational collapse of a very prolate spheroid or a cylinder whose axis of symmetry is defined by a strong magnetic field. Collapse perpendicular to the axis of symmetry would occur catastrophically, with a consequent ejection of material out of the ends. An important question to ask is: What effects would general relativity have on such a model? This question is most easily answered by studying "cylindrical model universes," which are idealizations of finite cylinders and which form the subject of the present paper and that of Thorne.[5] The results suggest that: (1) A strong magnetic field along the axis of symmetry may halt the cylindrical collapse of a finite cylinder before a singularity is reached. (2) Electromagnetic and gravitational waves will be profusely emitted by such a collapsing cylinder. So far as we know, these papers represent the first detailed study of a system which shows absolute stability against collapse to a singularity, and the first detailed treatment

TABLE IV. Asymptotic approximations for times of occurrence and magnitudes of extreme concentrations of the magnetic and electric perturbation fields b_{physical} and e_{physical} in a magnetic universe with superposed initial perturbations of pure g or h type. G, G_τ are the first moments of the initial perturbations $g(\rho',0)$, $g_\tau(\rho',0)$, and $\tilde{H}$, $\tilde{H}_\tau$ are the second moments of the initial perturbations $h(\rho',0)$, $h_\tau(\rho',0)$, respectively.

	Time τ for maximum of b_{physical}	Magnitude of maximum $b_{\text{physical}}\times(2/B_0)$	Magnitude of e_{physical} at maximum b_{physical}	Time τ for maximum of e_{physical}	Magnitude of maximum $e_{\text{physical}}\times(2/B_0)$	Magnitude of $b_{\text{physical}}\times(2/B_0)$ at maximum e_{physical}
g-type running waves	$\dfrac{2\tilde{G}}{\tilde{G}_\tau}$	$-\dfrac{4\rho^2}{(1+\rho^2)^2}\left(\dfrac{\tilde{G}_\tau}{2\tilde{G}}\right)^2\tilde{G}$	0	$\dfrac{3\tilde{G}}{\tilde{G}_\tau}$	$\dfrac{\rho}{(1+\rho^2)^2}\left(\dfrac{\tilde{G}_\tau}{3\tilde{G}}\right)^3\tilde{G}$	$-\dfrac{8\rho^2}{(1+\rho^2)^3}\left(\dfrac{\tilde{G}_\tau}{3\tilde{G}}\right)^2\tilde{G}$
h-type running waves	$\dfrac{4\tilde{H}}{\tilde{H}_\tau}$	$-\dfrac{1-3\rho^2}{(1+\rho^2)^3}\left(\dfrac{\tilde{H}_\tau}{4\tilde{H}}\right)^4\tilde{H}$	0	$\dfrac{5\tilde{H}}{\tilde{H}_\tau}$	$-\dfrac{3\rho}{2}\dfrac{1-\rho^2}{(1+\rho^2)^2}\left(\dfrac{\tilde{H}_\tau}{5\tilde{H}}\right)^5\tilde{H}$	$-\dfrac{2(1-3\rho^2)}{(1+\rho^2)^3}\left(\dfrac{\tilde{H}_\tau}{5\tilde{H}}\right)^4\tilde{H}$

of coupled electromagnetic and gravitational radiation in a situation relevant to quasistellar sources.

Specifically, consider the magnetic and electric perturbations b_{physical} and e_{physical} of a static magnetic universe as analyzed in this paper (Table III). Asymptotically, at a given point, for times large compared with the time required by light to travel to the given point from regions where the initial perturbation was appreciable, b_{physical} varies with time like $f_1/t^n - f_2/t^{n-1}$ and e_{physical} like $f_3/t^{n+1} - f_4/t^n$, where $n=2$ for g-type perturbations and $n=4$ for h-type perturbations. f_1, f_2, f_3, and f_4 are functions only of the radial coordinate ρ and the first and second moments of the distribution of the initial perturbation and its time derivative.

From these asymptotic expressions one can calculate the times of maximum concentration or "maximum collapse" (Table IV). This is in every case of the order of the ratio of the (first or second) moment of the initial perturbation to the corresponding moment of its time derivative. These times of maximum gravitational collapse are given in time units of the static magnetic universe [see Sec. 1.C]

$$\bar{a}/c = 2.32\times10^{14}(\text{G}/B_0)\ \text{sec}.$$

For an axis magnetic flux $B_0 \sim 10^5$ G, equal to the polar values observed in the most magnetic stars, $\bar{a}/c$ is about 75 years.

B. Einstein's Equations and Self-Coupling

Einstein's theory of the gravitational field associated with the presence of matter may be described as the theory of a 2-index tensor field g_{ij} with a remarkable geometric interpretation and with peculiar self-coupling properties. Formally, the theory comprises two laws, or sets of equations, which correspond respectively to the two laws characteristic of any field theory: (1) the field-generated-by-source law; (2) the force-generated-by-field-acting-on-source law. The physical content of these fundamental laws in the case of gravitation can be stated alternatively in Newtonian dynamical (Einsteinian geometrical) language as follows:

(1) There is a gravitational potential (space-time metric) g_{ij} whose space-time variation, as represented by the generalized d'Alembertian (Einstein tensor density) $\mathcal{G}_{ij}$ of g_{ij}, has as its source the energy-stress density $\mathcal{T}_{ij}$ associated with the presence of matter:

$$\mathcal{G}_j{}^i = -\kappa\mathcal{T}_j{}^i \quad (\kappa\equiv 8\pi G/c^4). \tag{1}$$

(2) The quantity[6] $\mathcal{T}_{j,}{}^u{}_u$ [which measures the excess: j-directed net stress force, generalized buoyancy, acting on a unit volume minus rate of increase of j momentum] is balanced by the gravitational pull (varying metric) $g_{uv,j}$ acting on the energy-stress density $\mathcal{T}^{uv}$:

$$\mathcal{T}_{j,}{}^u{}_u = \tfrac{1}{2}g_{uv,j}\mathcal{T}^{uv} = \tfrac{1}{2}g^{uw}g_{uv,j}\mathcal{T}_w{}^v. \tag{2}$$

The remarkable geometric interpretation, indicated everywhere in statements (1) and (2) by the expressions in parentheses, has the beautiful merit that Eq. (2) is an automatic consequence (geometric identity) following from Eq. (1).

In cases where the symmetry is high enough to permit an everywhere diagonal g tensor we set

$$g_{uu}\equiv\exp(2f_u), \quad \mathcal{T}_u{}^u\equiv\mathcal{T}^u \quad \text{(no summation)} \tag{3}$$

and Eq. (2) simplifies to

$$\mathcal{T}_{j,}{}^u{}_u = f_{u,j}\mathcal{T}^u, \tag{2a}$$

in which the resemblance to the Lorentz force equation in the case of the electromagnetic (vector) field is striking. It will be noted that, in the diagonal g case, the diagonal components alone of the energy-stress tensor play the role of gravitational charge-currents acted on ponderomotively by the force field.

The peculiar self-coupling property of the g field is apparent in that the energy-stress tensor of the material medium is the source of the gravitational field and is acted on by it. [$\mathcal{T}_j{}^u$ appears not only, as in other field theories, on the left side of Eq. (2) but also on the right sides of Eqs. (1) and (2).] It is the self-coupling property

[6] Differentiation with respect to a variable is labeled by a subscript comma followed by the appropriate letter or number index. Occasionally in the following, where economy and no confusion result, the comma will be omitted.

which is ultimately responsible for *gravitational collapse* —i.e., growth of $\mathcal{T}_j^u$ towards singular values under certain conditions—and, under other conditions, for the existence of *self-sustaining field structures*.

Wheeler[2] has proposed the investigation of such possible self-sustaining structures of sourceless electromagnetic fields contained by the curvature of space-time associated with their own energy density—in classical language we would say: "held together by their gravitational pull on themselves." With extraordinary intuitive power, Wheeler and his associates have made many interesting approximate calculations on such "geons" of various possible symmetries. By now there can hardly be any doubt of their potential existence—not as equilibrium but as near-equilibrium solutions of the combined Einstein-Maxwell system of equations.

C. Structure of the Static Magnetic Universe

The geon investigations refer only to dynamic (wave) structures; at any instant these do not correspond to rigorous solutions of the time-independent Einstein-Maxwell equations, but certain averages over the structures do. One might call this description on the average the "photon cloud picture" and say that *Wheeler's geons are in equilibrium in the photon cloud picture.*

Recently[3] it occurred to the present writer to analyze the simpler case of a static magnetic (or electric) field. There was found a rigorous *equilibrium* solution of the combined Einstein-Maxwell system without charges and currents. Actually, the solution is a specialization of a slightly more general solution obtained in Ref. 3, the general solution representing a *parallel cylindrically symmetric bundle of Faraday flux*[7] *held together by its own gravitational pull.* The gravitational description of the magnetic flux system depends on Eqs. (1) and (2a) which now reduce to $[\frac{1}{2}\ln g_{44}\equiv\psi]$:

$$(r\psi_r)_r/r=\kappa\mathcal{T}/r \quad \text{(Poisson equation)}, \qquad (1')$$

$$-(\mathcal{T}/r)_r=\psi_r(2\mathcal{T}/r) \quad \text{(Hydrostatic equilibrium equation)}. \qquad (2')$$

Here subscripts represent differentiation and we have:

$$-\mathcal{T}_1{}^1=\mathcal{T}_2{}^2=-\mathcal{T}_3{}^3=\mathcal{T}_4{}^4\equiv T_4{}^4\sqrt{|g|}\equiv\mathcal{T}, \qquad (3')$$

$$\begin{Bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{Bmatrix} \equiv \text{coordinate for} \begin{Bmatrix} \text{radius} \\ \text{translation parallel to symmetry axis} \\ \text{angle about symmetry axis} \\ c\cdot\text{time} \end{Bmatrix} = \begin{Bmatrix} r \\ z \\ \phi \\ t \end{Bmatrix}.$$

[7] A Faraday flux may be defined as one satisfying the simple condition that the radial stress density (e.g., the pressure) is equal and opposite in sign to the longitudinal stress density (e.g., tension). Such a condition is essential to Weyl's method of simplifying cylindrically symmetric problems so that they involve two rather than three gravitational potentials. A static magnetic or electric field automatically satisfies this condition, a fact which does not seem to have been referred to or applied in general relativity prior to the work in Ref. 3.

The physical content of Eqs. (1′) and (2′) can be stated simply in Newtonian dynamical language:

(1) The Newtonian gravitational potential $c^2\psi$ is generated by the quantity $2\mathcal{T}/rc^2$ associated with the presence of the flux; $2\mathcal{T}/r$, which is essentially the energy density plus the azimuthal pressure, is thus equivalent to gravitational mass density. [Because of the Faraday-Maxwell balance of stresses in a pure static field [Eq. (3′)] the contributions of the radial pressure $\mathcal{T}_1{}^1/r$ and of the longitudinal tension $\mathcal{T}_2{}^2/r$ to the mass just cancel each other.]

(2) The flux is held together by the balance between its outwardly directed pressure and its inwardly directed gravitational pull; again in the latter $2\mathcal{T}/rc^2$ represents the gravitating mass density. The essentially unique solution of Eqs. (1′) and (2′), subject only to the condition that ψ be regular at $r=0$, was found to be

First gravitational potential ≡ (Newtonian potential)$/c^2\equiv\psi=\ln(1+\rho^2)$,

Second gravitational potential $\equiv\gamma\equiv 2\psi$,

Energy density $=\mathcal{T}/r=\frac{1}{2}B_0{}^2e^{-2\psi}=B_0{}^2/2(1+\rho^2)^2$.

