in Eq. (B3). The contours C_1' and C_2' (Figs. 16 and 17) differ from those represented in Figs. 2 and 3 in that they are continued to ∞.

The function $F(t,t_1,t_2)$ has singularities at $t^{1/2}=t_1^{1/2}+t_2^{1/2}$, $t_1=0$, $t_2=0$. If $\mathrm{Re}t_2^{1/2}>\mathrm{Re}t^{1/2}$, the singularity $t_1^{1/2}=t^{1/2}-t_2^{1/2}$ of the function $F(t,t_1,t_2)$ is absent on the physical sheet of the plane t_1 represented in Fig. 16 since the point t_1 for which $\mathrm{Re}t_1^{1/2}<0$ lies below the cut made in Fig. 16 left of the singular point $t_1=0$.

Therefore at $\mathrm{Re}t^{1/2}<\mathrm{Re}t_2^{1/2}$ the singularity of the integral over t_1:

$$\phi_j(t,t_2)=\frac{1}{2}\frac{1}{2i}\int_{C_1'}\frac{F(t;t_1,t_2)}{j+1-\alpha(t_1)-\alpha(t_2)}dt_1$$

arises only for such j, t, and t_2 for which the zero of the denominator appearing across the cut $t_1>4\mu^2$ and deforming the contour C_1' of integration coincides with the points $t_1=0$. This singularity, given by the condition

$$j+1=\alpha(t_2)+\alpha(0)\,,$$

i.e., $j=\alpha(t_2)$, appears on the cut of the plane t_2, deforms the contour (as is indicated in Fig. 17) and reaching the line $\mathrm{Re}t_2^{1/2}=\mathrm{Re}t^{1/2}$ does not lead to the singularity of integral (B2)

$$f_j'(t)=\frac{1}{2i}\int_{C_2'}\phi_j(t,t_2)dt_2\,. \tag{B4}$$

This means that the singularity of this integral arises only from the region of small (or complex) t_2, $t_2\lesssim t$. Since t is small the quantity $t_1^{1/2}=t^{1/2}-t_2^{1/2}$ is also small (or if it is not, it is complex). In either case the particle masses cannot enter into the expression giving the location of the singularity. Actually the singularity of integral (B2), (B4) arises from the point $t_1=t_2=t/4$.

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Statistics of the Thermal Radiation Field*

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The characteristic functional is calculated for a system of bosons obeying linear field equations. The system is assumed to be in equilibrium, and the density matrix is taken to be of the form $\langle\{n\}|\rho|\{m\}\rangle=\Pi_\kappa\,\delta_{n_\kappa m_\kappa}(1-z_\kappa)z_\kappa^{n_\kappa}$, where κ labels the individual modes. From the characteristic functional, the moments and distribution functions of an arbitrary number of field components are derived. In addition, it is shown how to obtain the density matrix from the characteristic functional, and, for the system in question, the original density matrix is recovered. Explicit calculations are performed for the electromagnetic field in an unbounded domain and in a semi-infinite domain bounded by a perfectly conducting plane.

I. INTRODUCTION

USING the methods of quantum field theory, we shall compute the characteristic functional for an electromagnetic field in thermal equilibrium within an enclosure of arbitrary size and shape. From this functional, we shall compute the moments or correlation functions and the probability distributions for any number of field components at the same or different points in space-time.[1] We shall see that the probability distribution is a multivariate Gaussian function. Therefore, all correlation functions are expressible in terms of the two point correlation function. To exemplify the result, we shall explicitly calculate this correlation function for an unbounded domain and for a semi-infinite domain bounded by a perfectly conducting plane. For the unbounded domain our results agree with those of Sarfatt,[2] Bourret,[3] and Mehta and Wolf.[4] The correlation functions for a semi-infinite domain do not seem to have been calculated previously.

The deduction of the Gaussian distribution functions for black-body radiation in an unbounded domain has already been given by Glauber[5,6] and Holliday.[7] These distribution functions were used implicitly by Purcell[8] and explicitly by Mandel and Wolf[9] in order to analyze the intensity interferometry experiments of Hanbury-

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[1] Of course, the distribution functions are physically meaningful only when they refer to points at which the field components commute. For the electric and magnetic field components, this means that no two points lie on the same light cone.

[2] J. Sarfatt, Nuovo Cimento **27**, 1119 (1963).

[3] R. C. Bourret, Nuovo Cimento **18**, 347 (1960).

[4] C. L. Mehta and E. Wolf, Phys. Rev. **134**, A1143, A1149 (1964).

[5] R. J. Glauber, Phys. Rev. Letters **13**, 84 (1963); Phys. Rev. **130**, 2529 (1963); *Quantum Optics and Electronics: The 1964 Les Houches Lectures*, edited by C. DeWitt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach, Science Publishers, Inc., New York, 1965).

[6] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

[7] D. Holliday, Phys. Letters **8**, 250 (1964); see also D. Holliday and M. L. Sage, Ann. Phys. (N. Y.) **29**, 125 (1964).

[8] E. M. Purcell, Nature **178**, 1449 (1956).

[9] L. Mandel and E. Wolf, Phys. Rev. **124**, 1696 (1961).

Brown and Twiss.[10] Glauber[5] has also obtained the probability distribution for the number of photons counted by a detector.

Although the analysis described so far refers to the electromagnetic field, the same considerations apply to any boson field governed by linear equations. To illustrate this, we shall calculate also the characteristic functional and the two point correlation function for a scalar meson field in thermal equilibrium.

We next solve the inverse problem of determining the density matrix for a field when its characteristic functional is known. For this purpose, we make use of the representation developed by Glauber.[6] In the case of a characteristic functional which leads to Gaussian distribution functions, we shall show that the density matrix has the form of that for black-body radiation.

Our results shed some light on a question which has been raised concerning the applicability of classical physics to random electromagnetic fields. We know that all the statistical information concerning a field is contained in the characteristic functional. From this functional, both the quantum-mechanical density matrix and the probability distributions of the field components can be determined. Therefore, two completely equivalent descriptions of a random field can be given—one in terms of a quantum-mechanical density matrix, and the other in terms of a set of probability distributions for field components which may be regarded as classical quantities.[11] The two descriptions will give identical results for all quantities provided that they both correspond to the same characteristic functional. Of course, the correct probability distributions cannot be determined classically, but once they are determined, they can be used without any further reference to the quantum-mechanical nature of the fields.

II. DEFINITION OF THE CHARACTERISTIC FUNCTIONAL

The electromagnetic field in a domain V devoid of sources can be described in terms of a Hermitian vector potential operator $\mathbf{A}(\mathbf{r},t)$. Within V, $\mathbf{A}$ satisfies the wave equation

$$(\nabla^2-c^{-2}\partial_t^2)\mathbf{A}=0. \quad (1a)$$

In addition, some appropriate subsidiary condition is required to fix the gauge of $\mathbf{A}$, such as

$$\nabla\cdot\mathbf{A}=0, \quad (1b)$$

the transversality condition. On the boundary of V, we assume that $\mathbf{A}$ satisfies a real linear homogeneous boundary condition which makes the Laplacian operator Hermitian. From $\mathbf{A}$ the electric and magnetic field operators $\mathbf{E}(\mathbf{r},t)$, and $\mathbf{B}(\mathbf{r},t)$ can be obtained via the relations

$$\mathbf{E}=-c^{-1}\partial_t\mathbf{A}, \quad (2)$$

$$\mathbf{B}=\nabla\times\mathbf{A}. \quad (3)$$

It is convenient to introduce the product solutions of (1) of the form $\mathbf{v}_\kappa(x)=\mathbf{u}_\kappa(\mathbf{r})e^{i\omega t}$. From (1) it follows that $\mathbf{u}_\kappa(\mathbf{r})$ satisfies the reduced wave equation

$$(\nabla^2+c^{-2}\omega_\kappa^2)\mathbf{u}_\kappa(\mathbf{r})=0. \quad (4a)$$

The functions $\mathbf{u}_\kappa$ must also satisfy the transversality condition

$$\nabla\cdot\mathbf{u}_\kappa(\mathbf{r})=0. \quad (4b)$$

If $\mathbf{u}_\kappa$ satisfies the same boundary condition as $\mathbf{A}$, then $\mathbf{u}_\kappa$ is an eigenfunction of the domain and ω_κ, the corresponding eigenvalue, is real. The index κ labels the various eigenfunctions and corresponding eigenvalues. When V is bounded the eigenvalues are discrete and the index κ may be restricted to discrete values. Unbounded domains are covered by our treatment if we interpret κ as a continuous index, and understand that summation over κ means integration. Alternatively, we shall consider V to be bounded and pass to the limit of an unbounded domain in our final result. Since the equations and boundary conditions for $\mathbf{u}_\kappa$ are real, it follows that $\mathbf{u}_\kappa^*$, the complex conjugate of $\mathbf{u}_\kappa$, is also an eigenfunction corresponding to the same eigenvalue ω_κ. We assume that the $\mathbf{u}_\kappa$ have been chosen to satisfy the orthonormality conditions

$$\int_V \mathbf{u}_\kappa^*(\mathbf{r})\cdot\mathbf{u}_{\kappa'}(\mathbf{r})d\mathbf{r}=\delta_{\kappa\kappa'}. \quad (5)$$

