Moving Branch Points in j Plane and Regge-Pole Unitarity Conditions

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Moving branch points in the j plane are investigated on the basis of analysis of multiparticle terms of the unitarity condition in the t channel. A definite assumption about the form of an analytic continuation of these terms into complex j is used. It is shown that in this case in the j plane there arise branch points of the partial amplitude $f_j(t)$ corresponding to the production thresholds of two or more Regge poles with relative orbital momentum equal to -1. In the case of two zero-spin particles in the intermediate state, the partial wave has a singularity at negative integral values of the orbital momentum. Azimov has found that such singularities shift to the right if the particles in the intermediate state have nonzero spin. The branch points in the j plane result from the extension of this shift throughout the Regge trajectory. This mechanism of emergence of branch points has been indicated by Mandelstam for the case of Feynman diagrams of a certain class. The presence of these branch points at $j=j_n(t)$ where $j_n(t)=n\alpha(t/n^2)-n+1$ changes essentially the analytic properties of $f_j(t)$ in the t plane, leading to the emergence in the t plane of branch points at $t=t_n(j)$, where $t_n(j)$ is the solution of the equation $j=j_n(t)$. The discontinuity $\delta_t^{(n)}f_j(t)$ of the amplitude $f_j(t)$ on the singularity $t=t_n(j)$ corresponding to the n-Regge-pole production threshold (Regge-pole unitarity conditions) is calculated. It is shown that this discontinuity has a form similar to the conventional unitarity condition. $\delta_t^{(n)}f_j(t)=(1/2i)[f_j(t+i\epsilon)-f_j(t-i\epsilon)]$ being given by the product of the amplitudes $N_{j,n}$ of production of n Regge poles determined above and under a cut made in the t plane from the point $t=t_n(j)$. The discontinuity $\delta_t^{(n)}f_j(t)$ of the amplitude $f_j(t)$ across the cut connected with the branch point $t=t_n(j)$ is calculated at $t\to t_n(j)$. The discontinuity is shown to have the form $\delta_t^{(n)}f_j(t)=\pi B_n[t-t_n(j)]^{n-2}$. This means that the singularity of $f_j(t)$ has a logarithmic character, i.e., near it we have $f_j(t)=A_n+B_n[j-j_n(t)]^{n-2}\ln[j-j_n(t)]$, where A_n and B_n have no singularities at $j=j_n(t)$. The results obtained will be used elsewhere for analysis of the asymptotic behavior of the diffraction scattering amplitude in the region of not-large values of the momentum transfer.

I. INTRODUCTION

SEVERAL years ago it was found that the asymptotic behavior of the elastic scattering amplitude $A(s,t)$ as $s\to\infty$ can be determined by singularities[1-3] of partial-wave amplitudes $f_j(t)$ as a function of angular momentum j. Analysis of the asymptotic behavior was based on the hypothesis about a vacuum pole[4-6] whose trajectory $j=\alpha(t)$ passes at $t=0$ through the point $j=1$. The assumption about the presence of moving poles in $f_j(t)$ and in particular of a vacuum pole was natural since at integral physical j, the amplitude $f_j(t)$ has resonance poles on the unphysical sheets of the t plane, whose location depends on j. It is these resonance states that give rise to the poles of $f_j(t)$ in the j plane.

Until recently there were no reasons in evidence for the emergence in the j plane of any moving singularities except poles. Recently, however, Mandelstam[7] gave his arguments in favor of a possible emergence in relativistic theory of moving branch points resulting from singularities at integral negative j[8] and their shift[9] for nonzero spin particles. These singularities correspond to the thresholds of production of several resonance states (Regge poles) with integral negative orbital momenta $L=-1, -2,\cdots$. They can be regarded as continuation to complex j of the branch points which are located at integral physical j on the unphysical sheets of the t plane and which correspond to the thresholds of several resonances[10] with physical values of j.

The presence of moving branch points considered by Mandelstam cannot be regarded as rigorously proved. However, the arguments in favor of this point[7] are so serious that it seems necessary to us to investigate in detail these branch points and their effect on the asymptotic behavior of the amplitude.

This paper is the first part of this investigation. The branch points were obtained by Mandelstam from an asymptotic analysis of a class of perturbation diagrams. From Ref. 7 it is clear that these branch points are connected with multiparticle intermediate states. Therefore, the investigation of these singularities requires analysis

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of multiparticle unitarity conditions analytically continued into complex j.

This analytic continuation involves considerable difficulties and the problem has not yet been solved. An assumption is used in this paper about the form of this continuation near those values of j which are singular for the amplitude $f_j(t)$.

To understand the structure of this continuation, let us consider the terms of the unitarity condition [for $\text{Im}f_j(t)$] corresponding to the production in the intermediate state of two particles one of which has nonzero spin σ (and mass M). These terms can be written as

$$\sum_{m=-\sigma}^{\sigma} \frac{2p(t,M^2,\mu^2)}{t^{1/2}} \frac{\Gamma(j+1-m)}{\Gamma(j+1+m)} f_{jm}(t) f_{jm}^*(t),$$

where $f_{jm}(t)$ is the helical partial-wave amplitude of the production of the two particles and $p=p(t,M^2,\mu^2)$ is their relative momentum. As was noted by Azimov[9] this expression has a pole at $m=j+1$ and in particular at $j=\sigma-1$ (due to a pole of the Γ-function). Near the pole $j=\sigma-1$ the expression has the form

$$\frac{1}{\Gamma(2\sigma)} \frac{2p(t,M^2,\mu^2)}{t^{1/2}} \frac{f_{j\sigma}(t) f_{j\sigma}^*(t)}{j+1-\sigma}.$$

On the other hand, the contribution from Mandelstam's branch point to the unitarity condition for $\text{Im}f_j(t)$ has (as is clear from his paper[7]) this form:

$$\int^{(t^{1/2}-\mu)^2} \frac{2p(t,t_1,\mu^2)}{t^{1/2}} \frac{C(t,t_1)dt_1}{j+1-\alpha(t_1)}, \tag{1}$$

where $p(t,t_1,\mu^2)$ is the intermediate state relative momentum of a particle and a pair of particles having a Regge pole at $l=\alpha(t_1)$ and $C(t,t_1)$ is a certain function of t and t_1 [having at $p\to 0$ the form const/$(p^2(t,t_1,\mu^2)/t)$, see below, Eq. (45)].

Comparing the last two expressions we can see that the branch-point results from integration over the mass $t_1=M^2$ of the particle pair state with a variable spin $l=\sigma=\alpha(t_1)$. This circumstance can be interpreted as the fact that the Azimov singularity extends all along the Regge trajectory.

From the above comparison it is clear what is essential for investigating branch points in the multiparticle unitarity terms: We must know those of these terms which contain three-particle production amplitudes at m close to $j+1$ and an orbital momentum of the pair equal to m and close to its pole value $l=m=\alpha(t_1)$. The situation is similar when more than three particles are produced.

Accordingly, a method of analytic continuation of the unitarity conditions to complex j, corresponding to form (1) of the result is proposed in this paper and used for investigating the singularities in the j plane. We do not contend that this method is accurate in the general case. However, it reflects correctly the mechanism of generation of Mandelstam's branch points of the amplitude $f_j(t)$ and therefore seems to reproduce accurately the part of $f_j(t)$ singular in the j plane.

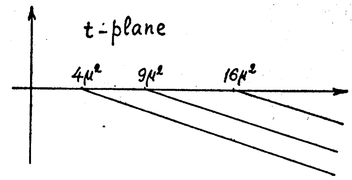

FIG. 1. Threshold singularities and corresponding cuts of the amplitude $f_j(t)$ in t plane.