Here the radial variable ρ equals $r/\bar{a}$ where the constant $\bar{a}$ is a length, the "range radius" of the flux structure. The constant B_0 is c times the square root of the gravitating mass density on the axis. When the structure is interpreted as a magnetic flux structure we have in rationalized units

energy density $\equiv\mathcal{T}/r=\frac{1}{2}HB_{\text{physical}}$,

contravariant magnetic field $\equiv H=B_0=$ constant everywhere,

physical magnetic field $\equiv B_0e^{-2\psi}\cong B_0$ in the vicinity of the axis $\cong B_0/\rho^4$ far away from the axis.

("Vicinity" here means a modest fraction of the "range radius" $\bar{a}$.)

The length $\bar{a}$ is related to B_0 (in gauss) inversely through the connection

$$\bar{a}=(1/B_0)2c^2/G^{1/2}=6.96\times10^{24}\ (\text{G}/B_0)\ \text{cm}$$

and the associated time is

$$\bar{a}/c=2.32\times10^{14}\ (\text{G}/B_0)\ \text{sec}.$$

For a magnetic flux $B_0\sim10^5$ G, equal to the polar values observed in the most magnetic stars, $\bar{a}$ is about one million times the diameter of the earth's orbit about the sun.

The static but curved geometry associated with the flux structure is

$$\begin{aligned}-&(\text{element of physical distance in units of } \bar{a})^2\\ &=+(\text{element of physical time in units of } \bar{a}/c)^2\\ &=d\sigma^2=(1+\rho^2)^2(d\tau^2-d\rho^2-d\zeta^2)-\rho^2(1+\rho^2)^{-2}d\phi^2, \qquad (4)\end{aligned}$$

where $\bar{a}\rho\equiv x^1$, $\bar{a}\zeta\equiv x^2$, $\phi\equiv x^3$, $\bar{a}\tau\equiv x^4$.

The total flux Φ is defined by

Total flux

$$= \int (\text{physical component of } B)$$

$$\times d(\text{physical distance in direction of increasing } \rho)$$

$$\times d(\text{physical distance in direction of increasing } \phi).$$

Φ comes out to be simply the product of B_0 by the coordinate area out to $\bar{a}$: $\pi\bar{a}^2 B_0$.

We define the "effective electromagnetic energy" E_{em} per unit physical length in the z direction by the integral of the physical magnetic energy density,

$$T_4{}^4 = \mathcal{T}_4{}^4/\sqrt{|g|} = \tfrac{1}{2}B_0{}^2 e^{-4\psi},$$

over the volume spanned by the entire physical area perpendicular to the z axis and by the physical distance corresponding to the z interval 0 to 1. We find

$$E_{\text{em}} = c^4/4G, \quad m_{\text{em}} = c^2/4G = 3.37\times 10^{27} \text{ g/cm}.$$

This value, which is that given in Ref. 3, is the appropriate energy quantity if we are interested in the result of adding up the measurements of a complete set of observers distributed throughout the space time, and each observing in a local Lorentz frame. The physical energy density and energy fluxes (in a time-dependent case) so defined are not however conserved (because of "interaction with the gravitational field"). If we are interested in a description using conserved quantities it is more appropriate to define the "physical" energy flux and energy densities not by $T_4{}^i$ but rather by the quantities $(\sqrt{g_{44}})T_4{}^i$. *These obey an ordinary conservation law if the gravitational field is stationary* ($g_{ik,4}=0$), *and the system of coordinates time-orthogonal* ($g_{\alpha 4}=0$, $\alpha\neq 4$). For, setting $S^i \equiv -(\sqrt{g_{44}})T_4{}^i$, Eq. (2) yields

$$\text{div}\mathbf{S} + \partial S^4/\partial t = 0, \quad \text{div}\mathbf{S} = (1/(\sqrt{{}^{(3)}g}))\{(\sqrt{{}^{(3)}g})S^i\}_{,i}.$$

The quantities S^4 and $\mathbf{S}$ are naturally interpreted as the energy density and energy flux, respectively. The extra factor $\sqrt{g_{44}} = 1+\rho^2$ in the integration to get the total electromagnetic energy or mass per unit length yields

$$(m_{\text{em}})_{\text{conserved}} = (E_{\text{em}}/c^2)_{\text{conserved}} = c^2/2G = 6.738\times 10^{27} \text{ g/cm},$$

where 2 occurs in the denominator instead of 4 as in Ref. 3. The results is equivalent to about one earth mass per cm.

A brief remark about the "stem-of-a-wineglass" nature of the equatorial plane spatial geometry in a static magnetic universe was made in Ref. 3. A fuller picture is given in the paper by Thorne[8] where the equatorial plane geometry is represented by an embedding diagram in a flat 3-dimensional Euclidean space. In order to understand the space-time geometry of the static magnetic universe somewhat better, we have also studied[9] its time-like and light-like geodesics; these are the orbits of electromagnetically neutral test particles with unit or zero rest mass. Since the density of magnetic flux—and energy and stress—is approximately uniform in the vicinity of the axis, the motion of test particles there is like that in a Newtonian simple harmonic oscillator field. As is to be expected from the universality of angular frequency ω_0 in the harmonic oscillator field, and the relation: orbital velocity$\cong\omega_0\rho$, no motion can get away too far from the axis. The strength of the attractive field is such that there is a "critical straddling radius" $\rho = 1/\sqrt{3}$. The fundamental reason for the critical radius is of course the limitation on orbital velocities by the velocity of light. Circular or circular helical light tracks occur only at the critical radius, and with $B_0 = 10^5$ G, the time required for light to circumnavigate the critical circle is about 200 years. The cylinder marked out by this radius plays a unique limiting role: All particles, whether of zero or nonzero mass, and no matter what their initial positions and velocities (except in the one singular subcase of light tracks parallel to the cylindrical axis), must have their orbits lying wholly or partially within the cylindrical region $\rho < 1/\sqrt{3}$; hence the use of the adjective "straddling." Constants of motion which correspond closely to ζ-component linear momentum, angular momentum, and energy in Newtonian mechanics are defined. Bounds are placed on these dynamical constants and on the apsidal radii by the requirement that the range of motion be real. Finally the magnetic universe is complete in the sense that "no news can enter or leave"—all orbits are of infinite duration.

D. Stability of the Magnetic Universe

A Newtonian energy argument similar to that for spherical geons[1(c),4] was given by Wheeler[1(c)] to show that the magnetogravitational equilibrium structure was unstable against collapse or explosion, but this time the result of an exact general relativity analysis (see Sec. 4 of this paper and the work of Thorne[8]) shows a radical difference from the Newtonian analysis: *While the equilibrium distribution follows equally well from a Newtonian model (with the one added idea that stress-energy acts as gravitating mass) and the exact general relativity analysis, the equilibrium is unstable according to the Newtonian model and stable according to the general-relativity analysis.*

There are two ways of analyzing stability of equilibrium of a system:

(1) The exact dynamical equations of motion, starting with a perturbation from the equilibrium configuration, are solved and it is determined explicitly whether or not the equilibrium configuration is departed

[8] K. S. Thorne, following paper, Phys. Rev. **139**, B244 (1965).

[9] M. A. Melvin and J. S. Wallingford, J. Math. Phys. (to be published).

from with time. In the case of the magnetic universe this is done analytically for small radial perturbations in Sec. 4 of this paper, and is done by computer methods for some large radial perturbations by Thorne[8] in his paper.

(2) The energy, or a similar constant of motion in the equilibrium configuration, is compared with the same quantity in nonequilibrium configurations to see if the nonequilibrium configurations are attainable starting from slight disturbed equilibrium. In the case of the magnetic universe this approach using the Newtonian energy concept[1(c)] failed to give the correct result, and the reason can be made clear as follows: The fact that the rigorous Einstein-Maxwell analysis leads to stability of the static system means that one can describe the system by saying that in it *the magnetic field spreads out to the maximum extent compatible with the geometry*.[10] If the geometry were asymptotically Euclidean this would lead to a total dissipation of the flux and the structure would be unstable as predicted by the Newtonian model. But in the actual geometry for large ρ values the circumferences of circles decrease[3]—distances measured in the ϕ direction "pinch in"—in a way completely contrasting with the widening circumferences of Euclidean geometry.

The strange geometry and the concentration of B flux *codetermine* each other, and one cannot reason adequately with the Newtonian energy concept which is not well defined for nonasymptotically flat space-times.[11] A different concept, indigenous to the theory and equations of cylindrically-symmetric systems, with properties paralleling in many of the most important respects the properties of Newtonian energy, has now been developed by Thorne[5] under the name of "C energy." The total C energy within a cylindrical region is closely related to the value of the second gravitational potential γ on the boundary of the region. A characteristic difference of C energy from its Newtonian antecedent is that whereas the contribution of the gravitational field to the Newtonian energy of a system is negative, its contribution to C energy is positive. Thorne shows that the behavior of C energy governs the *absolute* stability of cylindrically symmetric systems. Explosion of the magnetic-field distribution to infinite dilution, or collapse to a singularity, would bring about an *infinite increase* in the C energy of the system over its value in the static equilibrium distribution. From this result, and the uniqueness of the equilibrium solution, one can infer as well the *relative* stability of the equilibrium distribution, as is shown directly by the small perturbation analysis in Sec. 4 of this paper.

[10] It is this fact, that the distribution is stable (or spread out to the maximum extent compatible with its self-produced geometry) that leads us to change the name from "magnetic geon" as in Ref. 3 to "magnetic universe."

[11] For a review of hitherto unsuccessful attempts to define total energy is nonasymptotically flat space times see C. W. Misner, Phys. Rev. **130**, 1590 (1963).

2. THE GRAVITATIONAL EQUATIONS FOR WHOLE-CYLINDER SYMMETRY–UNSPECIFIED STRESSES

Our specific objective is to investigate the stability of the static magnetic universe by studying the dynamics of oscillation when the equilibrium is perturbed. For this and more general purposes it is useful and interesting to set up the general dynamical equations for the given symmetry.

We follow a procedure analogous to that which Weyl[12] and Levi-Civita[13] followed in studying static gravitational fields with polar[14] symmetry. We adapt the treatment however to nonstatic whole-cylinder symmetry, i.e., where there is a dependence of the surviving metric tensor components on a time variable x^4 and radial variable x^1, instead of on an altitude variable z and x^1. Such a treatment was followed by Einstein and Rosen in their work on cylindrical gravitational waves,[15] but they limited it to the case of space empty of material energy and stress. We treat the more general case where the space has material energy stress in it, specifically a distribution of electromagnetic fields, and we inquire after equilibrium configurations of such fields.

By our symmetry assumptions the only metric tensor components different from zero are

$$g_{44},\ g_{41},\ g_{11},\ g_{22},\ g_{33}\,,$$

where the indices 1, 2, 3 refer to x^1, z, and ϕ, respectively. Without loss of generality we simplify analysis by the two coordinate conditions

$$g_{14}=0\,,\quad g_{11}=-g_{44}\,. \tag{5}$$

Defining new potentials $\psi(x^1,x^4)$, $\gamma(x^1,x^4)$, and $r(x^1,x^4)$ we write

$$\begin{aligned} g_{11}&\equiv -g_{44}\equiv -e^{2(\gamma-\psi)}\,,\\ g_{22}&\equiv -e^{2\psi}\,,\\ g_{33}&\equiv -r^2/e^{2\psi}\,, \end{aligned} \tag{6}$$

from which are calculated the nonzero components of the Einstein tensor density

$$\mathfrak{G}_j{}^i=(R_j{}^i-\tfrac{1}{2}\delta_j{}^iR)\sqrt{|g|}\quad \text{(no summation)}$$

[12] H. Weyl, Ann. Physik **54**, 117 (1917); with R. Bach, Math. Z. **13**, 134 (1922).