By utilizing a complete set of $\mathbf{u}_\kappa(\mathbf{r})$ we can express $\mathbf{A}$ in the form

$$\mathbf{A}(\mathbf{r},t)=c\sum_\kappa(\hbar/2\omega_\kappa)^{1/2} \times[a_\kappa\mathbf{u}_\kappa(\mathbf{r})e^{-i\omega_\kappa t}+a_\kappa^\dagger\mathbf{u}_\kappa^*(\mathbf{r})e^{i\omega_\kappa t}]. \quad (6)$$

The coefficient a_κ and its Hermitian adjoint $a_\kappa^\dagger$ are the annihilation and creation operators for photons of the κth mode of the field. We assume that they satisfy the familiar commutation relations for independent harmonic oscillators,

$$[a_\kappa,a_{\kappa'}]=[a_\kappa^\dagger,a_{\kappa'}^\dagger]=0, \quad (7)$$

$$[a_\kappa,a_{\kappa'}^\dagger]=\delta_{\kappa\kappa'}. \quad (8)$$

From (6)–(8) the commutation relations of the components of $\mathbf{A}$ at two space-time points can be found.

The statistical description of any quantum-mechanical system, such as the electromagnetic field in V, is expressible in terms of a density operator ρ. In terms of ρ the expectation value $\langle O\rangle$ of any operator O is given by

$$\langle O\rangle=\mathrm{tr}(\rho O). \quad (9)$$

To find the expectation of $\mathbf{A}$ or of any function of $\mathbf{A}$ it is convenient to introduce the characteristic functional

[10] R. Hanbury-Brown and R. Q. Twiss, Nature **177**, 27 (1956); Proc. Roy. Soc. (London) **A242**, 300 (1957); **A243**, 291 (1957).

[11] See footnote 1.

$F[\lambda]$ defined by

$$F[\lambda]=\left\langle\exp\left(i\int \lambda(x)\cdot\mathbf{A}(x)dx\right)\right\rangle. \quad (10)$$

Here $x=(\mathbf{r},t)$ is a four vector, λ is an arbitrary real vector function and the integration extends over the domain $\mathbf{r}$ in V and $-\infty<t<\infty$.

Upon utilizing (6) for $\mathbf{A}$ the integral in (10) can be written in the form

$$\int \lambda(x)\cdot\mathbf{A}(x)dx=c\sum_\kappa\left(\frac{\hbar}{2\omega_\kappa}\right)^{1/2}[\lambda_\kappa a_\kappa+\lambda_\kappa^* a_\kappa^\dagger]. \quad (11)$$

The coefficient λ_κ in (11) is defined by

$$\lambda_\kappa=\int \lambda(x)\cdot\mathbf{u}_\kappa(\mathbf{r})e^{-i\omega_\kappa t}dx. \quad (12)$$

When (11) is used in (10) and the commutation relation (7) is recalled, (10) can be written as

$$F[\lambda]=\langle\prod_\kappa \exp[ic(\hbar/2\omega_\kappa)^{1/2}(\lambda_\kappa a_\kappa+\lambda_\kappa^* a_\kappa^\dagger)]\rangle. \quad (13)$$

To proceed further we must specify the density operator.

For thermal equilibrium, the density operator takes the form

$$\rho=e^{-\beta H}/\mathrm{tr}e^{-\beta H}, \quad (14a)$$

where the Hamiltonian H is given by

$$H=\sum_\kappa \tfrac{1}{2}\hbar\omega_\kappa(a_\kappa a_\kappa^\dagger+a_\kappa^\dagger a_\kappa)=\sum_\kappa \hbar\omega_\kappa(n_\kappa+\tfrac{1}{2}). \quad (14b)$$

Here $n_\kappa=a_\kappa^\dagger a_\kappa$ is the number operator for the κth mode and $\beta=1/kT$, where k is Boltzmann's constant and T is absolute temperature. It follows from (14) that

$$\rho=\prod_\kappa \rho_\kappa, \quad (15)$$

with

$$\rho_\kappa=\exp[-\beta\hbar\omega_\kappa(n_\kappa+\tfrac{1}{2})]/\mathrm{tr}\exp[-\beta\hbar\omega_\kappa(n_\kappa+\tfrac{1}{2})]. \quad (16)$$

More generally, let us consider operators with matrix elements of the form

$$(\rho_\kappa)_{n_\kappa m_\kappa}=\delta_{n_\kappa m_\kappa}(1-z_\kappa)z_\kappa^{n_\kappa}, \quad (17)$$

where z_κ is a scalar function of κ. It is clear that $[a_\kappa,\rho_{\kappa'}]=0$ and $[a_\kappa^\dagger,\rho_{\kappa'}]=0$ for $\kappa\neq\kappa'$. For density operators with this property, (13) becomes

$$F[\lambda]=\prod_\kappa\langle\exp\{ic(\hbar/2\omega_\kappa)^{1/2}(\lambda_\kappa a_\kappa+\lambda_\kappa^* a_\kappa^\dagger)\}\rangle. \quad (18)$$

The expectation value in (18) is just the characteristic function for a single mode κ and this is given by Bloch's theorem.[12] When we insert this characteristic function into (18) we obtain

$$F[\lambda]=\prod_\kappa \exp-\{c^2(\hbar/2\omega_\kappa)|\lambda_\kappa|^2\langle n_\kappa+\tfrac{1}{2}\rangle\}$$
$$=\exp\{-\sum_\kappa c^2(\hbar/2\omega_\kappa)|\lambda_\kappa|^2\langle n_\kappa+\tfrac{1}{2}\rangle\}. \quad (19)$$

From the expansion of $\mathbf{A}(x)$ according to Eq. (6), we notice that

$$\langle A_i(x)A_j(y)\rangle=\sum_\kappa c^2(\hbar/2\omega_\kappa)\{v_i(x)v_j^*(y)\langle n_\kappa+1\rangle+v_j(y)v_i^*(x)\langle n_\kappa\rangle\}. \quad (20)$$

By using (20) and the definition (12) of λ_κ we have, in dyadic notation,

$$\frac{1}{2}\int \lambda(x)\lambda(y):\langle\mathbf{A}(x)\mathbf{A}(y)\rangle dxdy=\sum_\kappa|\lambda_\kappa|^2c^2\left(\frac{\hbar}{2\omega_\kappa}\right)\langle n_\kappa+\tfrac{1}{2}\rangle. \quad (21)$$

Upon making use of (21), we can rewrite equation (19) in the form

$$F[\lambda]=\exp\left\{-\frac{1}{2}\int \lambda(x)\lambda(y):\langle\mathbf{A}(x)\mathbf{A}(y)\rangle dxdy\right\}. \quad (22)$$

This is our general result for the characteristic functional of an electromagnetic field described by a density matrix of the form (15) with the ρ_κ satisfying (17). In particular it applies to a field in thermal equilibrium for which ρ is given by (14).

III. MOMENTS OF THE RADIATION FIELD

Once the characteristic functional is known, the calculation of all moments and distribution functions is straightforward. The only difficulty arises from the noncommutativity of the field operators at different points in space-time. From the commutation relations imposed on the operators a_κ and $a_\kappa^\dagger$, it is possible to compute the commutator $[A_i(x),A_j(y)]$. When the field satisfies the transversality condition (1) it is found that the commutator vanishes only for time-like pairs of points x and y. (See, e.g., Heitler.[13]) The electric- and magnetic-field components commute more generally, but still do not commute for pairs of points lying on a light cone.