The location of the branch points and their character can be found with the aid of the analytic continuation method proposed. For simplicity let us consider only those branch points which result from a vacuum pole. A straightforward analysis made below shows that at $t\geqslant 16\mu^2$ in the j plane there are many singularities and the location of some of them depends on the masses μ of particles. However, at $t<16\mu^2$ on the physical sheet of the j plane there remain only singularities $j=j_n(t)$, where

$$j_n(t)=n\alpha(t/n^2)-n+1 \tag{2}$$

[or displaced with respect to $j_n(t)$ by an even number] whose location depends only on the $\alpha(t)$-pole trajectory. These singularities have been noticed by Amati, Fubini, and Stanghellini[11] (see also Ref. 12).

The moving branch points in the j plane lead to the partial-wave amplitude $f_j(t)$ at a fixed j as a function of t having on the physical sheet, apart from the normal threshold singularities, the branch points $t=t_n(j)$ whose location depends on j. Each of them is the threshold of production of a certain number n of Regge poles. The unitarity conditions determining the discontinuities of $f_j(t)$ across these singularities are found in the paper. These Regge-pole unitarity terms are analogs of the conventional ones in the sense that they are given by the integrals of the product of the production amplitudes of several Regge poles above the cut by the value of the same amplitude under the cut (associated with the corresponding singularity).

II. MULTIPARTICLE UNITARITY TERMS

To obtain the Regge singularities of partial-wave amplitudes we have to make an analysis of the multiparticle terms of the unitarity condition. Let us consider the partial-wave four-point amplitude divided by $k_0^{2j}=[(t/4)-\mu^2]^j$ so that it would, if continued into complex j, be real below the threshold $t=4\mu^2$. Let us denote it by $f_j(t)$.

The threshold singularities of the amplitude $f_j(t)$ at $t=t_n=(n\mu)^2$ where $n=2, 3, 4,\cdots$ are indicated in

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[12] I. A. Verdiyev, O. V. Kancheli, S. G. Matinyan, A. M. Popova, and K. A. Ter-Martirosyan, Zh. Eksperim. i Teor. Fiz. 46, 1700 (1964) [English transl.: Soviet Phys.—JETP 19, 1148 (1964)].

Fig. 1. Let $f_j^{(n)}(t)$ denote the value of this amplitude after enclosure in the plane of Fig. 1 of the singularity t_n and $\Delta_n f_j(t)=(1/2i)[f_j(t)-f_j^{(n)}(t)]$, its discontinuity across the corresponding cut. Unitarity gives this discontinuity in the form

$$\Delta_n f_j(t)=\sum_{\lambda_n}\int f_{j,\lambda_n}(t,\tau_n{}^+)\times f_{j,\lambda_n}{}^{(n)}(t,\tau_n{}^-)\rho_{j,\lambda_n}(t,\tau_n)d\tau_n\,, \quad (3)$$

where $f_{j,\lambda_n}(t,\tau_n{}^+)$ is the amplitude of the transition of two particles into n particles, λ_n are the angular momenta, τ_n are the energies characterizing (besides j and t) the state of n particles, and $\rho_{j,\lambda_n}(t,\tau_n)$ is the statistical weight of this state. By $f_{j,\lambda_n}{}^{(n)}(t,\tau_n{}^-)$ we denote the value $f_{j,\lambda_n}(t,\tau_n)$ after enclosure in the t plane of the singularity at $t=(n\mu)^2$ and change of the sign of the infinitesimal imaginary additions to the energies τ_n.

The state of a system of n particles can be determined by dividing the particles arbitrarily into groups and determining the energies, angular momenta, and helicities of these groups.[13–18]

For example, the state of four particles can be determined by dividing them arbitrarily into two pairs and determining the quantities: (1) l_1, m_1, and t_1, the orbital angular momentum, its projection, and total energy squared t_1 of the first pair in its c.m. system—the projection m_1 (usually called helicity) can be conveniently determined on the direction of the total momentum of both particles of this pair (in the over-all c.m. system); (2) l_2, m_2, and t_2 which are the same quantities for the second pair—here m_2 is the projection on the direction of the same momentum and the helicity in this case is not m_2 but $-m_2$; (3) j and t are the total angular momentum and total energy squared of all the four particles.

Therefore, when $n=4$, λ_n in Eq. (3) denotes the set of numbers l_1, m_1, l_2, m_2, and τ_n that of energies t_1 and t_2, i.e.,

$$f_{j,\lambda_4}(t,\tau_4)=f_{j;l_1,m_1;l_2,m_2}(t;\,t_1,t_2)\,.$$

Let us consider in detail the state with $n=4$ and the corresponding term of the unitarity condition. For $n=4$ the quantity ρ_{j,λ_4} has the form

$$\rho_{j,\lambda_4}(t,\tau_4)=\frac{1}{4!}C_j(l_1,m_1;\,l_2,m_2)\times\frac{2p(t;\,t_1,t_2)}{t^{1/2}}\frac{2k_1{}^{2l_1+1}}{t_1{}^{1/2}}\frac{2k_2{}^{2l_2+1}}{t_2{}^{1/2}}\,, \quad (4)$$

where[19]

$$p(t;\,t_1,t_2)=\tfrac{1}{2}t^{-1/2}[t^2-2t(t_1+t_2)+(t_1-t_2)^2]^{1/2} \quad (5)$$

is the relative momentum of the centers of inertial of both pairs of particles, $k_1=p(t_1,\mu^2,\mu^2)$ and $k_2=p(t_2,\mu^2,\mu^2)$ are the momenta of particles in the first and second pairs, and

$$C_j(\lambda_4)=C_j(l_1m_1,l_2m_2)=\frac{\Gamma(j+1-m_1-m_2)}{\Gamma(j+1+m_1+m_2)}\times\frac{(2l_1+1)\Gamma(l_1+1-m_1)}{\Gamma(l_1+1+m_1)}\frac{(2l_2+1)\Gamma(l_2+1-m_2)}{\Gamma(l_2+1+m_2)}\,. \quad (6)$$

This form of the statistical weight ρ_{j,λ_4} and, in particular, the factor $C_j(\lambda_4)$ results from the choice of normalization[18] of the amplitudes f_{j,λ_4}, viz., the quantities f_{j,λ_4} are connected with the production amplitude of three particles A with given momenta via the integrals of the form[13,18]:

$$f_{j,\lambda_4}(t;\,t_1,t_2)=\int A(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2;\,\mathbf{p}_0)\times P_{l_1m_1}(\mathbf{n}_1)P_{l_2m_2}(\mathbf{n}_2)P_{j,m_1+m_2}(\mathbf{n}_0)d\mathbf{n}_1d\mathbf{n}_2d\mathbf{n}_0\,,$$

where $\mathbf{n}_0=\mathbf{p}_0/p_0$, $\mathbf{n}_a=\mathbf{k}_a/k_a$, $a=1,2$. These integrals contain associated Legendre polynomials $P_{lm}(n)=P_{lm}(z)e^{im\phi}$ instead of the normalized spherical functions $Y_{lm}(n)$ differing from them by the factor $[(2l+1)\Gamma(l+1-m)/\Gamma(l+1+m)]^{1/2}$. If the amplitudes f_{j,λ_4} were determined through spherical functions, they would, if continued into complex j, l_1, and l_2 (which would be used below), have purely kinematic root singularities in these variables due to the poles of the Γ functions.