[13] Levi-Civita, Atti. Accad. Nazl. Lincei, Rend. **26**, 307 (1917); **27**, 3 (1918); **27**, 183, 220, 240, 283, 343 (1918); **28**, 3, 101 (1919).

[14] Our space symmetry terminology follows that indicated in M. A. Melvin, Rev. Mod. Phys. **32**, 477 (1960), Fig. 5; **28**, 18 (1956), Fig. 1. In polar symmetry the structure is invariant under (1) rotation through any azimuthal angle ϕ about an axis z; (2) reflection in meridian planes $\phi\to-\phi$.

$$\text{POLAR (1): } \partial g_{\lambda\mu}/\partial\phi=0 \text{ (all } \lambda,\mu);\quad \text{(2): } g_{\phi\mu}=0\ (\mu\neq\phi). \quad (\alpha)$$

In their papers Weyl and Bach call this kind of symmetry "axial symmetry," but consistency with vector and tensor nomenclature requires that we call this type of symmetry "polar" and use "axial" for a structure invariant both under (1) change of ϕ and (2) reflection in the equatorial plane at right angles to the rotation axis: $z\to-z$.

$$\text{AXIAL (1): } \partial g_{\lambda\mu}/\partial\phi=0 \text{ (all } \lambda,\mu);\quad \text{(2): } g_{z\mu}=0\ (\mu\neq z). \quad (\beta)$$

The whole-cylinder symmetry includes both conditions (α) and (β) and also independence of all surviving g's of z as well as ϕ.

[15] A. Einstein and N. Rosen, J. Franklin Inst. **223**, 43 (1937).

and Eqs. (1) become

$$\mathcal{G}_1{}^1/r=\kappa\mathcal{T}_1{}^1/r=\{r_{44}-(\gamma,r)\}/r+(\psi,\psi)\,, \tag{7a}$$

$$\mathcal{G}_2{}^2/r=\kappa\mathcal{T}_2{}^2/r=2\Box^2\psi+\{2[r,\psi]-\Box^2 r\}/r-\Box^2\gamma-[\psi,\psi]\,, \tag{7b}$$

$$\mathcal{G}_3{}^3/r=\kappa\mathcal{T}_3{}^3/r=-\Box^2\gamma-[\psi,\psi]\,, \tag{7c}$$

$$-\mathcal{G}_4{}^4/r=-\kappa\mathcal{T}_4{}^4/r=\{r_{11}-(\gamma,r)\}/r+(\psi,\psi)\,, \tag{7d}$$

$$\mathcal{G}_4{}^1/r=\kappa\mathcal{T}_4{}^1/r=\{r_{14}-\gamma_4 r_1-\gamma_1 r_4\}+2\psi_1\psi_4\,, \tag{7e}$$

where the differential operators $(,)$, $[,]$, and $\Box^2$ are defined by

$$(f,g)\equiv f_1g_1+f_4f_4\,,\quad [f,g]\equiv f_1g_1-f_4g_4\,,\quad \Box^2 f\equiv f_{11}-f_{44}\,.$$

We take the difference of the first and fourth equations —likewise the second and third—and combine. Further, we take linear combinations of the last two equations. We find

$$\Box^2 r=-\kappa(\mathcal{T}_1{}^1+\mathcal{T}_4{}^4)\,, \tag{8a}$$

$$\{(r\psi_1)_1-(r\psi_4)_4\}/r=\kappa(\mathcal{T}_2{}^2-\mathcal{T}_3{}^3-\mathcal{T}_1{}^1-\mathcal{T}_4{}^4)/2r\,, \tag{8b}$$

$$\gamma_1=\{r_1r_{11}-r_4r_{14}+rr_1(\psi,\psi)-2rr_4\psi_1\psi_4+\kappa(r_1\mathcal{T}_4{}^4+r_4\mathcal{T}_4{}^1)\}/[r,r]\,, \tag{8c}$$

$$\gamma_4=-\{r_4r_{11}-r_1r_{14}+rr_4(\psi,\psi)-2rr_1\psi_1\psi_4+\kappa(r_4\mathcal{T}_4{}^4+r_1\mathcal{T}_4{}^1)\}/[r,r]\,, \tag{8d}$$

$$\Box^2\gamma=\psi_4{}^2-\psi_1{}^2-(\kappa/r)\mathcal{T}_3{}^3\,. \tag{8e}$$

Equations (8) provide a complete system for solving the general problem of a cylindrically symmetric time-dependent field, provided the stresses are given. If the stresses are given as functions of the independent variables alone, then, with suitable boundary conditions, the system is *linear*, and the procedure is: Solve the first equation for r. Substitute this solution into the second equation. We arrive at a second-order linear differential equation which may now be solved for ψ. With ψ and r known the last pair of equations determine γ.

3. RESTRICTION ON THE STRESSES

A. The Stress Restriction $\mathcal{T}_1{}^1+\mathcal{T}_4{}^4=0$. Reduction of Number of Gravitational Potentials from Three to Two (γ and ψ)

In general however the stresses are not known, but rather quadratic functions of the components of some *stress-source field* (e.g., an elementary particle field) subject to a governing system of equations (e.g., Maxwell's equations). These field equations involve the metric tensor functions r, ψ, and γ, and likewise do the stresses. Thus we are faced with a complicated system of nonlinear equations. A considerable simplification results if we impose the condition

$$\mathcal{T}_1{}^1+\mathcal{T}_4{}^4=0 \tag{9}$$

corresponding for instance (see Sec. 2) to an electromagnetic field with zero-field components along the x^1 direction. [Note that $\mathcal{T}_4{}^4>0$ always, therefore $\mathcal{T}_1{}^1<0$ by Eq. (9).] The differential operator on the left-hand side of Eq. (8b) becomes the standard (r,t)-dependent D'Alembertian operator $\Box$ and Eq. (8a) then simplifies to

$$\Box^2 r\equiv r_{11}-r_{44}=0\,. \tag{10}$$

With a mild added restriction it follows that r itself can be chosen as the radial variable in place of x^1.

To show this we consider the complete set of invertible transformations which leave the coordinate conditions Eq. (5) invariant. For convenience designate

$$x^1\equiv u'\,,\quad x^4\equiv v'\,.$$

The general transformation

$$u'\to u(u',v')\,,\quad v'\to v(u',v')$$

which leaves Eqs. (5) invariant has to satisfy

$$u_{v'}{}^2-v_{v'}{}^2=-(u_{u'}{}^2-v_{u'}{}^2)\,,\quad u_{u'}u_{v'}-v_{u'}v_{v'}=0 \tag{11}$$

together with the Jacobian condition

$$u_{u'}v_{v'}-v_{u'}u_{v'}\neq 0\,. \tag{11$'$}$$

Taking sum and difference of the first and twice the second of Eqs. (11) we find as the only solutions compatible with (11′)

$$u_{v'}=v_{u'}\,,\qquad u_{u'}=v_{v'}\,; \tag{12}$$

$$u_{v'}=-v_{u'}\,,\quad u_{u'}=-v_{v'}\,. \tag{12$'$}$$

The transformations (12) have the Jacobian

$$v_{v'}{}^2-v_{u'}{}^2=u_{u'}{}^2-u_{v'}{}^2\neq 0\,.$$

They include in their set the identity, and are all sense preserving, whereas the transformations (12′) have the opposite sign of the Jacobian and are the sense-reversing transformations. In the case of either Eqs. (12) or (12′) both u and v satisfy the wave equation, e.g.,

$$u_{u'u'}-u_{v'v'}=0\,.$$

We see that this theory of u and v (having the wave equation as a basis) is analogous to the theory of functions of a complex variable having the Laplace equation as a basis. We may pick r as an arbitrary solution of this equation subject to the monotonicity condition $r_{u'}{}^2-r_{v'}{}^2\neq 0$ and then the "real conjugate function" $v\equiv t$ is determined by Eqs. (12) or (12′). We call the pair $u=r$ and $v=t$, chosen to satisfy Eqs. (12), a *hyperbolic canonical pair*.

B. Introduction of Dimensionless Independent Variables

It is convenient to introduce dimensionless independent variables

$$\rho\equiv r/\bar{a}\,,\quad \zeta\equiv z/\bar{a}\,,\quad \tau\equiv t/\bar{a}\,,\quad \sigma\equiv s/\bar{a}$$

based on the range radius $\bar{a}$ of the static magnetic universe solution. The line element is now

$$d\sigma^2 = e^{2(\gamma-\psi)}(d\tau^2 - d\rho^2) - e^{2\psi}d\zeta^2 - \rho^2 e^{-2\psi}d\phi^2. \quad (13)$$

Henceforth the symbols $\square$ and $[,]$ represent the differential operators written in terms of the dimensionless variables ρ and τ:

$$\square\psi \equiv (\rho\psi_\rho)_\rho/\rho - \psi_{\tau\tau},$$

$$[A,B] \equiv A_\rho B_\rho - A_\tau B_\tau, \quad [A,A] \equiv [A] \equiv A_\rho{}^2 - A_\tau{}^2. \quad (14)$$

(It will be noted that square brackets are not used for any other purpose throughout this paper.) The system of Eqs. (8), with or without the help of the divergence identity, Eq. (2a), yields the complete system of gravitational equations:

$$\begin{aligned} &\square\psi = k(\mathcal{T}_2{}^2 - \mathcal{T}_3{}^3)/2\rho \quad (k \equiv \kappa\bar{a}), \\ &\psi_\rho = (\mathcal{T}_2{}^2 - \mathcal{T}_3{}^3)^{-1}(\mathcal{T}_{1,}{}^4{}_4 - \mathcal{T}_{4,}{}^4{}_1 - \rho^{-1}\mathcal{T}_3{}^3), \\ &\psi_\tau = (\mathcal{T}_2{}^2 - \mathcal{T}_3{}^3)^{-1}(\mathcal{T}_{4,}{}^4{}_4 - \mathcal{T}_{1,}{}^4{}_1), \\ &\gamma_\rho = \rho(\psi_\rho{}^2 + \psi_\tau{}^2) + k\mathcal{T}_4{}^4, \\ &\gamma_\tau = 2\rho\psi_\rho\psi_\tau + k\mathcal{T}_1{}^4, \\ &\square^2\gamma = \psi_\tau{}^2 - \psi_\rho{}^2 - k\mathcal{T}_3{}^3/\rho, \end{aligned} \quad (15)$$

where the first three equations serve as integrability conditions for the next two. Thus, if the stresses are given as functions of the independent variables alone, with assigned boundary conditions, the potentials ψ and γ may be found by successive quadratures. In general, however, as we have remarked, the stresses themselves depend on the metric tensor through the equations which govern the field which is the seat of the stresses, and we have a coupled system of equations.

Combining the first and sixth and, alternatively, from the first, fourth, and sixth of Eqs. (15) we can also write

$$\begin{aligned} &\square^2(\gamma - 2\psi) = \psi_\tau{}^2 - \psi_\rho{}^2 + 2\psi_\rho/\rho - k\mathcal{T}_2{}^2/\rho, \\ &\square(\gamma - 2\psi) = 2\psi_\tau{}^2 + k(\mathcal{T}_4{}^4 - \mathcal{T}_2{}^2)/\rho, \end{aligned} \quad (15')$$

which will be useful later.