Nevertheless, we can define an nth-order moment as follows:

$$I_n^{i_1,\cdots,i_n}(x_1,\cdots x_n)\equiv\frac{1}{n!}\sum_{P(x_1\cdots x_n)}\langle A_{i_1}(x_1)\cdots A_{i_n}(x_n)\rangle. \quad (23)$$

The summation in (23) extends over the set $P(x_1\cdots x_n)$ of all permutations of $x_1\cdots x_n$. For a set of points

[12] F. Bloch, Z. Physik 74, 295 (1932).

[13] W. Heitler, *Quantum Theory of Radiation* (Clarendon Press, Oxford, England, 1954), p. 405.

$\{x_i\}$ such that $[A(x_i),A(x_j)]=0$ for all points in the set, (27) reduces to the usual nth-order moment $\langle A_{i_1}(x_1)\cdots A_{i_n}(x_n)\rangle$.

For an arbitrary set of points $\{x_i\}$, it follows from the definition of $F[\lambda]$ that I_n can be obtained by taking the nth-order functional derivative of $F[\lambda]$. That is

$$I_n{}^{i_1,\cdots i_n}(x_1,\cdots x_n)=i^{-n}\frac{\delta^n F[\lambda]}{\delta\lambda_{i_1}(x_1)\cdots\delta\lambda_{i_n}(x_n)}\bigg|_{\lambda(x)\equiv 0}. \quad (24)$$

When $F[\lambda]$ is given by (22), we find by using (24) that all moments for $n\geq 2$ can be expressed in terms of

$$I_2{}^{ij}=\tfrac{1}{2}[\langle A_i(x)A_j(y)\rangle+\langle A_j(y)A_i(x)\rangle]. \quad (25)$$

All odd moments vanish and the even moments (i.e., n even) are found to be given by

$$I_n{}^{i_1,\cdots i_n}(x_1,\cdots x_n)=\sum_{\text{partitions}}\prod_{\text{pairs}}\tfrac{1}{2}\langle A_{i_\alpha}(x_\alpha)A_{i_\gamma}(x_\gamma)+A_{i_\gamma}(x_\gamma)A_{i_\alpha}(x_\alpha)\rangle. \quad (26)$$

The summation in (26) extends over all partitions of the integers $1,\cdots,n$ into pairs, and the product extends over all pairs (α,γ) in each partition.

Let us write the symmetrized product $AB+BA$ as $\{A,B\}$. Then we have, in particular, from (26) and the vanishing of odd moments,

$$I_1{}^i(x_1)=\langle A_i(x_1)\rangle=0,$$
$$I_2{}^{i_1,i_2}(x_1,x_2)=\tfrac{1}{2}\langle\{A_{i_1}(x_1),A_{i_2}(x_2)\}\rangle,$$
$$I_3{}^{i_1,i_2,i_3}(x_1,x_2,x_3)=0,$$

$$\begin{aligned}I_4{}^{i_1,i_2,i_3,i_4}&(x_1,x_2,x_3,x_4)\\ &=\tfrac{1}{4}\langle\{A_{i_1}(x_1),A_{i_2}(x_2)\}\rangle\langle\{A_{i_3}(x_3),A_{i_4}(x_4)\}\rangle\\ &\quad+\tfrac{1}{4}\langle\{A_{i_1}(x_1),A_{i_3}(x_3)\}\rangle\langle\{A_{i_2}(x_2),A_{i_4}(x_4)\}\rangle\\ &\quad+\tfrac{1}{4}\langle\{A_{i_1}(x_1),A_{i_4}(x_4)\}\rangle\langle\{A_{i_2}(x_2),A_{i_3}(x_3)\}\rangle\\ &=I_2{}^{i_1i_2}(x_1,x_2)I_2{}^{i_3i_4}(x_3,x_4)+I_2{}^{i_1i_3}(x_1x_3)I_2{}^{i_2i_4}(x_2,x_4)\\ &\quad+I_2{}^{i_1,i_4}(x_1,x_4)I_2{}^{i_2,i_3}(x_2,x_3). \end{aligned} \quad (27)$$

The basic quantity $I_2{}^{i_1i_2}(x_1,x_2)$ can be obtained from (6). For a cubical domain of volume V with periodic boundary conditions

$$\mathbf{u}_\mathbf{k}(\lambda)=V^{-1/2}\mathbf{e}^{(\lambda)}e^{i\mathbf{k}\cdot\mathbf{r}}. \quad (28)$$

The vectors $\mathbf{e}^{(\lambda)}(\lambda=1,2)$ are unit polarization vectors orthogonal to $\mathbf{k}$, and $\mathbf{k}$ is a vector such that $V^{1/3}\mathbf{k}/2\pi$ has non-negative integers as components. Here and hereafter the index κ is replaced by a double label consisting of the vector $\mathbf{k}$ and the polarization index λ. We shall write ω_k instead of ω_κ since $\omega_\kappa=kc$ is independent of the polarization index λ and of the direction of $\mathbf{k}$.

On the basis of (6), (23), and (28), we have

$$I_2{}^{i_1i_2}(x_1,x_2)=c^2(2V)^{-1}\sum_{k,\lambda}\frac{\hbar}{2\omega_k}\langle n_{\mathbf{k}\lambda}+\tfrac{1}{2}\rangle\times(e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}+e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)})e_{i_1}{}^{(\lambda)}e_{i_2}{}^{(\lambda)}. \quad (29)$$

Here $e_i{}^{(\lambda)}$ denotes the ith component of $e^{(\lambda)}$, and $(x_1-x_2)=(\mathbf{r},t)$. In the limit $V\to\infty$, the sum $\sum_\mathbf{k}$ becomes $(2\pi)^{-3}\int d\mathbf{k}$, and (29) becomes

$$I_2{}^{i_1i_2}(x_1,x_2)=\tfrac{1}{2}c^2\hbar(2\pi)^{-3}\int d\mathbf{k}\sum_\lambda\langle n_{\mathbf{k}\lambda}+\tfrac{1}{2}\rangle(2\omega_k)^{-1}\times(e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}+e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)})e_{i_1}{}^{(\lambda)}e_{i_2}{}^{(\lambda)}. \quad (30)$$

The two point correlation function of the electric- and magnetic-field components can be obtained from (30) by means of the defining relations (2) and (3). If $\langle n_{\mathbf{k}\lambda}\rangle=\langle n_\mathbf{k}\rangle$, as is the case for thermal equilibrium, we can make use of the relation

$$\sum_{\lambda=1,2}e_i{}^{(\lambda)}e_j{}^{(\lambda)}=\delta_{ij}-k_ik_jk^{-2}, \quad (31)$$

where $k^2=\sum_i k_i^2$. We then find

$$\langle\{E_i(x_1),E_j(x_2)\}\rangle=\frac{\hbar}{(2\pi)^3}\int\omega_k d\mathbf{k}\langle n_\mathbf{k}+\tfrac{1}{2}\rangle\times e^{i\mathbf{k}\cdot\mathbf{r}}\cos\omega_k t\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right), \quad (32)$$

$$\langle\{B_i(x_1),B_j(x_j)\}\rangle=c^2\langle\{E_i(x_1),E_j(x_2)\}\rangle, \quad (33)$$

$$\langle\{E_i(x_1),B_j(x_2)\}\rangle=\frac{c\hbar}{(2\pi)^3}\int\omega_k d\mathbf{k}\langle n_\mathbf{k}+\tfrac{1}{2}\rangle\times e^{i\mathbf{k}\cdot\mathbf{r}}\cos\omega_k t(\epsilon_{ijl}k_l). \quad (34)$$

For thermal equilibrium it follows from (14) that

$$\langle n_\mathbf{k}\rangle=(e^{\hbar k\omega_k}-1)^{-1}. \quad (35)$$

In this case the above formulas can be explicitly evaluated. For $i\neq j$, we find from (32) that $\langle\{E_i,E_j\}\rangle=0$. For $i=j$ and E_i the component of $\mathbf{E}$ parallel to $\mathbf{r}$, we shall call the moment defined by the left side of (32) $\mathcal{E}_{\text{long}}(\mathbf{r},t)$. Then (32) yields

$$\mathcal{E}_{\text{long}}(\mathbf{r},t)\equiv\langle\{E_i(x_1),E_i(x_2)\}\rangle=\frac{c\hbar}{\pi^2r^2}\int_0^\infty kdk\frac{\cos kct}{e^{\alpha k}-1}\times\left[\frac{\sin kr}{kr}-\cos kr\right]+D'(x). \quad (36)$$