From the form (4) of ρ_{j,λ_4} it can also be noticed that the factors $k_a{}^{l_a}=[(t_a/4)-\mu^2]^{l_a/2}$ where $a=1,2$ have been isolated from the amplitudes.

The sum over λ_4 and the integral $\int d\tau_4$ in Eq. (3) at $n=4$ denote

$$\sum_{\lambda_4}\int d\tau_4=\sum_{l_1=0}^{\infty}\sum_{m_1=-l_1}^{l_1}\sum_{l_2=0}^{\infty}\sum_{m_2=-l_2}^{l_2}\int_{4\mu^2}^{(t^{1/2}-2\mu)^2}dt_2\int_{4\mu^2}^{(t^{1/2}-t_2{}^{1/2})^2}dt_1\,. \quad (7)$$

The amplitude $f_{j,\lambda_4}(t;\,t_1,t_2)$ satisfies unitarity not only in the t but in the t_1 and the t_2 channels; the latter channels correspond to the interaction of the produced pairs of particles with angular momenta l_1 or l_2. In the

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[18] A. M. Popova and K. A. Ter-Martirosyan, Nucl. Phys. **56**, 107 (1964).

[19] In the following we consider the intermediate states with generation of identical pions with neglect of isotopic variables. Hence the introduction of the factor 1/4! in Eq. (4). The problems involved in the identity of particles and symmetry in their permutations are discussed below in detail.

t_1 channel the (two-particle) unitarity condition has the form

$$\frac{1}{2i}[f_{j,\lambda_4}(t;t_1,t_2)-f_{j,\lambda_4}(t;t_1^{(2)},t_2)]=\frac{k_1^{2l_1+1}}{t_1^{1/2}}f_{j,\lambda_4}(t;t_1^{(2)},t_2)f_{l_1}(t_1),\quad (8)$$

where $f_{j,\lambda_4}(t;t_1^{(2)},t_2)$ is the amplitude $f_{j,\lambda_4}(t;t_1,t_2)$ after enclosure in the t_1 plane of the singularity at $t_1=4\mu^2$ and $f_{l_1}(t_1)$ the partial-wave scattering amplitude of the two particles produced. It satisfies the unitarity condition of the form

$$\Delta_2 f_{l_1}(t_1)=\frac{1}{2i}[f_{l_1}(t_1)-f_{l_1}^{(2)}(t_1)]=\frac{1}{2}\frac{2k_1^{2l_1+1}}{t_1^{1/2}}f_{l_1}^{(2)}(t_1)f_{l_1}(t_1)$$

which can be written as

$$\frac{1}{2i}[D_{l_1}(t_1)-D_{l_2}^{(2)}(t_2)]=\frac{1}{2}\frac{2k_1^{2l_1+1}}{t_1^{1/2}};\quad (9)$$

if this notation is introduced,

$$f_{l_1}(t_1)=-1/D_{l_1}(t_1).\quad (10)$$

The factor $\frac{1}{2}$ in Eq. (9) arises because of the identity of the particles.

Taking into account Eq. (8) and the same unitarity condition in the t_2 channel we notice that f_{j,λ_4} can be written as

$$f_{j,\lambda_4}=G_{j,\lambda_4}(t;t_1,t_2)/D_{l_1}(t_1)D_{l_2}(t_2),\quad (11)$$

where G_{j,λ_4} has no singularities at $t_1=4\mu^2$ and $t_2=4\mu^2$. Quite similarly we have

$$f_{j,\lambda_4}^{(4)}(t;t_1^-,t_2^-)=G_{j,\lambda_4}^{(4)}(t;t_1,t_2)/D_{l_1}^{(2)}(t_1)D_{l_2}^{(2)}(t_2).\quad (12)$$

Substituting Eqs. (11), (12), and (4) into integral (3), (7) and noticing that by virtue of Eq. (9) we have

$$\frac{1}{D_{l_1}(t_1)D_{l_2}(t_2)}\cdot\frac{1}{D_{l_1}^{(2)}(t_1)D_{l_2}^{(2)}(t_2)}\frac{2k_1^{2l_1+1}}{t_1^{1/2}}\frac{2k_2^{2l_2+1}}{t_2^{1/2}}=\frac{2^2}{(2i)^2}\left(\frac{1}{D_{l_1}(t_1)}-\frac{1}{D_{l_1}^{(2)}(t_1)}\right)\left(\frac{1}{D_{l_2}(t_2)}-\frac{1}{D_{l_2}^{(2)}(t_2)}\right),\quad (13)$$

we can write the right-hand side of Eq. (3) at $n=4$ in a form similar to that used by Mandelstam[7]:

$$\Delta_4 f_j(t)=\frac{2^2}{4!}\sum_{\lambda_4}C_j(\lambda_4)\frac{1}{(2i)^2}\int_{C_2}dt_2\int_{C_1}dt_1\times\frac{G_{j,\lambda_4}(t;t_1,t_2)G_{j,\lambda_4}^{(4)}(t;t_1,t_2)}{D_{l_1}(t_1)D_{l_2}(t_2)}\frac{2p(t;t_1,t_2)}{t^{1/2}},\quad (14)$$

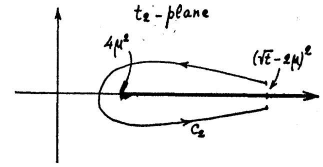

FIG. 2. The contour of integration over t_2 in the four-particle unitarity-condition term.

where C_2 and C_1 are the contours indicated[20] in Figs. 2 and 3.

Let us clarify the general principle of introduction of quantum numbers and form (3) of unitarity using as examples the cases where $n=6$ and $n=8$ (odd numbers except $n=3$ are of no interest to us in the following). The state of six particles can be determined by assigning, apart from j and t, the angular momenta and energies of some four-particle and two-particle group. Let them denote l_{12}, m_{12}, t_{12}, and l_3, m_3, t_3. To determine the state of four particles it is necessary to divide them as was done above (for $n=4$) into pairs and determine the quantum numbers l_1, m_1, t_1 and l_2, m_2, t_2 of each pair. Hence, in the case $n=6$ we have $\lambda_6=\{l_1,m_1;l_2,m_2;l_{12},m_{12};l_3,m_3\}$ and $\tau_6=\{t_1,t_2,t_{12},t_3\}$.

The state of eight particles can be determined by dividing them into two four-particle groups and determining, apart from j and t, the quantum numbers l_{12}, m_{12}, t_{12}, and l_{34}, m_{34}, t_{34} of both groups. Besides, we must assign the quantum numbers l_a, m_a, t_a where $a=1, 2, 3, 4$ of those pairs out of which the first and second four-particle groups are made.

Another method of describing the state of eight particles can be obtained by dividing them into groups of six and two particles and determining j,t and the quantum numbers l_{123}, m_{123}, t_{123} of the six-particle group and l_4, m_4, t_4 of the two-particle group. Besides, to describe the state of six particles (in their c.m. system) one has also to introduce the same quantum numbers as in the case $n=6$, i.e., λ_6 and τ_6.

The amplitudes f_{j,λ_6} and f_{j,λ_8} can be written similarly to (11)

$$f_{j,\lambda_6}=\frac{G_{j,\lambda_6}(t;t_{12},t_1,t_2,t_3)}{D_{l_1}(t_1)D_{l_2}(t_2)D_{l_3}(t_3)};\quad f_{j,\lambda_8}=\frac{G_{j,\lambda_8}}{\prod_{a=1}^{4}D_{l_a}(t_a)},\quad (12a)$$

G_{j,λ_6} and G_{j,λ_8} having no singularities at $t_a=4\mu^2$, where $a=1, 2, 3$ or $a=1, 2, 3, 4$.