C. Change of Dependent Variables (ψ and $\gamma \to W$ and G). Establishment of Gravitational Boundary Conditions

The foregoing choice of gravitational potentials, ψ and γ, is most natural for certain purposes in that ψ corresponds to the Newtonian potential in the static case (Ref. 3) and γ is related to the C energy or "mass" out to the radius in question (Thorne, Ref. 8). These are not however always the most convenient choices. It will prove valuable for the subsequent development to re-express Eqs. (15) in terms of two new gravitational potentials

$$G \equiv \gamma - 2\psi + \ln\rho, \quad W \equiv e^\psi/\rho. \quad (16)$$

In terms of G and W the line element takes the form

$$d\sigma^2 = W^2\{e^{2G}(d\tau^2 - d\rho^2) - \rho^2 d\zeta^2\} - d\phi^2/W^2. \quad (17)$$

From the Riemannian basis of general relativity, space time is locally Minkowskian everywhere. Thus as a natural regularity condition, since it refers to an infinitesimally small region, we require that the metric Eq. (17) [or Eq. (13)] immediately around the axis shall at all times remain Minkowskian:

(1) The ratio of circumference to radius of an infinitesimal circle around the axis equals 2π:

$$\text{for } \rho = d\rho \to 0: \; W^2 = e^{-G}/\rho \text{ or } \gamma(0,\tau) = 0 \text{ (all } \tau). \quad (17')$$

(2) The coordinate velocity of light in the ζ direction is bounded. The condition requires

$$\text{for } \rho \to 0: \; G(\rho,\tau) - \ln\rho \to -2\psi(0,\tau) < \infty \text{ (all } \tau). \quad (17'')$$

With these notations, and with the relation

$$\square W/W - [W]/W^2 = \square\psi \quad (18)$$

derived from the definition, Eq. (16), of W, we have

$$\square W/W - [W]/W^2 = (k/2\rho)(\mathcal{T}_2{}^2 - \mathcal{T}_3{}^3). \quad (19)$$

Equations (15′) may now be rewritten

$$\begin{aligned} &\square^2 G = -[W]/W^2 - (k/\rho)\mathcal{T}_2{}^2, \\ &\square G = 2(W_\tau/W)^2 + (k/\rho)(\mathcal{T}_4{}^4 - \mathcal{T}_2{}^2). \end{aligned} \quad (19')$$

The first of these combined with Eq. (19) and the remaining equations of the system (15) rewritten are

$$\square W/W = -\square^2 G - (k/2\rho)(\mathcal{T}_2{}^2 + \mathcal{T}_3{}^3), \quad (19a)$$

$$W_\rho/W = (\mathcal{T}_2{}^2 - \mathcal{T}_3{}^3)^{-1}(\mathcal{T}_{1,}{}^4{}_4 - \mathcal{T}_{4,}{}^4{}_1 - \rho^{-1}\mathcal{T}_2{}^2), \quad (19b)$$

$$W_\tau/W = (\mathcal{T}_2{}^2 - \mathcal{T}_3{}^3)^{-1}(\mathcal{T}_{4,}{}^4{}_4 - \mathcal{T}_{1,}{}^4{}_1), \quad (19c)$$

$$G_\rho = \rho(W,W)/W^2 + k\mathcal{T}_4{}^4, \quad (19d)$$

$$G_\tau = 2\rho W_\rho W_\tau/W^2 + k\mathcal{T}_1{}^4. \quad (19e)$$

Since the right-hand side of Eq. (19d) is always positive for any nonvanishing $\mathcal{T}_4{}^4$, it follows that G is a monotonic increasing function of ρ.

In the case of the pure electromagnetic field to be discussed in the next section, one simplification can be made immediately. The trace of the stress tensor is always zero:

$$\mathcal{T}_\mu{}^\mu = 0.$$

This combined with our condition Eq. (9) that the field be entirely lateral (zero radial component), leads to the result that

$$\mathcal{T}_2{}^2 + \mathcal{T}_3{}^3 = 0. \quad (20)$$

In this case Eqs. (19a) to (19e) and the second of Eqs. (19′) can be written in the form

$$\square W/W = -\square^2 G, \quad (20a)$$

$$W_\rho/W = (2\mathcal{T}_2{}^2)^{-1}(\mathcal{T}_{1,}{}^4{}_1 - \mathcal{T}_{4,}{}^4{}_1) - (2\rho)^{-1}, \quad (20b)$$

$$W_\tau/W = (2\mathcal{T}_2{}^2)^{-1}(\mathcal{T}_{1,}{}^4{}_4 - \mathcal{T}_{1,}{}^4{}_1), \quad (20c)$$

$$G_\rho = \rho(W,W)/W^2 + k\mathcal{T}_4{}^4, \quad (20d)$$

$$G_\tau = 2\rho W_\rho W_\tau / W^2 + k\mathcal{T}_1{}^4, \tag{20e}$$

$$\Box G = 2(W_\tau/W)^2 + (k/\rho)(\mathcal{T}_4{}^4 - \mathcal{T}_2{}^2). \tag{20f}$$

We note that in the first equation of this system the stresses have been eliminated entirely; this, together with Eq. (20f), will prove to be a key equation in the treatment of the dynamical problem of a perturbed magnetic universe in Sec. 4 of this paper.

4. GRAVITATING ELECTROMAGNETIC FIELDS

A. Metric-Independent Form of Maxwell's Equations and of the Stress Tensor

We consider now pure (sourceless) electromagnetic fields in the curved space-time defined by Eqs. (17) and (19a) to (19e). Actually we need not introduce the metric at first. It is worthwhile to spend a little time explaining this: We can write the general equations of electromagnetism in a form completely independent of any metric, i.e., in what may be called "amorphic" space time.[16] Formally this is possible because electromagnetism is expressible in terms of k-index multivectors (antisymmetric tensors with k indices) and multivector densities, and relations between multivectors are expressible in metric-independent form.[17] The development is most elegantly and economically made by introducing at first only the six-component *magnetoelectric flux* $\mathbf{F}\sim(\mathbf{B},\mathbf{E})$ and the four-component charge current $\mathbf{J}\sim(\rho,\mathbf{j})$ and stating the two integral principles of *conservation of charge* and *conservation of (magnetoelectric) flux:*

$$\oint \mathbf{J}\cdot d\mathbf{S}^{(3)} = 0, \tag{21}$$

$$\oint \mathbf{F}\cdot d\mathbf{S}^{(2)} = 0. \tag{22}$$

It is a conventional matter whether we choose to represent the four-component charge current by a contravariant vector density $\mathcal{J}^\alpha$ or its dual, a covariant axial[18] trivector. The former choice is almost invariably made. Likewise it is conventional whether we represent the magnetoelectric flux by a covariant bivector $F_{n4}\equiv E_n$, $F_{km}\equiv B_{km}$, or by a contravariant axial bivector density. The former choice is usually made.

By a general integral theorem[19]—the Gauss-Green theorem in both its space and space-time form—which has no reference to metric in it, Eqs. (21) and (22) imply respectively that $\mathcal{J}^\alpha$ can be written as the divergence of a contravariant bivector density $\mathcal{H}^{\alpha\mu}$,

$$\mathcal{H}^{km}\equiv H_n, \quad \mathcal{H}^{n4}\equiv -D^n,$$

and that $F_{\mu\nu}$ can be written as the curl of a covariant *vector potential* $(A_k, A_4\equiv-\phi)$, so that

$$\mathcal{J}^\alpha = \partial_\mu \mathcal{H}^{\alpha\mu} \equiv \mathcal{H}^{\alpha\mu}{}_{,\mu}, \tag{23}$$

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \tag{24}$$

$\mathcal{H}^{\alpha\mu}$, which plays the role of a "charge-current potential," is traditionally called the ***electromagnetic field.*** If we assume further that $\mathbf{J}$ and $\mathbf{F}$ are continuously differentiable in a region, Eqs. (23) and (24) imply that the divergence of $\mathbf{J}$ and the "cyclical divergence" or "rotation" of $\mathbf{F}$ will vanish everywhere in the region:

$$\mathcal{J}^\alpha{}_{,\alpha} \equiv \partial_\alpha \mathcal{J}^\alpha = 0, \tag{25}$$

$$\mathrm{Rot}\mathbf{F}\sim(\mathrm{Rot}\mathbf{F})_{\kappa\mu\nu} \equiv F_{\kappa\mu,\nu} + F_{\mu\nu,\kappa} + F_{\nu\kappa,\mu} = 0. \tag{26}$$

These are the differential equation equivalents of the integral conservation principles Eqs. (21) and (22).

Maxwell's equations in an Einsteinian gravitational field, i.e., with a Riemannian space-time metric prescribed so that tensor indices may be raised and lowered, are then obtained from Eqs. (23) and (26) by equating the bivector density $\mathcal{H}^{\alpha\mu}$ to that obtained from F:

$$\mathcal{H}^{\alpha\mu} = hF^{\alpha\mu}, \quad h\equiv\sqrt{(-g)} \equiv (-\mathrm{Det}g_{\mu\nu})^{1/2}. \tag{27}$$

---

[16] By the "general equations of electromagnetism" we mean the two Maxwell systems and specifically do not include the "constitutive relations" between **E** and **D** and **B** and **H** which characterize a particular medium. That the two Maxwell systems can be derived from the two integral principles of conservation of magnetic flux and conservation of electric charge was realized by R. Hargreaves [Trans. Cambridge Phil. Soc. **21**, 107 (1908)] and H. Bateman [Proc. London Math. Soc. **8**, 223 (1910)]. The full implications of this for a metric-independent (what we have called "amorphic") expression of electromagnetism were stated by F. Kottler [Sitzber. Akad. Wiss. Wien, Math. Naturw. Kl. Abt. IIa **131**, 119 (1922)] and developed further by D. van Dantzig [Proc. Acad. Sci. Amsterdam **39**, 126, 785 (1936), where reference to earlier work may be found] and others. M. A. Melvin, *Proceedings of the Second Canadian Mathematical Congress* (University of Toronto Press, 1951), p. 225, showed that the amorphic invariance of electromagnetism follows directly from a generalized symmetry analysis of the empirical phenomena where electric and magnetic fields are excited in systems in which there are initially no electromagnetic phenomena.

[17] The metric-independent nature of electromagnetism, as a formal possibility due to the expressibility of electromagnetism in terms of multivectors and multivector densities, was remarked before Kottler by H. Weyl in his book *Space—Time—Matter*, (Springer-Verlag, Berlin, 1918), 1st ed. [English transl. (Dover Publications, Inc., New York, 1950), 4th ed., p. 131.] It appears, however, that Weyl did not follow up this remark.

[18] "Axial" means that, besides undergoing tensor transformation, the geometric object is multiplied by the *sign* of the determinant of transformation coefficients. In former publications (M. A. Melvin, Ref. 16) the adjective "pseudo" was used for such entities, but it is better to switch to "axial." This switch is made without explanation in the article by C. Truesdell and R. Toupin, *Handbuch der Physik* (Springer-Verlag, Berlin, 1960), Vol. III/1, p. 661. We can give the following reasons for the change: In Ref. 14 we noted that the adjective "axial" is appropriate for any entity directed along an axis—e.g., the z axis—which is invariant: (1) under reflection in the equatorial plane $z\rightarrow -z$, and (2) under rotation about the z axis. It is significant that the second of these conditions is satisfied automatically for a k vector. This can be seen most easily by examining the rotated components starting from a frame in which the k vector is directed along the z axis. ("Along" means that for a 1-vector only the components V_z and V_t are different from zero; for a bivector only V_{xy} and V_{zt} are different from zero; for a trivector only V_{xyt} and V_{xyz} are different from zero.) If, in addition, it is specified that the k-vector be "axial" the condition (1), of invariance under $z\rightarrow -z$, is also fulfilled.