Here $\alpha=\hbar\beta c$, and

$$D'(x)\equiv\frac{ctr}{2\pi^2r^2}\int_0^\infty kdk\cos kct\left[\frac{\sin kr}{kr}-\cos kr\right] \quad (37)$$

is a singular contribution from the vacuum fluctuations. Equation (36) can be rewritten as

$$\mathcal{E}_{\text{long}}=\frac{c\hbar}{4\pi\alpha r^3}\left(1-\frac{r\partial}{\partial r}\right)\times\left[L\left(\frac{\pi}{\alpha}(r+ct)\right)+L\left(\frac{\pi}{\alpha}(r-ct)\right)\right]+D'(x). \quad (38)$$

Here $L(\tau)$ is the Langevin function

$$L(\tau)=\frac{2\alpha}{\pi}\int_0^\infty \frac{\sin(\pi^{-1}k\alpha\tau)}{(e^{\alpha k}-1)}dk=\coth\tau-\frac{1}{\tau}. \quad (39)$$

For $i=j$ and E_i the component of $\mathbf{E}$ perpendicular to $\mathbf{r}$, let us call the moment defined by the left side of (32) $\mathcal{E}_{\text{lat}}(\mathbf{r},t)$. Then (32) yields

$$\mathcal{E}_{\text{lat}}(\mathbf{r},t)=\langle\{E_i(x_1),E_i(x_2)\}\rangle =\frac{\hbar c}{2\pi^2}\int_0^\infty k^3dk\frac{\cos kct}{e^{\alpha k}-1} \times\left[\frac{\sin kr}{kr}-\frac{1}{k^2r^2}\left(\frac{\sin kr}{kr}-\cos kr\right)\right]+D''(x). \quad (40)$$

Here

$$D''(x)=\frac{\hbar c}{4\pi^2}\int_0^\infty k^3dk \times\cos kct\left[\frac{\sin kr}{kr}-\frac{1}{k^2r^2}\left(\frac{\sin kr}{kr}-\cos kr\right)\right] \quad (41)$$

is another singular contribution arising from the same vacuum fluctuations. Equation (40) can be rewritten as

$$\mathcal{E}_{\text{lat}}(\mathbf{r},t)=\left(1+\frac{r}{2}\frac{\partial}{\partial r}\right)\mathcal{E}_{\text{long}}(\mathbf{r},t) =\left(1+\frac{r}{2}\frac{\partial}{\partial r}\right)\left\{\frac{\hbar c}{4\pi\alpha r^3}\left(1-r\frac{\partial}{\partial r}\right) \times\left[L\left(\frac{\pi}{\alpha}(r+ct)\right)+L\left(\frac{\pi}{\alpha}(r-ct)\right)\right]\right\}+D''(x). \quad (42)$$

Apart from the singular terms $D'(x)$ and $D''(x)$, our formulas are the same, to within a numerical coefficient, as those of Sarfatt[1] and Bourett.[2]

A simple form can also be obtained for $\langle\{E_i(x_1), B_j(x_2)\}\rangle$ for E_i and B_j components of $\mathbf{E}$ and $\mathbf{B}$ perpendicular to $\mathbf{r}$. In that case, Eq. (34) yields

$$\langle\{E_i(x_1)^jB_j(x_2)\}\rangle=\frac{c^2\hbar}{4\pi^2r}\int_0^\infty k^2dk\cos kct\left(\frac{\sin kr}{kr}-\cos kr\right) \times[(e^{\alpha k}-1)^{-1}+\tfrac{1}{2}]. \quad (43)$$

From (34) we see that if $i=j$, then $\langle\{E_i(x_1),B_i(x_2)\}\rangle=0$.

Another case of interest is that of a semi-infinite domain bounded by a perfectly conducting wall at $x=0$. We will consider first a finite cubical domain of volume V with two perfectly conducting walls at $x=0$ and $x=V^{1/3}$, and later let $V\to\infty$. We assume that the field is periodic with period $V^{1/3}$ in the directions parallel to the conducting walls. The normalized eigenfunctions corresponding to the eigenvalue $\omega_k=kc$ are then

$$\mathbf{u}_{\mathbf{k}\lambda}=(2V)^{-1/2}\{\mathbf{n}(\mathbf{n}\cdot\mathbf{e}^{(\lambda)})(e^{i\mathbf{k}\cdot\mathbf{r}}+e^{i\cdot\mathbf{k}'\cdot\mathbf{r}}) +\mathbf{n}\times(\mathbf{e}^{(\lambda)}\times\mathbf{n})(e^{i\mathbf{k}\cdot\mathbf{r}}-e^{i\mathbf{k}'\cdot r})\}. \quad (44)$$

Here $\mathbf{n}$ is the unit vector normal to the conducting plane, and $\mathbf{k}'=\mathbf{k}-2(\mathbf{n}\cdot\mathbf{k})\mathbf{n}$. The vector $\mathbf{k}$ is restricted to those of the previous values for which $(\mathbf{n}\cdot\mathbf{k})<0$. Since $\mathbf{n}\times(\mathbf{e}^{(\lambda)}\times\mathbf{n})=\mathbf{e}^{(\lambda)}-(\mathbf{n}\cdot\mathbf{e}^{(\lambda)})\mathbf{n}$, $\mathbf{u}_{\mathbf{k}\lambda}$ may be rewritten as

$$\mathbf{u}_{\mathbf{k}\lambda}=(2V)^{-1/2}\{\mathbf{n}(\mathbf{n}\cdot\mathbf{e}^{(\lambda)})2e^{i\mathbf{k}'\cdot\mathbf{r}}+\mathbf{e}^{(\lambda)}(e^{i\mathbf{k}\cdot\mathbf{r}}-e^{i\mathbf{k}\cdot\mathbf{r}})\}. \quad (45)$$

From (6) and (27) we have

$$\langle\{A_i(x_1),A_j(x_2)\}\rangle=\hbar c^2\sum_k\frac{\langle n_k+\frac{1}{2}\rangle}{2\omega_k}[u_{\mathbf{k}\lambda,i}(\mathbf{r}_1)u^*_{\mathbf{k}\lambda,j}(\mathbf{r}_2)e^{-i\omega t}+u^*_{\mathbf{k}\lambda,i}(\mathbf{r}_1)u_{\mathbf{k}\lambda,j}(\mathbf{r}_2)e^{i\omega t}]=2I_2{}^{ij}(x_1,x_2). \quad (46)$$

We have chosen a coordinate system such that $\mathbf{n}$ lies along the x axis. Let us assume, as in blackbody radiation, $\langle n_{\mathbf{k}\lambda}\rangle=\langle n_{\mathbf{k}}\rangle$ and then consider separately the two possibilities $i=j$ and $i\neq j$.

First suppose $i=j$. Then Eq. (46) becomes

$$I_2{}^{ii}(x_1,x_2)=c^2\hbar(2V)^{-1}\,\mathrm{Re}\sum_{\mathbf{k}}\frac{\langle n_k+\frac{1}{2}\rangle}{2\omega_k}\left(1-\frac{k_i{}^2}{k^2}\right)[e^{i\mathbf{k}\cdot\mathbf{r}}+e^{i\mathbf{k}'\cdot\mathbf{r}}+(2\delta_{in}-1)(e^{i\mathbf{k}\cdot\mathbf{r}_1-i\mathbf{k}'\cdot\mathbf{r}_2}+e^{i\mathbf{k}'\cdot\mathbf{r}_1-i\mathbf{k}\cdot\mathbf{r}_2})]e^{-i\omega_kt}. \quad (47)$$

In the limit $V\to\infty$ the sum $V^{-1}\sum_{\mathbf{k}}$ becomes $(2\pi)^{-3}\int d\mathbf{k}$ where integration is performed over all $\mathbf{k}$ such that $(\mathbf{n}\cdot\mathbf{k})<0$. From (47) we see that $I_2{}^{ii}(x_1,x_2)$ can be considered as a sum of two terms P^{ii} and Q^{ii};

$$I_2{}^{ii}=P^{ii}(x_1,x_2)+Q^{ii}(x_1,x_2). \quad (48)$$

P^{ii} corresponds to what we have previously found in the absence of conducting walls and Q^{ii} results from the presence of the conducting walls. P^{ii} is given by

$$P^{ii}(x_1,x_2)\equiv\frac{c^2\hbar}{2}\,\mathrm{Re}(2\pi)^{-3}\int_{(\mathbf{n}\cdot\mathbf{k})<0}d\mathbf{k}\frac{\langle n_k+\frac{1}{2}\rangle}{2\omega_k}\left(1-\frac{k_i^2}{k^2}\right)(e^{i\mathbf{k}\cdot\mathbf{r}}+e^{i\mathbf{k}'\cdot\mathbf{r}})e^{-i\omega_kt} =\frac{c\hbar}{2(2\pi)^2}\int_0^\infty kdk\,\cos\omega_kt\langle n_k+\tfrac{1}{2}\rangle\left[\frac{(A+B)}{2}\frac{\sin kr}{kr}-\frac{B}{k^2r^2}\left(\frac{\sin kr}{kr}-\cos kr\right)\right]. \quad (49)$$

Here $A=[1+(r_i/r)^2]$ and $B=[1-3(r_i/r)^2]$, where r is the magnitude of $\mathbf{r}$.