Using the same normalization of these amplitudes as in the case $n=4$ we obtain for ρ_{j,λ_6} the value

$$\rho_{j,\lambda_6}=(1/6!)C_j(l_{12},m_{12};l_3m_3)C_{l_{12}}(l_1,m_1;l_2m_2)\times\frac{2p(t,t_{12},t_3)}{t^{1/2}}\frac{2p(t_{12},t_1,t_2)}{t_{12}^{1/2}}\frac{2k_1^{2l_1+1}}{t_1^{1/2}}\frac{2k_2^{2l_2+1}}{t_2^{1/2}}\frac{2k_3^{2l_3+1}}{t_3^{1/2}}.\quad (15)$$

[20] As Ya. Azimov pointed out to these authors, the integrand function in a separate term of the sum over λ_4 in Eq. (14) cannot be represented as a single analytic function throughout the region of variation of t_1 and t_2. However, this is not essential in our case since sum (14) will be understood in the sense that first the summation over all the values of the angular momenta (over $\lambda_4=l_1,m_1;l_2,m_2$) is performed whereupon the function obtained which is now analytic in t_1 and t_2 is integrated over the complex contours C_1 and C_2.

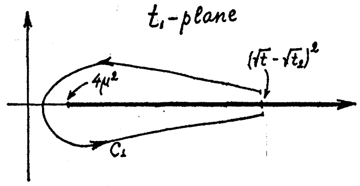

FIG. 3. The contour of integration over t_1 in the four-particle unitarity-condition term.

Therefore, for $n=6$ the unitarity condition (3) is

$$\Delta_6 f_j(t)=\sum_{\lambda_6}\frac{2^3}{6!}C_jC_{l_{12}}\frac{1}{(2i)^3}\int\int_{C_3C_2C_1}\int dt_a\int dt_{12}$$

$$\times\frac{G_{j,\lambda_6}G_{j,\lambda_6}{}^{(6)}}{D_{l_1}(t_1)D_{l_2}(t_2)D_{l_3}(t_3)}\frac{2p(t;t_{12},t_3)}{t^{1/2}}\frac{2p(t_{12};t_1,t_2)}{t_{12}{}^{1/2}},\quad(16)$$

where

$$\sum\nolimits_{\lambda_6}=\sum\nolimits_{l_{12}m_{12}}\sum\nolimits_{l_1m_1}\sum\nolimits_{l_2m_2}\sum\nolimits_{l_3m_3}.$$

The integration over t_{12} in Eq. (16) is performed within

$$(t_1^{1/2}+t_2^{1/2})^2<t_{12}<(t^{1/2}-t_3^{1/2})^2\quad(17a)$$

and over t_1, t_2, t_3 over contours similar to C_1 and C_2 (Figs. 2 and 3) around the point $t_a=4\mu^2$ to these points (located as in Figs. 2 and 3 on both sides of the cut):

$$\begin{aligned}&\text{over } t_3 \text{ to } t_3=(t^{1/2}-4\mu)^2,\\&\text{over } t_2 \text{ to } t_2=(t^{1/2}-t_3^{1/2}-2\mu)^2,\\&\text{over } t_1 \text{ to } t_1=(t^{1/2}-t_3^{1/2}-t_2^{1/2})^2.\end{aligned}\quad(17b)$$

In the general case of an arbitrary even number of particles, the unitarity condition (3) can be written quite similarly to Eqs. (14) and (16):

$$\Delta_{2n}f_j(t)=\sum_{\lambda_n}\frac{2^n}{(2n)!}C_j'(\lambda_n)\frac{1}{(2i)^n}$$

$$\times\int_{C_n}K_{j,n}'(\tau_n)\frac{G_{j,\lambda_n}G_{j,\lambda_n}{}^{(2n)}}{\prod_{a=1}^{n}D_{l_a}(t_a)}d\tau_n,\quad(18)$$

where $K_{j,n}'$ is the product of $n-1$ factors of the form

$$2p(t_{ab};t_a,t_b)/t_{ab}{}^{1/2},$$

one factors for each combination of two groups of particles with energies t_a and t_b into a group with energy t_{ab}. Similarly, C_j' is the product of $n-1$ factors (6), one factor for each such combination. The integration $\int_{C_n}d\tau_n$ over all the variables t_a, t_b, t_{ab}, etc., is performed over a region corresponding to energy conservation, of type (17a,b), and the integrals over t_a (over particle pair energies) are taken not over lengths of the real axis but over contours similar to Figs. 2 and 3 around these lengths and the points $t_a=4\mu^2$.

III. DIFFICULTIES OF ANALYTIC CONTINUATION INTO THE j PLANE

The unitarity condition (14), (16), or (18) has been written for integral j. Its continuation into complex j is an intricate problem the solution of which requires a knowledge of the analytic properties of the inelastic amplitudes f_{j,λ_n}.

For the sake of definiteness let us consider the four-particle term (14) of the unitarity condition. Even when the analytic properties of the amplitudes $f_{j,\lambda_4}=f_{j;l_1,m_1;l_2,m_2}$ are such that their analytic continuation into the j plane is unambiguous for integral l_1, m_1 and l_2, m_2 the following difficulty is encountered.[21] Since the quantity $|m_1+m_2|$ in the sum (14)–(16) varies from 0 to ∞ running through all integers, the function $\Gamma(j+1-(m_1+m_2))$ which enters as a factor into the coefficient $C_j(\lambda_4)$, Eq. (6), in the right-hand side of Eq. (14) has poles at all integral positive j. These poles would not be in evidence if the amplitudes f_{j,λ_4} (or G_{j,λ_4}) had direct physical meaning for all integral j since in this case G_{j,λ_4} would have to be zero for all integral j for which $|m_1+m_2|\geqslant j+1$.

However, the analytic continuation in the j plane [of the amplitudes $f_j(t)$ and f_{j,λ_n} alike] requires in any case the introduction of signature. In other words, the analytic continuation of $f_{j;l_1,m_1;l_2m_2}$ has the meaning of a physical amplitude only for even or only for odd j. Therefore, for integral j of "wrong" signature (odd for positive signature and even for negative signature), the function $f_{j;l_1m_1,l_2m_2}$ need not vanish.

A more detailed analysis of some perturbation diagrams shows that the partial-wave amplitude f_{j,λ_4} corresponding to these diagrams at the "wrong" signature points does not indeed vanish. For example, it is not zero for the case of those Feynman diagrams which, considered as the amplitudes of production of two particles with spins l_1, l_2 (with masses $t_1^{1/2}$ and $t_2^{1/2}$), have a nonzero spectral function $\rho(s,u)$. One of these diagrams is indicated in Fig. 4(a). [On the other hand, in the case of the diagrams of Fig. 4(b) for which only the spectral function $\rho(s,t)$ is nonzero, the amplitude $f_{j;l_1m_1,l_2m_2}$ vanishes not only at all integer j for which $j+1\leqslant m_1+m_2$ but also[19] if j and $m=m_1+m_2$ are not integers but the difference $(m_1+m_2)-(j+1)$ is an integer.]

Thus, in the form (14) [with sum (7) over m_1 and m_2 extended from $-\infty$ to $+\infty$] the unitarity condition cannot be continued into complex j since the right-hand side of Eq. (14) would have an infinite number of poles at all integral positive $j=n$ of "wrong" signature.[22]

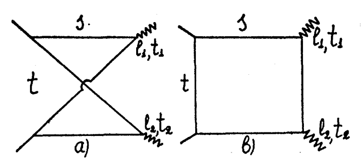

FIG. 4. Examples of diagrams of production of two particles with nonzero spin (l_1 and l_2) having (a) $\rho(s,u)\neq0$ and (b) $\rho(s,u)=0$, $\rho(s,t)\neq0$.