[19] See Truesdell and Toupin, Ref. 18, pp. 665, 666.

Postulating this identification of the two entities, $\mathbf{F}$ and $\mathcal{H}/\sqrt{(-g)}$, is equivalent to postulating that the usual form of the electromagnetic field Lagrange density $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\sqrt{(-g)}$ holds in a gravitational field. Maxwell's equations reduce then to

$$F_{\mu\nu,\kappa}+F_{\nu\kappa,\mu}+F_{\kappa\mu,\nu}=0 \quad \text{or} \quad F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}, \qquad (28)$$

$$(hF^{\alpha\mu})_{,\mu}=hJ^{\alpha} \quad \text{or} \quad (hg^{\alpha\mu}g^{\beta\nu}F_{\mu\nu})_{,\beta}=\mathcal{J}^{\alpha}, \qquad (29)$$

where $J^{\alpha}=\mathcal{J}^{\alpha}/h$ is the charge-current vector corresponding to the charge-current vector density $\mathcal{J}^{\alpha}$. If the system of space-time coordinates is "time-orthogonal,"[20] i.e., such that $g_{\alpha 4}=0(\alpha\neq 4)$, and if, besides, it is possible to diagonalize the space metric over a region, then the only terms which will appear in Eq. (29) will be those for which

$$\beta=\nu\neq\mu=\alpha$$

and Eq. (29) may be rewritten (suspending the summation convention)

$$\sum_{\beta} \operatorname{sgn}(\alpha\beta)(h^{\alpha}h^{\beta}h_{\gamma}h_{\delta}F_{\alpha\beta})_{,\beta}=\mathcal{J}^{\alpha}$$

$$(\alpha,\beta,\gamma,\delta=1,2,3,4 \text{ et cycl.}) \qquad (30)$$

where we have introduced the symbols

$$h_{\alpha}\equiv\sqrt{|g_{\alpha\alpha}|}, \quad h^{\alpha}\equiv\sqrt{|g^{\alpha\alpha}|}=\sqrt{(1/|g_{\alpha\alpha}|)}=1/h_{\alpha},$$
$$\operatorname{sgn}(\alpha\beta)\equiv(\operatorname{sign}g^{\alpha\alpha})\times(\operatorname{sign}g^{\beta\beta}).$$

Take the product of the four h's as a single quantity. There are then at most six terms on the left side of Eq. (30). In the special case where the g's are constants there are only the three derivatives of the $F_{\alpha\beta}$ and we have the usual Maxwell system.

The energy-stress tensor density is given in the general case by the equations

$$\mathcal{T}_{\nu}{}^{\mu}=-\mathcal{H}^{\mu\alpha}F_{\nu\alpha}+\tfrac{1}{4}\delta_{\nu}{}^{\mu}\mathcal{H}^{\alpha\beta}F_{\alpha\beta}. \qquad (31)$$

B. Simplifications in the Case of Cylindrical Symmetry. Introduction of Single-Component Vector Potential

We consider now the special case indicated by the symmetry requirements of Sec. 1, an electromagnetic field with zero field components along the r direction. Even more specifically we consider a cylindrical electromagnetic wave with magnetic field entirely along the z direction:

$$F_{31}(\rho,\tau)\equiv B,$$

and electric field entirely in the ϕ direction:

$$F_{34}(\rho,\tau)\equiv E.$$

Since, by Eq. (6), $h_4=h_1(g_{44}=-g_{11})$, and $h^3h_2=e^{2\psi}/\rho$, our Eqs. (28) and (30) become

$$B_{,\tau}-E_{,\rho}=0, \qquad (32)$$

$$((e^{2\psi}/\rho)B)_{,\rho}-((e^{2\psi}/\rho)E)_{,\tau}=\mathcal{J}^{3}. \qquad (32')$$

The nonvanishing components in the present case of axially cylindrical symmetry are (dimensional metric)

$$\mathcal{T}_1{}^4=-\mathcal{H}^{4\alpha}F_{1\alpha}=-hF^{4\alpha}F_{1\alpha}=EBe^{2\psi}/\rho\bar{a},$$
$$\mathcal{T}_2{}^2=-\mathcal{T}_3{}^3=(\tfrac{1}{2}h)(F^{43}F_{43}+F^{31}F_{31})$$
$$=\tfrac{1}{2}(B^2-E^2)e^{2\psi}/\rho\bar{a}, \qquad (31')$$
$$\mathcal{T}_4{}^4=-\mathcal{T}_1{}^1=(\tfrac{1}{2}h)(-F^{43}F_{43}+F^{31}F_{31})$$
$$=\tfrac{1}{2}(B^2+E^2)e^{2\psi}/\rho\bar{a}.$$

As Eq. (32) shows, the vector potential in this case can be represented by a single covariant component $\mathcal{A}_3$ such that

$$\bar{a}B=-\mathcal{A}_{3,1}, \quad \bar{a}E=-\mathcal{A}_{3,4}.$$

For the length parameter on which we base the dimensionless length and time variables, we choose the range of the static solution.[3]

$$\bar{a}=\frac{1}{B_0}\left(\frac{8}{\kappa}\right)^{1/2}\equiv\frac{1.108\times10^{24}}{B_0}\text{ cm}$$

($B_0\equiv$ flux density on axis in Heaviside units)

$$=\frac{2c^2}{B_0\sqrt{G}}\cong\frac{6.96\times10^{24}}{B_0}\text{ cm} \quad (B_0 \text{ in gauss}).$$

Correspondingly we may introduce a dimensionless single-component vector potential

$$A\equiv\mathcal{A}_3(2/B_0\bar{a}^2) \qquad (33)$$

so that

$$F_{31}\equiv B=-\tfrac{1}{2}(B_0\bar{a})A_{\rho}, \quad F_{43}\equiv E=-\tfrac{1}{2}(B_0\bar{a})A_{\tau}. \qquad (33')$$

The Maxwell equation (32′) with current set equal to zero then becomes

$$(\rho W^2A_{\rho})_{\rho}-(\rho W^2A_{\tau})_{\tau}=0. \qquad (34)$$

C. Electromagnetic Regularity Conditions

The stresses and the energy density must not become singular at the axis. It follows then from Eqs. (31′) that for some time—at least near the beginning—we have the boundary conditions

$$\rho\to 0: \quad A_{\rho}\leq O(1)\sqrt{\rho}, \quad A_{\tau}\leq O(1)\sqrt{\rho},$$

where $O(1)$ stands for "some finite value." We obtain a stronger condition by imposing the "*electromagnetic regularity conditions*":

(1) On the axis the physical magnetic field shall remain finite, at least within a time range $T_i<\tau<T_f$; the physical electric field, which "does not know which way to point," shall be zero on the axis.

(2) At $\rho=\infty$, at all times, the physical electromagnetic field—and the covariant potential—shall be zero.

[20] This can always be arranged by suitable choice; cf. Chr. Møller, *The Theory of Relativity* (Oxford University Press, New York, 1952), p. 296.

The relations between the physical components of the fields and the covariant components for any diagonal metric are

$$B_{\rm phys}=F_{31}/h_3h_1=-B_0(\tfrac{1}{2}e^{-G})A_\rho\,,$$
$$E_{\rm phys}=F_{34}/h_3h_4=-B_0(\tfrac{1}{2}e^{-G})A_\tau\,. \qquad (34')$$

Conditions (1) and (2) lead to [$O(1)$ stands for "some finite value"]

$$\rho\to 0:\quad A_\rho\leq O(1)\rho\,,\quad A_\tau<O(1)\rho$$
$$\rho\to\infty:\quad A=0$$

or

$$A(0,\tau)=\text{constant}\,,\quad A(\infty,\tau)=0\,. \qquad (34'')$$

These conditions are related to one of the two fundamental physical principles with which we started the analysis of electromagnetism in an arbitrary space: conservation of flux, Eq. (22). As will now be shown, the constant in Eq. (34″) has the interpretation

$$A(0,\tau)\propto\text{total flux}\equiv\Phi\,. \qquad (34''')$$

Here we have an explicit statement in our particular case (where there is only one vanishing component A_3) of that aspect of Eq. (22) which asserts the existence of a "constant of motion"—the total flux. This becomes evident when we write out the integral in Eq. (22) explicitly for the space-time "cylinder" whose boundary comprises the entire ρ, ϕ plane at two different times τ_1 and τ_2. We have (normals along positive τ and ρ directions)

$$\oint \mathbf{F}\cdot d\mathbf{S}^{(2)}=\iint_{\tau_2} F_{31}d\phi d\rho-\iint_{\tau_1} F_{31}d\phi d\rho+\iint_{\rho=\infty} F_{34}d\phi d\tau$$
$$=\tfrac{1}{2}(B_0\bar{a}^2)\left[\iint_{\tau_1} A_\rho d\phi d\rho-\iint_{\tau_2} A_\rho d\phi d\rho-\iint_{\rho=\infty} A_\tau d\phi d\tau\right]=0\,,$$

which implies

$$\text{total flux}=\pi\bar{a}^2A(0,\tau_2)B_0=\pi\bar{a}^2A(0,\tau_1)B_0$$

as stated.

The constant of motion implied by the other fundamental physical principle—conservation of charge—is trivial in our case since the electric current is everywhere zero.

D. Canonical Form of the Coupled Gravitational-Electromagnetic Equations

It is helpful to introduce instead of the covariant component A of the vector potential, the (dimensionless) physical component $A_{\rm phys}$ which is related to the covariant component by the equation

$$A_{\rm phys}=(g_{33})^{-1/2}A=(e^\psi/\rho)A$$

or

$$A_{\rm phys}=WA\,. \qquad (35)$$

Upon substituting in Eq. (34) this yields the remarkably simple relation which we may call the first *canonical electromagnetic-gravitational equation*

$$\Box W/W=\Box A_{\rm phys}/A_{\rm phys}\,. \qquad (36)$$

One can combine Eq. (36) with Eq. (20a), relating the gravitational potentials W and G among themselves. Three further equations relating W, G, and A are obtained from Eqs. (20d)–(20f), with the values of $\mathcal{T}_4{}^4$ and $\mathcal{T}_1{}^4$ from Eqs. (31′) rewritten in terms of dimensionless quantities.

$$G_\rho/\rho=(W,W)/W^2+(A,A)W^2\,,$$
$$G_\tau/\rho=2\{W_\rho W_\tau/W^2+A_\rho A_\tau W^2\}\,, \qquad (37)$$
$$\Box G=2\{(W_\tau/W)^2+A_\tau{}^2W^2\}\,.$$

This basic system of equations can be rewritten in terms of W, G, and $A_{\rm phys}$:

$$\Box W/W=\Box A_{\rm phys}/A_{\rm phys}=-\Box^2G\,,$$
$$\frac{G_\rho}{\rho}=(\ln W,\ln W)+A_{\rm phys}{}^2\left(\ln\frac{A_{\rm phys}}{W},\ln\frac{A_{\rm phys}}{W}\right),$$
$$\frac{G_\tau}{\rho}=2\left\{(\ln W)_\rho(\ln W)_\tau+A_{\rm phys}{}^2\left(\ln\frac{A_{\rm phys}}{W}\right)_\rho\left(\ln\frac{A_{\rm phys}}{W}\right)_\tau\right\}, \qquad (38)$$
$$\Box G=2\left\{[(\ln W)_\tau]^2+A_{\rm phys}{}^2\left[\left(\ln\frac{A_{\rm phys}}{W}\right)_\tau\right]^2\right\}.$$

E. The Static Solution and Its Total Magnetic Energy per Unit Length

The static solution,[3] which we shall designate by superior bars, is readily verified to correspond to

$$\bar{G}=\ln\rho\,,\quad \bar{W}=(1+\rho^2)/\rho\,,\quad \bar{B}_{\rm phys}=B_0/(1+\rho^2)^2\,,$$
$$\bar{A}=\frac{1}{1+\rho^2}\,,\quad \bar{A}_{\rm phys}=\frac{1}{\rho}\,,\quad \frac{k}{\rho}\bar{\mathcal{T}}_2{}^2=\frac{k}{\rho}\bar{\mathcal{T}}_4{}^4=\frac{4}{(1+\rho^2)^2} \qquad (39)$$

with the total flux

$$\Phi=\pi\bar{a}^2B_0\bar{A}(0)=\pi\bar{a}^2B_0$$

as in Ref. 3.