To obtain an explicit expression for $Q^{ii}(x_1,x_2)$ it is convenient to specify the orientation of $\mathbf{r}$. Let us first suppose $\mathbf{r}$ is parallel to $\mathbf{n}$. Then we find from (47)

$$Q_{(\mathbf{r}\|\mathbf{n})}{}^{ii}(x_1,x_2)=\frac{c\hbar}{2(2\pi)^2}(2\delta_{in}-1)\int_0^\infty kdk\ \cos\omega_k t\langle n_k+\tfrac{1}{2}\rangle \times\left\{\left(\frac{A+B}{2}\right)\frac{\sin k(r-2s)}{k(r-2s)}-\frac{B}{k^2(r-2s)^2}\left[\frac{\sin k(r-2s)}{k(r-2s)}-\cos k(r-2s)\right]\right\}, \quad (50)$$

where s is the perpendicular distance between $\mathbf{r}_1$ and the wall. A and B are defined as above.

For $\mathbf{r}$ perpendicular to $\mathbf{n}$, (47) yields:

(a) $(\hat{\imath}\|\mathbf{n})$

$$Q^{nn}(x_1,x_2)=(2\pi)^{-2}\hbar c\int_0^\infty kdk\left(1+k^{-2}\frac{\partial^2}{\partial s^2}\right)D(k,r,s)\langle n_k+\tfrac{1}{2}\rangle,$$

(b) $(\hat{\imath}\|\mathbf{r})$

$$Q^{rr}(x_1,x_2)=(2\pi)^{-2}\hbar c\int_0^\infty kdk\left(-1-k^{-2}\frac{\partial^2}{\partial r^2}\right)D(k,r,s)\langle n_k+\tfrac{1}{2}\rangle, \quad (51)$$

(c) $(\hat{\imath}\perp\mathbf{n}\perp\mathbf{r})$

$$Q^{\perp\perp}(x_1,x_2)=(2\pi)^{-2}\hbar c\int_0^\infty \frac{dk}{k}\left(\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial s^2}\right)D(k,r,s)\langle n_k+\tfrac{1}{2}\rangle.$$

Here $\hat{\imath}$ is a unit vector in the direction of the i axis. The superscripts nn, rr, and $\perp\perp$ indicate that both the i and j components of the field have been chosen in the directions parallel to $\mathbf{n}$ and $\mathbf{r}$ in the first two cases, and perpendicular to both $\mathbf{n}$ and $\mathbf{r}$ in the third. $D(k,r,s)$ is given by

$$D(k,r,s)\equiv\cos kct\int_{-1}^1 d\mu\ \cos kr\mu\ J_0(ks(1-\mu^2)^{1/2}). \quad (52)$$

Considering now the case $i\neq j$, we find that $I_2{}^{ij}(x_1,x_2)=0$ unless $\hat{\imath}$ is parallel to $\mathbf{n}$ and $\hat{\jmath}$ parallel to $\mathbf{r}$. In that case we find

$$I_{(\mathbf{r}\perp\mathbf{n})}{}^{nr}(x_1,x_2)=-\frac{\partial}{\partial r}\frac{\partial}{\partial s}\frac{\hbar c}{(2\pi)^2}\int_0^\infty\frac{dk}{k^3}\langle n_k+\tfrac{1}{2}\rangle D(k,r,s). \quad (53)$$

It is interesting to note that in computing the characteristic functional we have made no use of the specific nature of the electromagnetic field. In fact, the same functional will describe any linear boson field whose density matrix has the required form. The only modification which must be made is that the eigenfunctions $\mathbf{u}_\kappa(\mathbf{r})$ defined by (4) must be replaced by the eigenfunctions of the equation which describes the field in question. For example, let us consider the thermal equilibrium of a scalar meson field φ satisfying the Klein-Gordon equation $(\nabla^2-c^{-2}\partial_t{}^2-m^2)\varphi(\mathbf{r},t)=0$.

The characteristic functional for this field is [cf., (22)]

$$F[\lambda]=\exp\left\{-\frac{1}{2}\int\lambda(x)\lambda(y)\langle\varphi(x)\varphi(y)\rangle dxdy\right\}, \quad (54)$$

where $\lambda(x)$ is now an arbitrary real scalar function. As in the case of the electromagnetic field, all moments can be expressed in terms of the second-order moment, as in (26). It is therefore useful to compute the second-order moment explicitly. For a cubical domain of volume V with periodic boundary conditions, the Klein-Gordon equation has the plane wave solutions $e^{-i\omega_k t}u_k(\mathbf{r}) = V^{-1/2}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_k t}$. By using them, and letting V become infinite, we find the two point correlation function $I_2=\frac{1}{2}\langle\{\varphi(x),\varphi(y)\}\rangle$ to be

$$I_2(x,y)=\hbar c(4\pi r)^{-1}\int_0^\infty\langle n_\kappa+\tfrac{1}{2}\rangle\frac{kdk}{(k^2+m^2)^{1/2}} \times[\sin(kr+\omega_k t)+\sin(kr-\omega_k t)]. \quad (55)$$

When $m=0$ and $\langle n_k\rangle=(e^{-\beta\hbar ck}-1)^{-1}$ we can evaluate this integral and express the result in terms of Langevin functions. If we ignore singular contributions on the light cone due to vacuum fluctuations, we obtain from (55)

$$I_2(x,y)=\frac{\hbar c}{16\pi\alpha r}\left[L\left(\frac{\pi}{\alpha}(r+ct)\right)+L\left(\frac{\pi}{\alpha}(r-ct)\right)\right]. \quad (56)$$

IV. DISTRIBUTION FUNCTIONS

If $[A_{i_\alpha}(x_\alpha),A_{i_\beta}(x_\beta)]=0$ for all α, $\beta=1,\cdots,n$, then it is possible to introduce the distribution functions $P_n(A_{i_1}'(x_1),A_{i_2}'(x_2),\cdots,A_{i_n}'(x_n))$. These are defined to give the joint probability of the operators $A_{i_1}(x_1)$, $\cdots,A_{i_n}(x_n)$ taking on the values $A_{i_1}'(x_1),\cdots,A_{i_n}'(x_n)$, respectively. From them, all moments can be obtained in the usual way. That is,

$$\langle f(A_{i_1}'(x_1),A_{i_2}'(x_\alpha),\cdots,A_{i_n}'(x_n))\rangle =\int\cdots\int P_n(A_{i_1}'(x_1),\cdots,A_{i_n}'(x_n)) \times f(A_{i_1}'(x_1),\cdots,A_{i_n}'(x_n))dA_{i_1}'(x_1)\cdots dA_{i_n}'(x_n). \quad (57)$$

Here f is an arbitrary function of the components A_{i_α}'.