[21] The authors are indebted to Ya. I. Azimov, G. S. Danilov, and I. T. Dyatlov who have directed their attention to this circumstance.

[22] This would contradict that well-known fact that the partial-wave amplitude has no singularities in the right-hand part of the j plane.

To obviate this difficulty let us determine the analytic continuation into complex j of the right-hand side of Eq. (14) not as a sum over integral l_1, m_1, l_2, m_2, as in Eq. (7), but as contour integrals over these variables.

IV: ANALYTIC CONTINUATION OF THREE-PARTICLE UNITARITY CONDITION

To avoid cumbersome operations let us first describe the transition to the contour integrals over l and m as exemplified by the unitarity condition term corresponding to the production of three spinless particles in the intermediate state. We assume that two of these particles are produced in a state with angular momentum l and helicity m. The corresponding term in the unitarity condition for $f_j(t)$ has a form analogous to Eq. (14):

$$\Delta_3 f_j(t)=2\sum_{m=0}^{\infty}{}'\sum_{l=m}^{\infty}\frac{2}{3!}C_j(l,m;0,0)\times\frac{1}{2i}\int_{C_1}\frac{G_{j;lm}(t,t_1)G_{j;lm}{}^{(3)}(t,t_1)}{D_l(t_1)}\times\frac{2p(t;t_1,\mu^2)}{t^{1/2}}dt_1,\qquad(19)$$

$$C_j(l,m;0,0)=\frac{\Gamma(j-m+1)}{\Gamma(j+m+1)}\frac{(2l+1)\Gamma(l-m+1)}{\Gamma(l+m+1)},$$

the contour C_1 being exactly the same as in Fig. 3, but ending at the point $t_1=(t^{1/2}-\mu)^2$. Equation (19) takes into account the fact that the product $C_jG_{j;lm}G_{j;lm}{}^{(3)}$ does not change[23] if m is replaced by $-m$, i.e., the sum $\sum_{l=0}^{\infty}\sum_{m=-l}^{l}$ can be written in the form $2\sum'_{m=0}{}^{\infty}\sum_{l=m}{}^{\infty}$ where the prime means that the term with $m=0$ contains a factor of $\frac{1}{2}$. For the three-particle production amplitude $f_{j;\lambda_3}=f_{j;lm}(t,t_1)$ (Fig. 7) in Eq. (19) a value analogous to Eq. (11)

$$f_{j;lm}(t,t_1)=G_{j,lm}(t,t_1)/D_l(t_1)\qquad(20)$$

was substituted and Eq. (9) was used for $D_l(t_1)$.

To continue analytically the right-hand side of Eq. (19) to complex j we can write the sum over l, m as contour integrals

$$\sum_{lm}=2\sum_{m=0}^{\infty}\sum_{l=m}^{\infty}\rightarrow\frac{2}{(2i)^2}\int_M\frac{dm}{\tan\pi m}\int_L\frac{dl}{\tan\pi(l-m)}.\qquad(21)$$

The contours L and M are indicated in Figs. 5 and 6; the contour L encloses the point $l=m$, $m+1$, $\cdots$, and

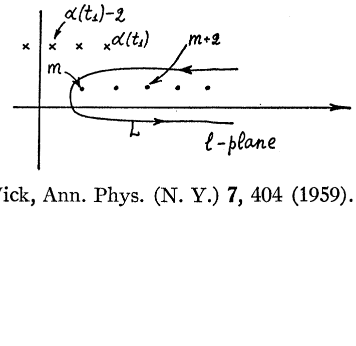

FIG. 5. The contour of integration over l in (21), (19), (14).

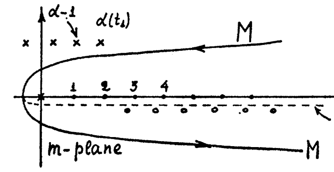

FIG. 6. The contour of integration over m in (21), (19).

the contour M encloses not only the poles of $1/\tan\pi m$ [at $m=0,1,2,\cdots$; the term with $m=0$ is reproduced by the right-hand side of Eq. (21) without the factor $\frac{1}{2}$, yet it is not essential for the following], but also the poles of the function $\Gamma(j-m+1)/\Gamma(j+m+1)$ entering into $C_j(l,m;0,0)$. The latter condition is necessary for integral (21) to have no singularities at arbitrarily large positive integral j for which the poles of the function $\Gamma(j+1-m)$ indicated by circles in Fig. 6 (at the points $m=j+1+\nu$, $\nu=0,1,2,\cdots$) coincide with the poles of $1/\tan\pi m$.

The continuation of Eq. (19) directly as the sum over integral l, m corresponds to the form of Eq. (21) with such a contour M' which would not enclose the poles $\Gamma(j+1-m)$ (see Fig. 6). In this case integral (21) has the singularities indicated above: poles at integral j since as $j\rightarrow n$ the singularities $\Gamma(j+1-m)$ and $1/\tan\pi m$ will pinch the contour M'.

It is worth noting that since the contour of integration M over m in Eq. (21) encloses the poles $\Gamma(j+1-m)$ the expression of the unitarity condition (19) in the form (21) is ambiguous. Indeed, if $1/\tan\pi m$ in Eq. (21) is replaced by $(1/\tan\pi m)+\chi(j,l,m)$ where χ is any function of j, l, m without singularities inside the contours L and M in Figs. 5 and 6, then integral (21) will change[24] due to the contribution of the poles of the function $1/\tan\pi m$. The function $\chi(j,l,m)$ must actually be taken into account in the right-hand side of Eq. (21), its form being given unambiguously by the limiting condition for integral (21) to be smaller than $\exp(i\pi j/2)$ at $\mathrm{Im}\,j\rightarrow\pm\infty$. To obtain an explicit form of $\chi(j,l,m)$ we must know the analytical properties of the functions in the right-hand side of Eq. (19) in the planes of the variables j, l, and m. However, we shall only be interested in the singular part of integral (21) as a function of j. Therefore the possibility (or necessity) of adding $\chi(j,l,m)$ to $1/\tan\pi m$ is ignored in the following under the assumption that the factor $[1+\chi(j,l,m)\tan\pi m]^{1/2}$ is included in Eq. (19) in the definition of the functions $G_{j;lm}$.

Let us show how the singularity which has been discovered by Mandelstam[7] follows from Eqs. (19) and (21). Let us assume that when $l=\alpha(t_1)$ the function $D_l(t_1)$ vanishes, i.e.,

$$D_l(t_1)=-[1/g^2(t_1)](l-\alpha(t_1))\qquad(22)$$

when $l\rightarrow\alpha(t_1)$. This corresponds to the Regge pole of the five-point amplitude of Fig. 7 in the channel t_1

[23] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).

[24] Addition of an analogous term to $1/\tan\pi(l-m)$ in Eq. (21) does not change integral (19), (21).

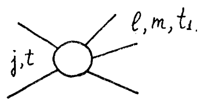

FIG. 7. The five-point amplitude.

(Fig. 9) and corresponds to the conventional form of the elastic scattering amplitude (10)

$$f_l(t_1)=g^2(t_1)/[l-\alpha(t_1)],\quad l\to\alpha(t_1),\qquad(23)$$

i.e., the contribution from the Regge pole, Fig. 8.