In the stability analysis of the static solution we shall be interested only in perturbations which leave the total flux constant. It follows from Eqs. (34″) and (34‴) that under these circumstances

$$A(0,\tau)=\bar{A}(0)=1\,,\quad A_\rho(0,\tau)=0\,,$$
$$A(\infty,\tau)=0\,. \qquad (39'')$$

Because of these limitations on the variation of A it may be conjectured that no gravitational collapse will occur and that the static solution may be stable. That this is actually the case for small radial perturbations is verified analytically in the next part of this paper.[21]

5. DYNAMICS OF A PERTURBED MAGNETIC UNIVERSE

A. First-Order Equations for a Purely Radial Perturbation. Reduction of Problem to Finding Two Functions, *g* and *h*

To get first-order perturbation equations, we expand with the perturbation parameter ϵ in the neighborhood of the static solution. Setting

$$\begin{aligned} G&=\bar{G}+\epsilon g(\rho,\tau)+\cdots\,,\\ W&=\bar{W}(\rho)+\epsilon w(\rho,\tau)+\cdots\,,\\ A_{\text{phys}}&=\bar{A}_{\text{phys}}(\rho)+\epsilon a(\rho,\tau)+\cdots\,,\end{aligned}\tag{40}$$

where $\cdots$ stands for "terms of order ϵ^2 and higher," we find, from Eqs. (38) and (35), using (39), after some calculation,

$$\Box g=0 \quad\text{or}\quad \Box^2 g=-g_\rho/\rho\,,\tag{40a}$$

$$\Box w-w/\rho^2=g_\rho(1+\rho^2)/\rho^2\,,\tag{40b}$$

$$\Box a-a/\rho^2=g_\rho/\rho^2\,,\tag{40c}$$

$$g_\rho=\frac{2}{\rho^2+1}\left\{\frac{\rho^2-1}{\rho^2+1}(2a-w)-\rho(2a_\rho-w_\rho)\right\}\,,\tag{40d}$$

$$g_\tau=-[2\rho/(\rho^2+1)](2a_\tau-w_\tau)\,.\tag{40e}$$

From Eqs. (40d) and (40e) we immediately find the integral

$$g=[2\rho/(\rho^2+1)](w-2a)+\text{constant}\,.\tag{41}$$

We will solve the first three of Eqs. (40) for g, a, and w in terms of Laplace- or Fourier-type integrals with cylinder-function kernels, but before doing this we reduce the problem to one involving only homogeneous equations. Equations (40b) and (40c) being linear, each has as its solution the general solution to the corresponding homogeneous equation ($w^{(h)}$ and $a^{(h)}\equiv h$) plus a particular solution to the inhomogeneous equation ($w^{(\text{part})}$ and $a^{(\text{part})}$):

$$w=w^{(h)}+w^{(\text{part})}\,,\quad a=h+a^{(\text{part})}\,.\tag{42}$$

As a trial form we set $w^{(\text{part})}=gf(\rho)\equiv gf$ where g is the general solution to the Eq. (40a), whose ρ derivative appears on the right side of the equations for w and a. Noting that $\Box w\equiv\Box(fg)=\Delta(fg)-f\ddot{g}$, where Δ is the Laplacian in ρ, and making use of a familiar identity for the Laplacian of a product, we find

$$2g_\rho f_\rho+g(\Delta f-f/\rho^2)=g_\rho(1+\rho^2)/\rho^2\,.$$

Setting coefficients of g and g_ρ separately equal to zero, we find

$$f=\tfrac{1}{2}(\rho-1/\rho)\quad\text{or}\quad w^{(\text{part})}=\tfrac{1}{2}(\rho-1/\rho)g\,.\tag{42$'$}$$

Similarly, even more easily, we find

$$a^{(\text{part})}=-g/2\rho\,.\tag{42$''$}$$

Substituting Eqs. (42), (42′), and (42″) into (41), we find the remarkably simple result

$$w^{(h)}=2h+\text{const}(\rho+1/\rho)\,.\tag{43}$$

Thus we have only to determine solutions of the equations

$$\Box g=0\,,\tag{44a}$$

$$\Box h-h/\rho^2=0\,,\tag{44b}$$

subject to given side and regularity conditions, and the complete solution to the perturbation problem is immediately given by

$$\begin{aligned} w&=2h+\text{const}(\rho+1/\rho)+\tfrac{1}{2}(\rho-1/\rho)g\,,\\ a&=\tfrac{1}{2}(2h-g/\rho)\,.\end{aligned}\tag{44c}$$

We shall see from the regularity conditions derived in the next section that the constant in the first of Eqs. (44c) is zero.

B. Regularity Conditions and Solutions for *g*

The regularity requirements are made evident if we expand the metric about its static form. The added term in $d\sigma^2$, of order ϵ, is

$$2\rho^2\bar{W}w\{(1+g\bar{W}/w)(d\tau^2-d\rho^2)-d\zeta^2+(\bar{W}^4\rho^2)^{-1}d\phi^2\}\,.\tag{45}$$

If we are to allow for all admissible developments, including possible instability, we must be careful not to impose excessively strong restrictions on the behavior of the perturbed magnetic universe. The natural restrictions are the *gravitational regularity conditions* [Eqs. (17′) and (17″)] extended to include behavior at infinity; and the *electromagnetic regularity conditions* Eq. (34″) [including the constant total flux condition, Eq. (39′)]. Spelled out with respect to the perturbed magnetic universe we have:

Gravitational Regularity Conditions

(1) At all times τ, to a first order in ϵ, the locally flat nature of the metric on the axis $\rho=0$ shall not be altered. This implies

for $\rho\to 0$:

$$g(\rho,\tau)\to-2\psi(0,\tau)<\infty\,,\tag{46a}$$

$$w(\rho,\tau)\to-g/2\rho(1+\rho^2)\to\psi(0,\tau)/\rho\,.\tag{46b}$$

[21] Since this was written, stability under large radial perturbation has been proved by K. Thorne with the method of C energy (private communication Ref. 8). Our perturbation method retains however the value of giving the explicit dynamical solutions, including the representation in normal modes.

Therefore, from Eq. (44c) we have

$$\text{for } \rho\to 0: \qquad h(\rho,\tau)\to -\tfrac{1}{2}\,\text{const}/\rho. \tag{46c}$$

(2) At all times to a first order in ϵ, the static metric at $\rho=\infty$ shall not be altered; in other words the ratio of the added ϵ term to the static metric shall go to zero as $\rho\to\infty$. This implies

$$\text{for } \rho\to\infty: \qquad w/\rho\to 0. \tag{47}$$

We now consider the implications of the electromagnetic regularity conditions which also hold in our case: The relations between physical and covariant fields for any diagonal metric are [Eqs. (34′)]

$$B_{\text{phys}}=(-B_0e^{-G}/2)A_\rho=\bar{B}_{\text{phys}}+\epsilon b_{\text{phys}}+\cdots,$$
$$E_{\text{phys}}=(-B_0e^{-G}/2)A_\tau=\epsilon e_{\text{phys}}+\cdots,$$

where we calculate, expressing everything in terms of g and h,

$$(2/B_0)b_{\text{phys}}=\{-4\rho^2g+\rho(\rho^2+1)g_\rho+(\rho^4-6\rho^2+1)h/\rho-(\rho^4-1)h_\rho\}/(1+\rho^2)^3, \tag{48}$$
$$(2/B_0)e_{\text{phys}}=\{-\rho g_\tau+(\rho^2-1)h_\tau\}/(1+\rho^2)^2.$$

Electromagnetic Regularity Conditions

(1) At all times, b_{phys} shall be finite and e_{phys} shall be zero on the axis $\rho=0$. This implies that the constant in Eqs. (46c) and (44c) is zero:

$$\text{for } \rho\to 0: \qquad h(\rho,\tau)\to 0. \tag{49}$$

(2) At all times b_{phys} and e_{phys} shall go to zero as $\rho\to\infty$. This implies, with the help of Eq. (44c) (in which the constant is to be set equal to zero) and the condition Eq. (47):

$$\text{for } \rho\to\infty: \qquad g_\rho/\rho\to 0 \quad\text{or}\quad g/\rho^2\to 0. \tag{50}$$

We now solve Eq. (44a) on the basis of the regularity conditions. Separating the variables with the arbitrary real or complex separation constant $-\omega^2$ (or $-\Omega^2$) the general admissible solution of Eq. (44a) in terms of the modified Bessel function kernel is

$$g(\rho,\tau)=\mathbf{S}(\alpha(\omega)e^{i\omega\tau}+\beta(\omega)e^{-i\omega\tau})J_0(\omega\rho), \tag{51}$$

where the neutral symbol $\mathbf{S}$ indicates possible summation or integration over all eigenvalues ω, whose admissibility will be decided on the basis of the regularity conditions. Possible additional terms of the form

$$\mathbf{S}(\eta(\Omega)e^{i\Omega\tau}+\xi(\Omega)e^{-i\Omega\tau})Y_0(\Omega\rho)$$

can be eliminated on the basis of the regularity conditions requiring $g(0,\tau)$ to be bounded, Eq. (46a). Since

$$\lim_{\rho\to\infty}J_0(\omega\rho)$$

blows up exponentially for every value of ω except those purely real, and this would violate requirement (50), we are limited to the real axis of the ω plane for admissible eigenvalues. *Thus we see that the combined requirements of local flatness at the axis, and zero deviation from the static solution at infinity, force the g solution to be stable (purely oscillatory).* It is clear by a simple regrouping that only the positive real ω axis need be taken into account.

Equation (51) then takes the form of the sum of two integral transforms which may be regarded as inverse Fourier transforms or Hankel transforms[22] according to the point of view

$$\rho^{1/2}g(\rho,\tau)=\int_0^\infty\{F(\omega)\sin\omega\tau+G(\omega)\cos\omega\tau\}\times J_0(\omega\rho)(\omega\rho)^{1/2}d\omega. \tag{52}$$

$G(\omega)$ and $\omega F(\omega)$ are, respectively, the Hankel transforms of initial distributions $g(\rho,0)$ and $g_\tau(\rho,0)$:

$$G(\omega)=\int_0^\infty\omega^{1/2}\rho' g(\rho',0)J_0(\omega\rho')d\rho',$$

$$F(\omega)=\int_0^\infty\omega^{-1/2}\rho' g_\tau(\rho',0)J_0(\omega\rho')d\rho'.$$

The mathematical theory is of course consistent with the idea that any initial first-order perturbation is finally shaken off, the distribution subsiding to its static value: For any integrable g and g_τ, i.e., such that

$$\int\rho g(\rho,0)d\rho \quad\text{and}\quad \int\rho g_\tau(\rho,0)d\rho$$

exist, we have

$$\lim_{\omega\to0}G(\omega)=c_1\omega^{1/2},\quad \lim_{\omega\to0}F(\omega)=c_2/\omega^{1/2}, \tag{52′}$$

and by general Abel-Tauber theorems it follows that $g(\rho,\infty)=0$.