Because of the commutativity of all the operators in question, it is possible to choose a representation in which all $A_{i_\alpha}(x_\alpha)$ are diagonal. In that case, the expectation value of a product of operators $A_{i_1}(x_1)\cdots A_{i_n}(x_n)$ obtained by taking the trace, as in (9), is the same as the expectation value obtained from (23). The distribution functions $P_n(A_{i_1}'(x_1),\cdots,A_{i_n}'(x_n))$ can be obtained from the characteristic functional $F[\lambda]$ as follows. Let

$$\boldsymbol{\lambda}(x)=\sum_{\alpha=1}^{n}\lambda_\alpha\delta(x-x_\alpha)\mathbf{n}_{i_\alpha}, \quad (58)$$

where $\mathbf{n}_{i_\alpha}$ is the unit vector corresponding to i_α. Then

$$F[\lambda]=\langle\exp\{i[\lambda_1A_{i_1}(x_1)+\cdots+\lambda_nA_{i_n}(x_n)]\}\rangle \equiv\mathrm{tr}\{\rho\exp[i\sum_\alpha\lambda_\alpha A_{i_\alpha}(x_\alpha)]\}. \quad (59)$$

Using the formula for $F[\lambda]$ given by (22), we find

$$\langle\exp\{i[\lambda_1A_{i_1}(x_1)+\cdots+\lambda_nA_{i_n}(x_n)]\}\rangle =\exp\{-\tfrac{1}{2}\sum_{\alpha,\beta=1}^{n}\lambda_\alpha\lambda_\beta\langle A_{i_\alpha}(x_\alpha)A_{i_\beta}(x_\beta)\rangle\}. \quad (60)$$

But $\langle\exp\{i[\lambda_1A_{i_1}(x_1)+\cdots+\lambda_nA_{i_n}(x_n)]\}\rangle$ can also be obtained from (23):

$$\langle\exp\{i[\lambda_1A_1(x_1)+\cdots+\lambda_nA_{i_n}(x_n)]\}\rangle =\int\cdots\int dA_{i_1}'(x_1)\cdots dA_{i_n}'(x_n) \times P_n(A_{i_n}'(x_1),\cdots,A_{i_n}'(x_n))\exp[i\sum_\alpha\lambda_\alpha A_{i_\alpha}'(x_\alpha)]. \quad (61)$$

Therefore, the distribution function $P_n(A_{i_n}'(x_1),\cdots,A_{i_n}'(x_n))$ can be obtained by taking the Fourier transform of (26). We have

$$P_n(A_{i_1}'(x_1),\cdots,A_{i_n}'(x_n)) =\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}d\lambda_1\cdots\int_{-\infty}^{\infty}d\lambda_n\exp[-i\sum_{\alpha=1}^{n}\lambda_\alpha A_{i_\alpha}'(x_\alpha)] \times\exp[-\tfrac{1}{2}\sum_{\alpha,\beta=1}^{n}\lambda_\alpha\lambda_\beta\langle A_{i_\alpha}(x_\alpha)A_{i_\beta}(x_\beta)\rangle]. \quad (62)$$

This integral can be evaluated, being simply the Fourier transform of a multivariate Gaussian. The answer can be written as

$$P_n(A_{i_1}'(x_1),\cdots,A_{i_n}'(x_n)) =\frac{(2\pi)^{-n/2}}{(\det G_{\alpha\beta})^{1/2}} \times\exp\{-\tfrac{1}{2}\sum_{\alpha,\beta=1}^{n}(G^{-1})_{\alpha\beta}A_{i_\alpha}'(x_\alpha)A_{i_\beta}'(x_\beta)\}, \quad (63)$$

where G is a matrix with components

$$G_{\alpha\beta}=\langle A_{i_\alpha}(x_\alpha)A_{i_\beta}(x_\beta)\rangle,$$

and $(G^{-1})_{\alpha\beta}$ is the (α,β) element of the inverse matrix.

For $n=1$, G is a scalar

$$G=\langle A_i^2(x)\rangle. \quad (64)$$

For a uniform, isotropic system in equilibrium,

$$\langle A_i^2(x)\rangle=\tfrac{1}{3}\langle A^2(0)\rangle. \quad (65)$$

Then G is independent of the choice of component i and space-time vector x, and (63)–(65) yield

$$P[A_i'(x)]=(\tfrac{2}{3}\pi\langle A^2\rangle)^{-1/2}\exp[-\tfrac{3}{2}A_i'^2(x)/\langle A^2\rangle]. \quad (66)$$

For $n=2$

$$G=\begin{pmatrix}\langle A_i^2(x)\rangle & \langle A_i(x)A_j(y)\rangle\\ \langle A_i(x)A_j(y)\rangle & \langle A_j^2(y)\rangle\end{pmatrix} \quad (67)$$

and

$$G^{-1}=\frac{1}{\langle A_i^2(x)A_j^2(y)\rangle-\langle A_i(x)A_j(y)\rangle^2} \times\begin{pmatrix}\langle A_i^2(x)\rangle & -\langle A_i(x)A_j(y)\rangle\\ -\langle A_i(x)A_j(y)\rangle & \langle A_j^2(y)\rangle\end{pmatrix}. \quad (68)$$

Again, for a uniform isotropic system in equilibrium G depends only on $(i-j)$, and $(x-y)$. That is

$$G=\begin{pmatrix}g(0) & g(x-y)\delta_{ij}\\ g(x-y)\delta_{ij} & g(0)\end{pmatrix}, \quad (69)$$

where $g(x)=\langle A_1(x)A_1(0)\rangle$. The distribution function is then

$$P_2[A_i(x),A_j(y)]=\frac{1}{2\pi(g^2(0)-\delta_{ij}g^2(x-y))^{1/2}}\exp\left\{-\frac{1}{2}\frac{g(0)[A_i^2(x)+A_j^2(y)]-2\delta_{ij}g(x-y)A_i(x)A_j(y)}{g^2(0)-\delta_{ij}g^2(x)}\right\}. \quad (70)$$

It is possible to extend the definition of the distribution functions $P_n[A_{i_1}'(x_1),\cdots,A_{i_n}'(x_n)]$ to include those points where $[A_{i_\alpha}(x_\alpha),A_{i_\beta}(x_\beta)]\neq 0$. This is done simply by replacing the product $\langle A_{i_\alpha}(x_\alpha)A_{i_\beta}(x_\beta)\rangle$ by the corresponding symmetrized product $\frac{1}{2}\langle\{A_{i_\alpha}(x_\alpha), A_{i_\beta}(x_\beta)\}\rangle$ wherever it appears. Then Eq. (63) remains unchanged provided that the matrix $G_{\alpha\beta}$ is appropriately symmetrized. That is, $G_{\alpha\beta}=\langle\frac{1}{2}\{A_{i_\alpha}(x_\alpha),A_{i_\beta}(x_\beta)\}\rangle$. It is then possible to compute all the symmetrized moments from these distribution functions at all points in space-time. Caution must be taken, however, in the interpretation of the distribution function at those points where noncommutativity occurs. Since it does not make sense to speak of joint probability distributions for noncommuting operators, the usual interpretation of P_n must be abandoned at these points.

Nevertheless, since the above extension of the definition of P_n permits the calculation of the symmetrized moments everywhere, the full set of functions $\{P_n\}$ provides sufficient information to re-obtain the characteristic functional. This is because the functional Taylor expansion of $F[\lambda]$ involves only symmetrized moments. That is

$$F[\lambda]=\sum_{n=0}^{\infty}\frac{i^n}{n!}\sum_{i_1,\cdots,i_n}\int\lambda_{i_1}(x_1)\cdots\lambda_{i_n}(x_n)\times\frac{\delta^nF[\lambda]}{\delta\lambda_{i_1}(x_1)\cdots\delta\lambda_{i_n}(x_n)}\bigg|_{\lambda=0}$$
$$=\sum_{n=0}^{\infty}\frac{i^n}{n!}\sum_{i_1,\cdots,i_n}\int\lambda_{i_1}(x_1)\cdots\lambda_{i_n}(x_n)\times I_n^{(i_1,\cdots i_n)}(x_1,\cdots,x_n),\quad(71)$$

where $I_n^{i_1,\cdots i_n}(x_1,\cdots x_n)$ are defined by Eq. (27). Consequently, the full set of distribution functions $\{P_n\}$, defined everywhere, provides a complete description of the system.

For comparison with experiment, it is convenient to obtain the distribution functions of the electric- and magnetic-field components. For the electric field, Eq. (58) is replaced by

$$\lambda(x)=-\frac{1}{c}\frac{\partial}{\partial t}\sum_{\alpha=1}^{N}\lambda_\alpha\delta(x-x_\alpha)\mathbf{n}_{i_\alpha},\quad(72)$$

while for the magnetic field we would take

$$\lambda(x)=-\nabla\times\sum_{\alpha=1}^{N}\lambda_\alpha\delta(x-x_\alpha)\mathbf{n}_{i_\alpha}.\quad(73)$$

For any mixture of the two, the appropriate combination of (72) and (73) would be employed. In the same way as above it is found that the distribution functions of the electric- and magnetic-field components are the same multivariate Gaussian functions with the appropriate replacement of A_{i_α} by E_{i_α} or H_{i_α}. Because of the wider domain of commutativity of the electric- and magnetic-field operators, the distribution functions of these fields have a correspondingly wider range over which they can be physically interpreted.