It is clear from Fig. 5 that integral (21) over l has poles in m at the points $m=\alpha(t_1)$, $\alpha(t_1)-1$, $\alpha(t_1)-2,\cdots$ (indicated by crosses in Fig. 6) because of coincidence in Eqs. (19) and (21) of the zeros of $\tan\pi(l-m)$ with those of $D_l(t_1)$. These poles must be taken into account in subsequent integration over m. In particular, the poles of the function $\Gamma(j+1-m)$ may coincide with these poles, as j is varied in the complex plane. As a result the double integral (21) over l and m will have poles in the j plane at the points $j=\alpha(t_1)-1-k$, where $k=0, 1, 2,\cdots$.

Let us consider the extreme right singularity of integral (19), (20) in the j plane corresponding to $k=0$. Substituting Eq. (22) into Eqs. (19) and (21), we can readily calculate the singular part of the integrals over l, m resulting from the poles (crosses in Figs. 5 and 6) of the integrand functions at $m=\alpha(t)$ and $l=\alpha(t)$:

$$\frac{1}{2i}\int_L\frac{dl}{\tan\pi(l-m)}\frac{G_{j;lm}G_{j;lm}{}^{(3)}}{D_l(t_1)}\frac{(2l+1)\Gamma(l-m+1)}{\Gamma(l+m+1)}=\frac{g^2}{\alpha(t_1)-m}\frac{G_{j,\alpha m}G_{j\alpha m}{}^{(3)}(2\alpha+1)\Gamma(\alpha-m+1)}{\Gamma(\alpha+m+1)},$$

$$2\int_M\frac{dm/2i}{\tan\pi m}\frac{g^2G_{j;\alpha m}G_{j;\alpha m}{}^{(3)}}{\alpha(t_1)-m}\frac{(2\alpha+1)\Gamma(\alpha-m+1)}{\Gamma(\alpha+m+1)}\times\frac{\Gamma(j-m+1)}{\Gamma(j+m+1)}=\Lambda^2(j,\alpha)\frac{g^2G_{j;\alpha\alpha}G_{j;\alpha\alpha}{}^{(3)}}{j+1-\alpha(t_1)},$$

where

$$\Lambda^2(j,\alpha)=\frac{(2\alpha+1)}{\Gamma(2\alpha+1)}\frac{1}{\Gamma(j+\alpha+1)}\cdot\frac{2\pi}{\tan\pi\alpha(t_1)}\qquad(24)$$

is a function having no singularities in the region $\alpha\neq0$, $j+1\simeq\alpha(t_1)$ essential in the following.

Using this result, we can write the singular part $\Delta_3'f_j(t)$ of interest to us in this form

$$\Delta_3'f_j(t)=\frac{2}{3!}\frac{1}{2i}\int_{C_1}\frac{\Lambda^2(j,\alpha)(gG_{j\alpha})(gG_{j\alpha}{}^{(3)})}{j+1-\alpha(t_1)}\frac{2p(t;t_1,\mu^2)}{t^{1/2}},$$

where $G_{j\alpha}$ is written instead of $G_{j,\alpha\alpha}$ for the sake of brevity.

It is worth noting that the quantity $gG_{j\alpha}$ has the meaning of the amplitude for the transition of two particles into a Regge-pole and particle. If, indeed, the production of three particles is regarded as occurring through a virtual Regge state, it is natural to confront this process with the diagram of Fig. 9 and the quantity

$$N_j'\frac{1}{l-\alpha(t_1)}g(t_1),\qquad(25)$$

where $N_{j\alpha}'$ is the Regge-pole and particle-production amplitude.

On the other hand, when $l\to\alpha(t_1)$ the amplitude $f_{j\lambda_3}$ just has, according to Eqs. (20) and (22), the form (25) with $N_{j\alpha}'=-gG_{j\alpha}$. Therefore, if the amplitude

$$N_{j\alpha}=\Lambda(j,\alpha)gG_{j\alpha},\qquad(26)$$

differing from $N'(j,\alpha)$ only by the standard factor $-\Lambda(j,\alpha)$ is introduced in the right-hand side of Eq. (24), the constant g, of the decay of a Regge pole into two particles in the intermediate state drops altogether out of the right-hand side of the expression for $\Delta_3'f_j(t)$:

$$\Delta_3'f_j(t)=\frac{2}{3!}\frac{1}{2\pi i}\times\int_{C_1}\frac{N_{j\alpha}(t,t_1)N_{j\alpha}{}^{(3)}(t,t_1)}{j+1-\alpha(t_1)}\frac{2p(t;t_1,\mu^2)}{t^{1/2}}dt_1.\qquad(27)$$

It can readily be noticed that after integration over t_1 the right-hand side of Eq. (27) proves a singular function of the variable j. The branch point in j of integral (27) results from the coincidence of the zero of the denominator with the singularity of $p(t;t_1,\mu^2)$ at $t_1=(t^{1/2}-\mu)^2$ (with the upper limit of the integral over t_1), i.e., at

$$j=\alpha[(t^{1/2}-\mu)^2]-1.$$

This branch point has been discovered by Mandelstam (this singularity was considered in detail by Simonov.[25])

In conclusion of this section let us discuss this problem. We have found only one branch point resulting from a Regge pole in the amplitude of interaction of only one pair of particles (let us denote them by 1 and 2). Obviously, identical branch points must result because of Regge poles in the interaction amplitude for particles 1 and 3 or 2 and 3. However, no such singularities are in evidence in our expression of unitarity in the form (19) and (21) in which the state of three particles is described in particular by the angular momentum l and helicity m of the pair of particles 1 and 2.

It might be helpful to recall that a similar problem arises for integer j if we want to find the contribution

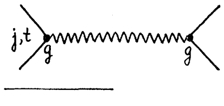

FIG. 8. Regge-pole diagram for $f_j(t)$.

[25] Yu. A. Simonov, Zh. Eksperim. i Teor. Fiz. 48, 242 (1965) [English transl.: Soviet Phys.—JETP (to be published)].

to the three-particle unitarity condition from the real physical resonance interactions of not only particles 1 and 2 but also 1, 3 and 2, 3 as well.

Obviously, if we choose as the variables, the angular momentum and the helicity of one pair, e.g., l_{12}, m_{12} of particles 1 and 2, a resonance in the state of this pair with spin s_{12} will appear only in those terms of the sum over l_{12} and m_{12} for which $l_{12}=s_{12}$, $s_{12}\leqslant m_{12}\leqslant s_{12}$. The other resonance interactions will show in that the sum over l_{12} will prove divergent, if the energy of particles 1 and 3 (or 2 and 3) is made to tend to its resonance value. This is clear from the fact that the expression of the unitarity conditions as the sum over l_{12} corresponds to the expansion of the three-particle production amplitude in the variables t_{13} and t_{23} at fixed t and t_{12}. In this case the singularities in t_{13} and t_{23} (resonances) must show in divergence of the sum.

Instead of the investigation of the divergence of the sums, it is more convenient to find the contribution from all the resonances, rewriting the unitarity condition in all possible (three, in our case) ways and adding up the results.

These considerations are also fully applicable to the case where we are interested in the contribution from Regge poles due to the interaction of a certain pair of particles in the intermediate state, to the singular part of the quantity $\Delta_3 f_{j\lambda_3}$.

Thus in the case of identical particles the right-hand side of Eq. (27) should be multiplied by 3 in order to take into account the contribution from all the three interactions of 1 and 2, 1 and 3, and 2 and 3. In this case the factor 2/3! in the right-hand side of Eq. (27) changes to unity.