The regularity conditions are sufficient to guarantee stability (small oscillation) of the h potential as well as of the g potential.

The solution of Eq. (44b) for h is found to be

$$\rho^{1/2}h(\rho,\tau)=\int_0^\infty\{I(\omega)\sin\omega\tau+H(\omega)\cos\omega\tau\}\times J_1(\omega\rho)(\omega\rho)^{1/2}d\omega.$$

$$H(\omega)=\int_0^\infty\omega^{1/2}\rho' h(\rho',0)J_1(\omega\rho')d\rho', \tag{53}$$

$$I(\omega)=\int_0^\infty\omega^{-1/2}\rho' h_\tau(\rho',0)J_1(\omega\rho')d\rho'.$$

---

[22] For the theory of Hankel transforms, see *Tables of Integral Transforms, Bateman Manuscript Project* (McGraw-Hill Book Company, Inc., New York, 1954), Vol. 2, p. 3.

Possible additional terms involving $Y_1(\Omega\rho)$ can be eliminated on the basis of the regularity condition at the axis, Eq. (49). The spectrum of ω is restricted to the real ω axis on the basis of the regularity condition at infinity, Eq. (47).

C. Electromagnetic and Gravitational g-Type and h-Type Normal Modes

It is clear from Eqs. (52) and (53) that there are fundamentally two types of waves, those of g type [$h(\rho,0)$ and $h_\tau(\rho,0)$ equal to zero] and those of h type [$g(\rho,0)$ and $g_\tau(\rho,0)$ equal to zero]. For a given circular frequency ω the amplitudes of these two mode types can be represented by $\{\sin\omega\tau,\ \cos\omega\tau\}\omega^{1/2}J_0(\omega\rho)$ and $\{\sin\omega\tau,\ \cos\omega\tau\}\omega^{1/2}J_1(\omega\rho)$, respectively. It is of special interest to examine the electromagnetic and gravitational waves associated with these two types of modes. Their amplitudes can be deduced simply from Eqs. (52) and (53) from the relations

$$\delta\psi = w/\bar{W} = \frac{1}{\rho+1/\rho}\{\tfrac{1}{2}(\rho-1/\rho)g+2h\},$$

$$\delta\gamma = g+2\delta\psi = \frac{2}{\rho+1/\rho}\{\rho g+2h\},$$

$$\frac{2}{B_0}e_{\rm phys} = \frac{1/\rho}{(\rho+1/\rho)^2}\{-g_\tau+(\rho-1/\rho)h_\tau\}, \tag{54}$$

$$\frac{2}{B_0}b_{\rm phys} = \frac{1/\rho}{(\rho+1/\rho)^3}\left\{-4g+\left(\rho+\frac{1}{\rho}\right)g_\rho + \left(\rho^2-6+\frac{1}{\rho^2}\right)\frac{h}{\rho} - \left(\rho^2-\frac{1}{\rho^2}\right)h_\rho\right\},$$

which follow directly from the definitions of W and G and from Eq. (48). In this way we find the results tabulated in Tables I and II, and plotted in Figs. 1 and 2 for three different representative values of ω.

D. Shaking Off of Perturbations and Causality

Suitable superpositions of standing waves (normal modes) give running waves. After analyzing the possible standing waves which can occur, it is interesting to examine the manner in which an initial perturbation is shaken off and radiated away in the form of outgoing waves, the entire field subsiding in time to the static distribution. The general situation is well illustrated by the case where the initial perturbation differs from zero only in the region contained within the range radius of the static magnetic universe ($0\leqq\rho\leqq1$). Here we can verify immediately that "casuality" holds, i.e., that the initial perturbation is propagated outward with the velocity of light. (No disturbance outside the light cone of the range radius!)

We illustrate with the case of a pure g-type perturbation. Any initial distribution between 0 and 1 may be represented by a superposition of Legendre polynomials; i.e., by a sum over nonnegative integral values of n of terms of the form $a_nP_n(1-2\rho^2)$. For each such term the integral in Eq. (52) vanishes outside the light cone of the event ($\rho=1$, $\tau=0$). This is verified as follows:

For

$$\begin{aligned} &g(\rho,0)=P_n(1-2\rho^2), && 0<\rho<1,\\ &g(\rho,0)=0, && 1<\rho<\infty,\\ &g_\tau(\rho,0)=0, && \text{everywhere}\end{aligned}$$

the transform is[23]

$$G(\omega)=\omega^{-1/2}J_{2n+1}(\omega),\quad H(\omega)=0,$$

and

$$g(\rho,\tau)=\int \cos\omega\tau\ J_{2n+1}(\omega)J_0(\omega\rho)d\omega. \tag{52''}$$

Writing the cosine in terms of a Bessel function of order $-\frac{1}{2}$ we have on the right-hand side of Eq. (52″) an integral involving besides the factor $(\pi\tau\omega/2)^{1/2}$ the product of three Bessel functions of orders p, q, r such that for our case

$$p+q+r=2n+\tfrac{1}{2}.$$

The integral can be evaluated by a general formula of Bailey[24]

$$\left(\frac{\pi\tau}{2}\right)^{1/2}\int_0^\infty \omega^{\lambda-1}J_p(a\omega)J_q(b\omega)J_r(c\omega)d\omega = \left(\frac{\pi\tau}{2}\right)^{1/2}\frac{2^{\lambda-1}a^pb^qc^r\Gamma(\sigma)}{c^{2\sigma}\Gamma(p+1)\Gamma(q+1)\Gamma(r-\sigma+1)} \times F_4\left[\sigma-r,\sigma;p+1,q+1;\frac{a^2}{c^2},\frac{b^2}{c^2}\right]$$

$$\lambda+p+q+r=2\sigma,\quad c>a+b, \tag{55}$$

which applied to our case gives

$$\frac{\pi^{1/2}\Gamma(n+1)}{c^{2(n+1)}\Gamma(p+1)\Gamma(q+1)\Gamma(r-n)} \times F_4\left[n+1-r,n+1;p+1,q+1;\frac{a^2}{c^2},\frac{b^2}{c^2}\right].$$

Here F_4 is the fourth of the original hypergeometric series in two variables defined by Appell.[25] We have four different regions of physical interest according to the assignment of 1, ρ, and τ to satisfy the inequality $c>a+b$ (Fig. 3), and we must assign the values $-\frac{1}{2}$, 0,

[23] Reference 22, p. 13, Eq. (1).

[24] Integrals of products of three Bessel functions by a power of the variable of integration were evaluated by W. N. Bailey, Proc. London Math. Soc. 40, 37 (1936), Eq. (7.1), from which our expression may be derived.

[25] See *Higher Transcendental Functions, Bateman Manuscript Project* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, pp. 222–245.

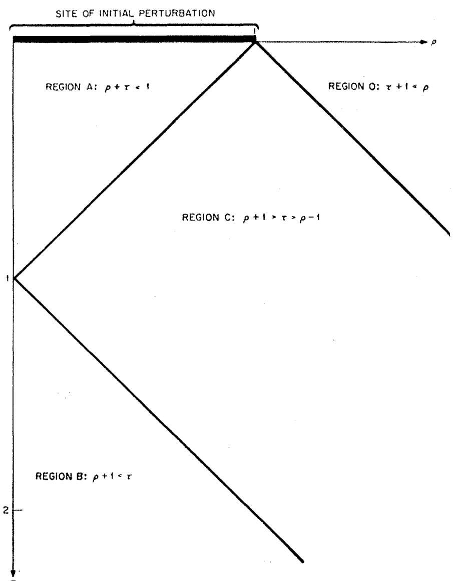

FIG. 3. Case of an initial perturbation of the magnetic universe differing from zero only in the region between the axis ($\rho=0$) and the range radius ($\rho=1$). Represented here are the four regions of physical interest in the space-time diagram for the "shaken-off wave." For any form of the initial perturbation between 0 and 1, the amplitude of the wave is always zero in Region O, i.e., outside the light cone; hence causality is verified.

$2n+1$ to the different indices p, q, r of the Bessel functions correspondingly.

Region O: $\tau+1<\rho$.—This is the region outside the light cone of $\rho=1$, $\tau=0$. Here the requirement $a+b<c$ means that we must take the index r to be that of the Bessel function associated with the argument $\omega\rho$, i.e., $r=0$. But then the factor $\Gamma(-n)$ in the denominator makes the integral vanish for every non-negative integral value of n. *This is precisely the causality condition stated above.*

E. Causal Green's Functions or Propagators

We can verify the causality condition quite generally, and at the same time reduce the problem of finding the general running-wave solution to a single integration over the initial distributions, by finding the Green's functions or propagators K_g, $K_{g\tau}$ and K_h, $K_{h\tau}$ connected, respectively, with $g(\rho,\tau)$ and $h(\rho,\tau)$. These we define by

$$g(\rho,\tau)=\int K_g(\rho';\rho,\tau)\rho' g(\rho',0)d\rho' + \int K_{g\tau}(\rho';\rho,\tau)\rho' g_\tau(\rho',0)d\rho', \quad (55a)$$

$$h(\rho,\tau)=\int K_h(\rho';\rho,\tau)\rho' h(\rho',0)d\rho' + \int K_{h\tau}(\rho';\rho,\tau)\rho' h_\tau(\rho',0)d\rho'. \quad (55b)$$

From the Eqs. (52) and (53) and those immediately following, assuming that interchange of order of integration is permissible, we find with the help of Bailey's general integration formula, Eq. (55), that each of the four propagators can be expressed as an Appell F_4 function with gamma function coefficients. Here too, as in Sec. 5.D, we have to satisfy a triangle inequality and, correspondingly, we have four different regions of physical interest. A figure similar to Fig. 3, with the location of the point $\rho'=1$ generalized to any value ρ', applies here.