V. OBTAINING THE DENSITY MATRIX FROM THE CHARACTERISTIC FUNCTIONAL

In Sec. II we have derived the characteristic functional for a system described by a density matrix of the form (17). In this section, we will solve the inverse problem for an arbitrary system. We will show that, given the characteristic functional $\langle F[\lambda]\rangle=\langle\exp[i\int\boldsymbol{\lambda}(x)\cdot\mathbf{A}(x)dx]\rangle$, it is possible, by judicious choice of $\boldsymbol{\lambda}(x)$, to obtain the density matrix ρ.

It will be convenient for this purpose to use the basis employed by Glauber.[5] In particular, we will obtain the matrix elements $\langle\alpha|\rho|\beta\rangle$, where $|\alpha\rangle=\prod_\kappa|\alpha_\kappa\rangle$ and $|\beta\rangle=\prod_\kappa|\beta_\kappa\rangle$. The kets $|\alpha_\kappa\rangle$ and $|\beta_\kappa\rangle$ are eigenstates of the annihilation operators a_κ. That is,

$$a_\kappa|\alpha_\kappa\rangle=\alpha_\kappa|\alpha_\kappa\rangle,\quad a_\kappa|\beta_\kappa\rangle=\beta_\kappa|\beta_\kappa\rangle,\quad(74)$$

where α_κ and β_κ are complex numbers. It follows that $\langle\alpha_\kappa|$ and $\langle\beta_\kappa|$ are eigenstates of the creation operator $a_\kappa^\dagger$:

$$\langle\alpha_\kappa|a_\kappa^\dagger=\langle\alpha_\kappa|\alpha_\kappa^*,\quad\langle\beta_\kappa|a_\kappa^\dagger=\langle\beta_\kappa|\beta_\kappa^*.\quad(75)$$

Glauber has shown that these states, although not orthogonal, do provide a complete basis in terms of which any state of the system can be expressed. We will call this representation the "α" representation. In order to simplify the calculations, we will consider a single mode of oscillation. That is, we will show how to find $\langle\alpha_\kappa|\rho|\beta_\kappa\rangle$. The results for a full set of modes are obtained by straightforward generalization of the results for a single mode.

The matrix elements of ρ in any other representation can be obtained from the appropriate transformation formula. In particular, the matrix elements $\rho_{m_\kappa n_\kappa}\equiv\langle m_\kappa|\rho|n_\kappa\rangle$, (where $|m_\kappa\rangle$ and $|n_\kappa\rangle$ are eigenstates of the photon-number operators $a_\kappa^\dagger a_\kappa$), are obtained as follows:

$$\rho_{m_\kappa n_\kappa}=\frac{1}{\pi^2}\int d^2\alpha_\kappa\int d^2\beta_\kappa\langle\alpha_\kappa|\rho|\beta_\kappa\rangle\times\exp[-\tfrac{1}{2}|\alpha_\kappa|^2-\tfrac{1}{2}|\beta_\kappa|^2]\frac{\alpha_\kappa^{\,n}}{\sqrt{n!}}\frac{(\beta_\kappa^*)^m}{\sqrt{m!}}.\quad(76)$$

By $\int d^2\alpha$ we understand the double integration

$$\int_{-\infty}^{\infty}d(\mathrm{Re}\alpha)\int_{-\infty}^{\infty}d(\mathrm{Im}\alpha);$$

that is, integration is over the entire complex plane.

In the "α" representation

$$\left\langle \exp\left(i\int \boldsymbol{\lambda}(x)\cdot\mathbf{A}(x)dx\right)\right\rangle$$

$$\equiv \mathrm{tr}\left[\rho \exp\left(i\int \boldsymbol{\lambda}(x)\cdot\mathbf{A}(x)dx\right)\right]$$

$$=\frac{1}{\pi}\int d^2\alpha\langle\alpha|\rho \exp\left(i\int \boldsymbol{\lambda}(x)\cdot\mathbf{A}(x)dx\right)|\alpha\rangle . \quad (77)$$

Let us take $\boldsymbol{\lambda}(x)$ to be

$$\begin{aligned}\boldsymbol{\lambda}(x)&=i(\gamma_1{}^*\boldsymbol{\lambda}_{\kappa+}-\gamma_1\boldsymbol{\lambda}_{\kappa-})+(\gamma_2\boldsymbol{\lambda}_{\kappa+}+\gamma_2{}^*\boldsymbol{\lambda}_{\kappa-})\\&=\boldsymbol{\lambda}^*(x),\end{aligned} \quad (78)$$

where γ_1 and γ_2 are arbitrary complex numbers, and $\lambda_\kappa\pm$ are functions defined by

$$\boldsymbol{\lambda}_{\kappa}\pm(x)=\mp\lim_{\epsilon\to 0}c^{-1}(2\omega_\kappa/\hbar)^{1/2}\frac{1}{2\pi i}\frac{u_\kappa\pm(r)}{t_0-t\mp i\epsilon}\exp(\pm i\omega_\kappa t_0)$$

$$=\boldsymbol{\lambda}_\kappa\mp{}^*(x). \quad (79)$$

Here $u_\kappa^-(r)\equiv u_\kappa(r)$ and $u_\kappa^\dagger(r)\equiv u_\kappa^*(r)$. Because

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\alpha\tau}d\tau}{\tau-i\epsilon}=1,\alpha>0$$
$$=0,\alpha<0,$$

it follows from the plane wave expansion of $\mathbf{A}(x)$ (6) that

$$\begin{aligned}&\text{(a)}\quad \int\boldsymbol{\lambda}_{\kappa+}(x)\cdot\mathbf{A}(x)dx=a_\kappa,\\&\text{(b)}\quad \int\boldsymbol{\lambda}_{\kappa-}(x)\cdot\mathbf{A}(x)dx=a_\kappa{}^\dagger.\end{aligned} \quad (80)$$

From our choice of $\boldsymbol{\lambda}(x)$ [Eq. (27)], we then have

$$i\int\boldsymbol{\lambda}(x)\cdot\mathbf{A}(x)dx=(\gamma_1a_\kappa{}^\dagger-\gamma_1{}^*a_\kappa)+i(\gamma_2a_\kappa+\gamma_2{}^*a_\kappa{}^\dagger). \quad (81)$$

We now use the fact that, if A and B are operators such that $[A,B]$ is a "c" number, then

$$e^{A+B}=e^Ae^Be^{-1/2[A,B]}. \quad (82)$$

Then we can write $\langle\exp(i\int\boldsymbol{\lambda}(x)\cdot\mathbf{A}(x)dx)\rangle$ (now a function of γ_1 and γ_2) as follows:

$$I(\gamma_1,\gamma_2)\equiv\left\langle\exp\left(i\int\boldsymbol{\lambda}(x)\cdot\mathbf{A}(x)dx\right)\right\rangle$$

$$=\mathrm{tr}[\rho\exp(\gamma_1a_\kappa{}^\dagger-\gamma_1{}^*a_\kappa)\exp(i\gamma_2a_\kappa)$$
$$\times\exp(i\gamma_2{}^*a_\kappa{}^\dagger)]Z(\gamma_1,\gamma_2), \quad (83)$$

where

$$Z(\gamma_1,\gamma_2)=\exp\{\tfrac{1}{2}[(\gamma_2)^2+i(\gamma_1\gamma_2+\gamma_1{}^*\gamma_2{}^*)]\}.$$

Making a cyclic permutation of the operators under the trace, we have

$$I(\gamma_1,\gamma_2)=\frac{1}{\pi}\int d^2\alpha_\kappa\langle\alpha_\kappa|\exp(i\gamma_2{}^*\alpha_\kappa{}^\dagger)\rho$$
$$\times\exp(\gamma_1a_\kappa{}^\dagger-\gamma_1{}^*a_\kappa)\exp(i\gamma_2a_\kappa)|\alpha_\kappa\rangle Z \quad (84)$$
$$=\frac{1}{\pi}\int d^2\alpha_\kappa\langle\alpha_\kappa|\rho\exp(\gamma_1a_\kappa{}^\dagger-\gamma_1{}^*a_\kappa)|\alpha_\kappa\rangle$$
$$\times\exp[i(\gamma_2\alpha_\kappa+\gamma_2{}^*\alpha_\kappa{}^*)]\rangle Z.$$

But $\exp(\gamma_1a_\kappa{}^\dagger-\gamma_1{}^*a_\kappa)$ is just the displacement operator $D(\gamma_1)$, introduced by Glauber,[6] which has the property

$$D(\gamma_1)|\alpha_\kappa\rangle=|\alpha_\kappa+\gamma_1\rangle\exp\tfrac{1}{2}(\gamma_1\alpha_\kappa{}^*-\alpha_\kappa\gamma_1{}^*). \quad (85)$$