V. ANALYTIC CONTINUATION OF FOUR-PARTICLE UNITARITY TERMS

We continue analytically the four-particle terms (14) of the unitarity condition similarly to Eq. (21), replacing by integration the summation over λ_4 (i.e., over l_1, m_1, l_2, m_2) in Eq. (14).

To this end let us write the sum (7) over λ_4 in the form

$$\sum_{l_1m_1l_2m_2}\equiv 2\sum_{m_1m_2>0}{}'\sum_{l_1=m_1}^{\infty}\sum_{l_2=m_2}^{\infty}+2\sum_{\substack{m_1>0\\ m_2<0}}\sum_{l_1=m_1}^{\infty}\sum_{l_2=-m_2}^{\infty}, \quad (28)$$

which, just as Eq. (21), takes into account the fact that the expression in Eq. (14) under the sign of summation over λ_4 does not change as m_1 and m_2 are replaced simultaneously by $-m_1$ and $-m_2$. The formulas obtained in the following for the four-particle case will have a real and not illustrative (as in the three-particle case) meaning. Therefore, it should be borne in mind that in any case the amplitudes $f_{j,\lambda_4}\equiv f_{j;l_1m_1,l_2m_2}$ cannot be continued as functions on l_1 and l_2 from all integral values l_1 and l_2 and it is necessary to introduce a signature with respect to these variables. In the following, with respect to both l_1 and l_2, we are interested exclusively in a

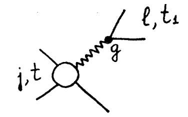

FIG. 9. The five-point amplitude with a Regge pole corresponding to the interaction of a pair of particles produced (with c.m. system energy $t_1^{1/2}$).

vacuum pole which occurs only in a state with positive signature. Therefore, we consider merely that part of the sums (28) in which l_1 and l_2 are even numbers.

In this case the sum over l_1 and m_1, for example, in the first term of Eq. (28) can be represented as

$$\sum_{m_1>0}^{\infty}\sum_{\substack{l_1\,\text{even}\\ l_1\geqslant m}}\equiv\sum_{m_1=0,2,4,\cdots}^{\infty}\sum_{l_1=m_1}^{\infty}+\sum_{m_1=1,3,5,\cdots}^{\infty}\sum_{l_1=m_1+1}^{\infty}$$

or, passing to contour integrals as in Eq. (21), we can write

$$\frac{1}{(4i)^2}\int_{M_1}\frac{dm_1}{\tan(\pi m_1/2)}\int_{L_1}\frac{dl_1}{\tan[\pi(l_1-m_1)/2]}+\frac{1}{(4i)^2}\int_{M_1}\frac{dm_1}{\cot(\pi m_1/2)}\int_{L_1}\frac{dl_1}{\cot[\pi(l_1-m_1)/2]} \quad (29)$$

where M_1 and L_1 are the contours such as indicated in Figs. 5 and 6.

In the previous section it was shown that the singularity of the total integral results from values of l close to m. In the second term of (29) the quantity l_1 is always larger than m_1 at least by unity. Taking this into account we can note (if we perform on this term all those operations that are performed below for the first term) that this term has a singularity in j removed in the j plane to the left by unity as compared with the singularity of the first term. We shall not be interested in such singularities and hence we omit altogether the second term in Eq. (29). Similarly we can ignore the entire second term in Eq. (28) since it does not lead to singularities essential in the following (as is shown in detail in Appendix A).

Thus we can write the unitarity condition (14) as

$$\Delta_4'f_j=\frac{2^2}{4!}\frac{1}{(4i)^4}\int_{M_1}\frac{dm_1}{\tan(\pi m_1/2)}\int_{L_1}\frac{dl_1}{\tan[\pi(l_1-m_1)/2]}\times\int_{M_2}\frac{dm_2}{\tan(\pi m_2/2)}\int_{L_2}\frac{dl_2}{\tan[\pi(l_2-m_2)/2]}\times 2C_j(l_1,m_1;l_2,m_2)\int_{C_2}\frac{dt_2}{2i}\int_{C_1}\frac{dt_1}{2i}\times\frac{G_{j,\lambda_4}G_{j,\lambda_4}{}^{(4)}}{D_{l_1}(t_1)D_{l_2}(t_2)}\frac{2p(t;t_1,t_2)}{t^{1/2}}, \quad (30)$$

where the factor $C_j(l_1,m_1;l_2,m_2)$ was determined above [see Eq. (6)], the contours L_1 and L_2 coincide with the contour of Fig. 5, and the contour M_2 includes the

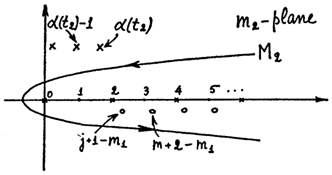

FIG. 10. The contour of integration over m_2 in the four-particle unitarity-condition term (30).

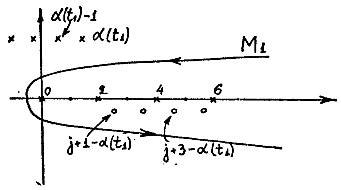

FIG. 11. The contour of integration over m_1 in (30).

singularities of the function $\Gamma(j+1-m_1-m_2)$ contained in $C_j(l_1,m_1;l_2m_2)$ (Fig. 10).

Repeating the treatment of the previous section we can come to the conclusion that after integration over l_1, l_2, and m_2 which must take into account the Regge poles in l_1 and l_2 [i.e., the zeros of $D_{l_1}(t_1)$ and $D_{l_2}(t_2)$ at $l_1=\alpha(t_1)$, $l_2=\alpha(t_2)$] the integrand function in the integral over m_1 has poles at the points: (1) $m_1=0, 2, 4, \cdots$ because of the zeros of $\tan(\pi/2)m_1$, (2) $m_1=\alpha(t_1)$, $\alpha(t_1)-2$, etc., resulting from integration over l_1 and (3) $m_1=j+1-\alpha(t_2)+k$, $k=0, 1, 2, \cdots$ resulting from integration over l_2 and m_2. The mechanism of emergence of these poles is perfectly similar to that considered in detail in the previous section. These poles result as the poles of the function $\Gamma(j+1-m_1+m_2)$, contained as a factor in the coefficient of Eq. (6), are taken into account.

All these singularities are indicated in Fig. 11. The contour of integration M_1 must be chosen so that the poles of the third type lie inside, as indicated in Fig. 11. Otherwise, the right-hand side of Eq. (30) has singularities (poles) at an arbitrarily large j because of coincidence of the third-type poles with the singularities of $\cot(\pi/2)m_1$. Note that the possibility of adding the function χ to $\cot\pi m$ discussed in the previous section holds here for both $\cot(\pi m_1/2)$ and $\cot(\pi m_2/2)$.

The final integration in Eq. (30) over the contour M_1 of Fig. 11 leads to a singularity because of coincidence of the poles of the second and third types at the following values of j

$$j=\alpha(t_1)+\alpha(t_2)-1-k, \quad k=0, 1, 2, \cdots.$$

If the trajectory $\alpha(t)$ at a certain $t=m^2$ passes through a physical value (e.g., $\alpha=0$), this formula corresponds to the Azimov displacement of the singularity at $j=-1$. However, when the particle production amplitude satisfies the Mandelstam representation it is known that k is to be even ($k=0, 2, 4, \cdots$) because of the symmetry properties of $\rho(s,u)$ with respect to replacement of s by u. If a symmetry of this kind holds for Regge-pole production amplitudes as well, the coefficients of the singularity are zero for odd k. If it is assumed that this is the case in reality we can, using another choice of function χ (see Sec. IV), write the latter so that only even k figure explicitly. This can be done, for example, replacing in (30) $\cot(\pi m_1/2)\cot(\pi m_2/2)$ by the expression

$$\frac{\sin[\pi(j-m_1-m_2)/2]}{\sin(\pi m_1/2)\sin(\pi m_2/2)\sin(\pi j/2)},$$

which is no worse than the initial one. This procedure is used in the following.