Regions O and A: $\tau+\rho'<\rho$ and $\tau+\rho<\rho'$.—These are the regions outside the light cone of the event at the radial location ρ' at time $\tau=0$ and, correspondingly, a factor $\Gamma(0)$ in the denominators of each of the four propagators makes each vanish in these regions. *Causality is verified.*

Region B: $\rho+\rho'<\tau$.—This is the region between the axis and a light-wave front, starting at $(\rho',0)$ and propagating inward, *after* this wave front has reached the axis and has started outward again. In all four cases the F_4 function reduces to the ordinary hypergeometric function of Gauss[26] and we have

$$K_g(\rho';\rho,\tau)=-(1/\tau^2)(1-x)^{3/2} \times(1-y)^{3/2}F(\tfrac{3}{2},\tfrac{3}{2};1;xy), \quad (56a)$$

$$K_{g\tau}(\rho';\rho,\tau)=\tau^{-1}(1-x)^{1/2} \times(1-y)^{1/2}F(\tfrac{1}{2},\tfrac{1}{2};1;xy), \quad (56b)$$

$$K_h(\rho';\rho,\tau)=(3\rho\rho'/2\tau^4)(1-x)^{5/2} \times(1-y)^{5/2}F(\tfrac{5}{2},\tfrac{3}{2};2;xy), \quad (56c)$$

$$K_{h\tau}(\rho';\rho,\tau)=-(\rho\rho'/2\tau^3)(1-x)^{3/2} \times(1-y)^{3/2}F(\tfrac{3}{2},\tfrac{1}{2};2;xy), \quad (56d)$$

where

$$-\frac{x}{(1-x)(1-y)}=\left(\frac{\rho'}{\tau}\right)^2, \quad \frac{-y}{(1-x)(1-y)}=\left(\frac{\rho}{\tau}\right)^2. \quad (57)$$

Equation (57) can be inverted to give x and y, or better yet the product

$$k^2\equiv xy,$$

which is all we need, in terms of ρ and τ. Thus we find

$$k_\pm=R\pm(R^2-1)^{1/2}, \quad R\equiv(\tau^2-\rho^2-\rho'^2)/2\rho\rho',$$
$$(1-x)(1-y)=k\tau^2/\rho\rho',$$

so that we have

$$K_g=-\tau(k/\rho\rho')^{3/2}F(\tfrac{3}{2},\tfrac{3}{2};1;k^2), \quad (58a)$$

$$K_{g\tau}=(k/\rho\rho')^{1/2}F(\tfrac{1}{2},\tfrac{1}{2};1;k^2), \quad (58b)$$

$$K_h=\tfrac{3}{2}k\tau(k/\rho\rho')^{3/2}F(\tfrac{5}{2},\tfrac{3}{2};2;k^2), \quad (58c)$$

$$K_{h\tau}=-\tfrac{1}{2}k(k/\rho\rho')^{1/2}F(\tfrac{3}{2},\tfrac{1}{2};2;k^2). \quad (58d)$$

[26] Reference 25, p. 238, Eq. (7).

Since $R>1$ (as we see from the condition $\tau>\rho+\rho'$), and since the hypergeometric series diverges for $k^2>1$, only k_- is acceptable.

It is possible further to express the Gauss function with these parameters in terms of the associated toroidal harmonics of the second kind $Q_{-1/2,1/2}{}^{1,0}(R)$; but there is no particular advantage for our present purposes in doing this. Except for preparing one general case, of value in understanding the effect of magnetic fields upon gravitational collapse, we shall leave all further mathematical discussion for another communication where the wave amplitude for particular forms of the initial perturbation will be studied in detail.

We will limit ourselves to values of τ not merely larger, but much larger, than $\rho+\rho'$. We then have

$$k=k_-\cong(2R)^{-1}\cong\rho\rho'/\tau^2\ll1,$$

and Eqs. (58) give approximately

$$K_g\cong-1/\tau^2, \tag{59a}$$

$$K_{g\tau}\cong\tau^{-1}, \tag{59b}$$

$$K_h\cong3\rho\rho'/2\tau^4, \tag{59c}$$

$$K_{h\tau}\cong-\rho\rho'/2\tau^3. \tag{59d}$$

Fortunately the region to which we are limiting ourselves covers the space and time domains in which we are interested, i.e., the space around the axis, stretching as far out as we like, for sufficiently large times. This space-time domain is adequate for our purpose of understanding: (1) how the initial perturbation is shaken off and radiated out to infinity; and (2) the effect of a (very) strong magnetic field upon a "collapsing," i.e., incoming, field of radiation.

F. Asymptotic Time Dependence of a Radial Perturbation and State of Maximum Concentration

From Eqs. (59) inserted in Eqs. (55a) and (55b) we find

$$g(\rho,\tau)=-(1/\tau^2)\tilde{G}+\tau^{-1}\tilde{G}_\tau, \tag{60a}$$

$$h(\rho,\tau)=\tfrac{1}{2}\rho\{(3/\tau^4)\tilde{H}-(1/\tau^3)\tilde{H}_\tau\}, \tag{60b}$$

where $\tilde{G}$, $\tilde{G}_\tau$ and $\tilde{H}$, $\tilde{H}_\tau$ are constants representing, respectively, the first moments of $g(\rho',0)$, $g_\tau(\rho',0)$ and the second moments of $h(\rho',0)$ and $h_\tau(\rho',0)$, respectively. We see that after a long time g is independent of ρ and subsides with time like τ^{-1}, whereas h is proportional to ρ and subsides like ρ/τ^3. (For the behavior of b_{phys} and e_{phys} see Table III.)

Item (2) in the last paragraph of Sec. 5.E pertains to the possibility of investigating a collapsing radiation field, superposed on a magnetic universe, with these simplified formulas. The most interesting question here is that of the time of occurrence of extrema ("last crests") of the amplitudes of the fields e_{phys} and b_{phys}, and of the magnitudes of these extreme amplitudes. For either the pure g-type or the pure h-type case, with given initial conditions (i.e., given $\tilde{G}$, $\tilde{G}_\tau$ or $\tilde{H}$, $\tilde{H}_\tau$), this extremum is unique—but different for e_{phys} and b_{phys}. The conditions at the extrema are immediately obtained with the help of the time derivatives of Eqs. (54) and (60) and are tabulated in Table IV. The cases of mixed g and h waves would lead to more complicated conditions.

We note two remarkable physical facts about the behavior of the last crests in the region of asymptotic subsidence ($\tau\gg\rho+\rho'$). First: *The electric field is zero when the magnetic field is at a maximum.* Because g_ρ, as given by Eq. (60a), is zero, and $h_\rho=h/\rho$, the time derivative of b_{phys} is a linear combination of time derivatives of $g(\rho,\tau)$ and $h(\rho,\tau)$ and—*for the pure g or h case*—b_{phys} reaches its maximum value at a time when $g_\tau(\rho,\tau)=0$ or $h_\tau(\rho,\tau)=0$, i.e., when $e_{\mathrm{phys}}=0$. Thus this condition, which holds in each normal mode, holds also for the general superposition of g modes alone or, alternatively, h modes alone (but not for a general mixture of g and h modes). The reciprocal relation does not hold: The magnetic field is not zero when the electric field is at a maximum for a general g- or h-type wave.

The second noteworthy fact is the interpretation provided by our system of the constants in the general Abel-Tauber-type theorem relating the behavior of a function after long times to that of its Fourier transform at very low frequencies [see Eq. (52′)]. We see, from the expansions of $J_0(\omega\rho)$ and $J_1(\omega\rho)$ in the neighborhood of the origin, that

$$\lim_{\omega\to0}G(\omega)/\omega^{1/2}=\tilde{G},\qquad \lim_{\omega\to0}\omega^{1/2}F(\omega)=\tilde{G}_\tau,$$

$$\lim_{\omega\to0}H(\omega)/\omega^{3/2}=\tilde{H}/2,\quad \lim_{\omega\to0}I(\omega)/\omega^{1/2}=\tilde{H}_\tau/2,$$

and we have interpreted the Abel-Tauber constants as first and second moments.

6. CONCLUSION

The main outcome of the foregoing investigation may be expressed in the following simple physical terms:

At one time it was thought that one could have a distribution of magnetic lines of force only if electric currents are present. It was shown in a preceding publication,[3] on the basis of Maxwell's theory of electromagnetism combined with Einstein's theory of gravitation, that a self-sustaining collection of magnetic lines of force can exist. The present investigation shows that not only is this self-sustaining distribution an equilibrium distribution but it is a *stable equilibrium distribution*, with respect to cylindrically symmetric perturbations.

An important theoretical reason for emphasizing the perturbed cylindrical case is that one can study in this case the very important physical problem of coupled gravitational and electromagnetic radiation whereas there is rigorously no radiation in problems with spherical symmetry.

The relevance of the magnetic universe to the problem of understanding actual gravitational collapse has already been indicated at the end of the Introduction, Sec. 1. It may be summarized as follows: It is shown by Thorne[5] and by the writer, in different and physically illuminating ways, that the magnetic universe is stable against disturbance. If it is altered by squeezing the magnetic field together or by expanding it away from the equilibrium distribution, it will oscillate, shaking off the modification in the field by radiating it away in the form of electromagnetic and gravitational waves. Thus the only presently known physical system, which would not gravitationally collapse, if subjected to very large pressures, is a sufficiently extended pure magnetic (or electric) field. It is of considerable interest that the observed quasistellar sources last much longer (1000 to 1 000 000 years) than they ought to according to the present theories of spherically symmetric gravitational collapse of material systems not including magnetic fields. On the one hand, from the observational evidence of double structure, one would infer that there is a tendency to cylindrical rather than spherical symmetry in the quasistellar systems undergoing gravitational collapse. On the other hand, it is well known that extended magnetic fields do exist in the universe.[27] Thus, while of course it cannot be claimed that anything of the extent of the magnetogravitational structure we have discussed has been observed, it is possible that extended magnetic fields play a role in retarding and finally halting the process of gravitational collapse.

The foregoing suggestion may be spelled out more explicitly: Cylindrical geometry appears more relevant than spherical geometry to an important feature of quasistellar sources: Such objects eject matter and magnetic fields in two opposite directions.[1(a)] Exterior to the supposed region of collapse the observed fields are of course fantastically small compared to any field whose self-gravitating action would be appreciable.[27] Even in the interior of a quasistellar source the gravitational pull of magnetic lines of force may be small compared to the gravitational action of matter. Therefore the present analysis of a cylindrical magnetogravitational structure may not apply *directly* to quasistellar objects. In contrast, however, magnetic fields of roughly cylindrical symmetry *accompanied by matter* provide a model for gravitational collapse of the very greatest interest and importance.[28] As the matter falls inward and carries the magnetic field with it, does this magnetic field inhibit the collapse—as it does in the case of the present pure magnetogravitational structure? Does the magnetic field give rise to a counterpressure of decisive importance to the dynamics of collapse? These questions about gravitational collapse are not answered by the present investigation. However, one has arrived here at a new point, of which it would seem desirable to take cognizance in all future investigations of collapse: *A pure magnetic field has a remarkable and previously unsuspected ability to stabilize itself against gravitational collapse.*

[27] Fields extending over ten thousand light years seem to have been observed in the explosion at the center of M82; C. R. Lynds and A. R. Sandage, Astrophys. J. **137**, 1005 (1963). We note that if this range were that of a magnetic universe the corresponding axis field—or effective field within the range radius—would be $B_0 \approx 700$ G in contrast with $<2\times10^{-6}$ G for M82 estimated by Lynds and Sandage. The factor of the order of 100 million points up the difference between actual astrophysical fields and the hypothetical ones needed to make up a magnetic universe. It is only in the extreme conditions of gravitational collapse that we might expect our considerations to be of possible relevance.

ACKNOWLEDGMENTS

The author wishes to acknowledge his deep appreciation to J. A. Wheeler, K. S. Thorne, and B. K. Harrison for carefully reading and commenting most helpfully on this paper. It is a special pleasure to acknowledge the inspiration we have all derived from J. A. Wheeler. To J. D. Melvin go the author's great thanks for thoroughly reviewing the manuscript and for the computer work leading to Figs. 1 and 2.

[28] V. L. Ginzburg, Dokl. Akad. Nauk SSSR **156**, 43 (1964) [English transl.: Soviet Phys.—Doklady **9**, 329 (1964)]; V. L. Ginzburg and L. M. Ozernoi, Zh. Eksperim. i Teor. Fiz. **47**, 1030 (1964) [English transl.: Soviet Phys.—JETP (to be published)]; I. Novikov, Astron. Tsirkular, No. 290 (1964).