Hence

$$I(\gamma_1,\gamma_2)=\frac{1}{\pi}\int d^2\alpha_\kappa\langle\alpha_\kappa|\rho|\alpha_\kappa+\gamma_1\rangle\exp[i(\gamma_2\alpha_\kappa+\gamma_2{}^*\alpha_\kappa{}^*)]$$
$$\times\exp[\tfrac{1}{2}(\gamma_1\alpha_\kappa{}^*-\alpha_\kappa\gamma_1{}^*)]Z(\gamma_1,\gamma_2). \quad (86)$$

Letting $\gamma_1=\beta_\kappa-\alpha_\kappa$, we can obtain $\langle\alpha_\kappa|\rho|\beta_\kappa\rangle$ by taking the Fourier transform of

$$[I(\gamma_1,\gamma_2)\exp(-\tfrac{1}{2}\gamma_1\alpha_\kappa{}^*+\tfrac{1}{2}\alpha_\kappa\gamma_1{}^*)Z^{-1}(\gamma_1,\gamma_2)].$$

That is:

$$\langle\alpha_\kappa|\rho|\beta_\kappa\rangle=\pi^{-1}\int d^2\gamma_2(I(\gamma_1,\gamma_2)$$
$$\times\exp[-\tfrac{1}{2}(\gamma_1\alpha_\kappa{}^*-\alpha_\kappa\gamma_1{}^*)]Z^{-1}(\gamma_1,\gamma_2))$$
$$\times\exp[-i(\gamma_2{}^*\alpha_\kappa{}^*+\gamma_2\alpha_\kappa)]. \quad (87)$$

Letting $h\equiv\alpha_\kappa+\beta_\kappa=h_1+ih_2$, $\gamma_1\equiv\beta_\kappa-\alpha_\kappa=g_1+ig_2$, and $\gamma_2=s_1+is_2$, Eq. (87) can be rewritten as

$$\langle\alpha_\kappa|\rho|\beta_\kappa\rangle=\pi^{-1}\exp[-\tfrac{1}{2}(\beta_\kappa\alpha_\kappa{}^*-\alpha_\kappa\beta_\kappa{}^*)]$$
$$\times\int ds_1ds_2e^{-i(h_1s_1-h_2s_2)}$$
$$\times\exp(-\tfrac{1}{2}s_1{}^2-\tfrac{1}{2}s_2{}^2)I(g_1+g_2,s_1+is_2). \quad (88)$$

Using (83) to define $I(\gamma_1,\gamma_2)$, we now have an explicit formula for obtaining the density matrix ρ from the characteristic functional $F[\lambda]$. To actually compute $\langle\alpha_\kappa|\rho|\beta_\kappa\rangle$ it is of course necessary to know $F[\lambda]$ for the system in question.

In particular, for a system described by a characteristic functional of the form

$$F[\lambda]=\exp\left(-\frac{1}{2}\int\lambda(x)\lambda(y):\langle\mathbf{A}(x)\mathbf{A}(y)\rangle\right), \tag{89}$$

we can show that the most general equilibrium density matrix is

$$\rho_{mn}=\delta_{mn}(1-z)z^n. \tag{90}$$

This corresponds to black-body radiation if we take $z=\exp(-\beta\hbar\omega)$. In order to show this, we substitute $\lambda(x)$ as defined by (78) into (89), and find

$$\begin{aligned}I(\gamma_1,\gamma_2)&=\exp(-\langle n_\kappa+\tfrac{1}{2}\rangle|\gamma_2+i\gamma_1^*|^2)\\&=\exp[-\langle n_\kappa+\tfrac{1}{2}\rangle\{s_1^2+s_2^2+g_1^2+g_2^2\\&\qquad+2(g_2s_1+g_1s_2)\}].\end{aligned} \tag{91}$$

Inserting this into (88) and performing the integration indicated, we find, after some algebraic manipulation,

$$\begin{aligned}\langle\alpha_\kappa|\rho|\beta_\kappa\rangle=\langle n_\kappa+1\rangle^{-1}&\exp[\alpha_\kappa^*\beta_\kappa\langle n_\kappa\rangle/(1+\langle n_\kappa\rangle)]\\&\times\exp[-\tfrac{1}{2}|\alpha_\kappa|^2-\tfrac{1}{2}|\beta_\kappa|^2].\end{aligned} \tag{92}$$

Using (76) to obtain $\rho_{n_\kappa m_\kappa}$, we find

$$\rho_{n_\kappa m_\kappa}=\delta_{n_\kappa m_\kappa}(1-z)z^{n_\kappa}, \tag{93}$$

where $z=\langle n_\kappa\rangle/(1+\langle n_\kappa\rangle)$.

The procedure outlined above can be immediately generalized to obtain the matrix elements of ρ corresponding to all modes of oscillation. It is simply necessary to consider an infinite set of complex numbers $\{\gamma_{1\kappa},\gamma_{2\kappa}\}$ in place of the numbers γ_1 and γ_2. Then (78) is replaced by

$$\lambda(x)=\sum_\kappa[i(\gamma_{1\kappa}^*\lambda_{\kappa+}-\gamma_{1\kappa}\lambda_{\kappa-})+(\gamma_{2\kappa}\lambda_{\kappa+}+\gamma_{2\kappa}^*\lambda_{\kappa-})]. \tag{94}$$

Equation (86) becomes

$$\begin{aligned}I(\{\gamma_{1\kappa},\gamma_{2\kappa}\})=\prod_\kappa\pi^{-1}&\int d^2\alpha_\kappa\langle\alpha_\kappa|\rho|\alpha_\kappa+\gamma_{1\kappa}\rangle\\&\times\exp[i\sum_\kappa(\gamma_{2\kappa}\alpha_\kappa+\gamma_{2\kappa}^*\alpha_\kappa^*)]\\&\times\exp\{\tfrac{1}{2}\sum_\kappa[(\gamma_{2\kappa})^2+i(\gamma_{1\kappa}\gamma_{2\kappa}+\gamma_{1\kappa}^*\gamma_{2\kappa}^*)\\&\qquad+(\gamma_{1\kappa}\alpha_\kappa^*-\alpha_\kappa\gamma_{1\kappa}^*)]\}.\end{aligned} \tag{95}$$

Then, taking $\gamma_{1\kappa}=\beta_\kappa-\alpha_\kappa$, we obtain

$$\begin{aligned}\langle\alpha|\rho|\beta\rangle\equiv\langle\{\alpha_\kappa\}|\rho|\{\beta_\kappa\}\rangle=\prod_\kappa\frac{1}{\pi}&\int d^2\alpha_\kappa I(\{\gamma_{1\kappa},\gamma_{2\kappa}\})\\&\times\exp[-i\sum_\kappa(\gamma_{2\kappa}^*\alpha_\kappa^*+\gamma_{2\kappa}\alpha_\kappa)]\\&\times\exp\{-\tfrac{1}{2}\sum_\kappa[(\gamma_{1\kappa}\alpha_\kappa^*-\alpha_\kappa\gamma_{1\kappa}^*)+|\gamma_{2\kappa}|^2\\&\qquad+i(\gamma_{1\kappa}\gamma_{2\kappa}+\gamma_{1\kappa}^*\gamma_{2\kappa}^*)]\}.\end{aligned} \tag{96}$$

For $F[\lambda]$ given by (89), we find

$$\begin{aligned}I(\{\gamma_{1\kappa},\gamma_{2\kappa}\})&=\exp(-\tfrac{1}{2}\sum_\kappa\langle n_\kappa+\tfrac{1}{2}\rangle|\gamma_{2\kappa}+i\gamma_{1\kappa}^*|^2)\\&=\prod_\kappa I(\gamma_{1\kappa},\gamma_{2\kappa})\end{aligned} \tag{97}$$

and

$$\langle\{\alpha_\kappa\}|\rho|\{\beta_\kappa\}\rangle=\prod_\kappa\langle\alpha_\kappa|\rho|\beta_\kappa\rangle. \tag{98}$$

Consequently, the full density matrix, in the occupation number representation, is given by

$$\begin{aligned}\langle\{n_\kappa\}|\rho|\{m_\kappa\}\rangle=\prod_\kappa\delta_{n_\kappa m_\kappa}&(1+\langle n_\kappa\rangle)^{-1}\\&\times[\langle n_\kappa\rangle/(1+\langle n_\kappa\rangle)]^{n_\kappa}.\end{aligned} \tag{99}$$