We are only interested in the extreme right-hand singularity corresponding to $k=0$. The singular part of the integrals over m_1, m_2, l_1, l_2 is calculated just as was done in the previous section, using for $D_{l_1}(t_1)$ and $D_{l_2}(t_2)$ near the pole the value (22)

$$\frac{1}{(4i)^4}\int_{M_1}\frac{dm_1}{\sin(\pi m_1/2)}\int_{M_2}\frac{dm_2}{\sin(\pi m_2/2)}\int_{L_1}\frac{dl_1}{\tan[\pi(l_1-m_1)/2]}\int_{L_2}\frac{dl_2}{\tan[\pi(l_2-m_2)/2]}\frac{2b(l_1,m_1;l_2,m_2)}{[l_1-\alpha(t_1)][l_2-\alpha(t_2)]}$$

$$\times\frac{\Gamma(j+1-m_1-m_2)}{\Gamma(j+1+m_1+m_2)}\frac{\sin[\pi(j-m_1-m_2)/2]}{\sin(\pi j/2)}=\frac{\Lambda^2(j,\alpha_1,\alpha_2)}{j+1-\alpha(t_1)-\alpha(t_2)},\quad j+1\to\alpha(t_1)+\alpha(t_2),$$

$$\Lambda^2(j,\alpha_1,\alpha_2)=\frac{-2\pi^2}{\sin(\pi j/2)\sin(\pi\alpha_1/2)\sin(\pi\alpha_2/2)}\frac{2\alpha_1+1}{\Gamma(2\alpha_1+1)}\frac{2\alpha_2+1}{\Gamma(2\alpha_2+1)}\frac{1}{\Gamma(2j+2)},$$

which is a function having no singularities in the region of values of j close to $\alpha_1+\alpha_2-1$ ($\alpha_1=\alpha(t_1)$, $\alpha_2=\alpha(t_2)$). The expression $b(l_1,m_1;l_2,m_2)$ in the integral denotes all the factors in Eq. (6) except

$$(2j+1)\Gamma(j+1-m_1-m_2)/\Gamma(j+1+m_1+m_2)$$

and

$$b(\alpha_1,\alpha_1,\alpha_2,\alpha_2)=(2\alpha_1+1)(2\alpha_2+1)/\Gamma(2\alpha_1+1)\Gamma(2\alpha_2+1).$$

Hence for the singular part of Eq. (30) we obtain

$$\Delta_4'f_j(t)=\frac{1}{2!}\frac{1}{(2i)^2}\int_{C_2}dt_2\int_{C_1}dt_1 \times\frac{N_{j\alpha_1\alpha_2}(t;t_1,t_2)N_j^{(4)}(t;t_1,t_2)}{j+1-\alpha(t_1)-\alpha(t_2)}\frac{2p(t;t_1,t_2)}{t^{1/2}} \quad (31)$$

while

$$N_{j\alpha_1\alpha_2}(t;\,t_1,t_2)=\Lambda(j,\alpha_1,\alpha_2)g_1(t_1)g_2(t_2)G_{j\alpha_1\alpha_2}(t;\,t_1,t_2) \quad (32)$$

has the meaning of the two-Regge-pole production amplitude. The latter can be noticed by considering the production of four particles (just as in the case of three particles in the previous section) as occurring through two virtual Regge states (Fig. 12). In the region $l_1\to\alpha_1$, $l_2\to\alpha_2$, its amplitude has the form

$$f_{j\lambda_4}=N_{j\alpha_1\alpha_2}'\frac{1}{(l_1-\alpha_1)(l_2-\alpha_2)}g_1g_2\,,$$

where $N_{j\alpha_1\alpha_2}'$ is the amplitude of the transition of two particles into two Regge poles. According to Eqs. (11) and (22) we have $N_{j\alpha_1\alpha_2}'=g_1g_2G_{j\alpha_1\alpha_2}$, i.e., $N_{j\alpha_1\alpha_2}$ differs from the amplitude $N_{j\alpha_1\alpha_2}'$ by only a factor $\Lambda(j,\alpha_1\alpha_2)$ (with standard dependence on t_1, t_2, and j).

Consequently, just as in the three-particle case the constants g_1 and g_2 of the decay of both Regge poles into particles drop out of the result (31) altogether. This circumstance is a general property of relativistic theory in which Regge poles act as real particles.

Equation (31) takes into account the fact that four particles can be grouped in three different ways into two pairs with definite l_1m_1 and l_2m_2. Since all the particles are identical all these ways yield (just as in the three-particle case) the same contribution and hence the factor 3 was introduced in Eq. (31). Instead of the coefficient $2^2/4!$ in Eq. (30), the factor $1/2!=(2^2/4!)\times 3$ appears in Eq. (31). The factor $1/2!$ in Eq. (31) corresponds to the identity of both Regge poles.

Integral (31) has singularities of several types. The integral over t_1 over the contour C_1 of Fig. 3 [denoted by $\phi_j(t_1t_2)$] at a fixed t_2 may have singularities if the zero of the denominator in Eq. (31) coincides with the integration edges $t_1=4\mu^2$ and $t_1=(t^{1/2}-t_2^{1/2})^2$. In other words, the conditions for the emergence of a singularity of this integral are

$$j+1=\alpha(4\mu^2)+\alpha(t_2) \quad (33)$$

and

$$j+1=\alpha((t^{1/2}-t_2^{1/2})^2)+\alpha(t_2)\,. \quad (34)$$

In the second integration over t_2 the singularities of the integral $\phi_j(t,t_2)$ may coincide either with the integration edges $t_2=(t^{1/2}-2\mu)^2$ and $t_2=4\mu^2$ or they may pinch the integration contour C_1 from two sides. Therefore, the singularities of the first and second types of integral (31) are given[26] by the conditions

$$j=2\alpha(4\mu^2)-1\,, \quad (35)$$

$$j=\alpha((t^{1/2}-2\mu)^2)+\alpha(4\mu^2)-1\,. \quad (36)$$

The singularities of the third type are the most interesting since it is only these singularities that remain in the j plane as t is decreased and the region of negative t

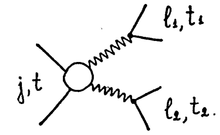

FIG. 12. The four-particle production amplitude with Regge poles in t_1 and t_2 channels.

reached. The location of these singularities is given by that value of j for which the two solutions $t_2=t_2^{(+)}$ and $t_2=t_2^{(-)}$ of Eq. (34) with respect to t coincide. The condition for this coincidence is the vanishing of the derivative in t_2 of the right-hand side of Eq. (34).

$$\alpha'(t_2)=\alpha'((t^{1/2}-t_2^{1/2})^2)((t^{1/2}/t_2^{1/2})-1)\,. \quad (37)$$

This equation has in any case the solution

$$t_2^{1/2}=t^{1/2}/2\,, \quad (38)$$

the substitution of which into Eq. (34) gives the location of the corresponding singularity

$$j=j_2(t)=2\alpha(t/4)-1\,. \quad (39)$$

The problem of the possibility of other solutions requires a special study of the properties of the trajectory $\alpha(t)$.

It can be shown that there is a certain way of movement in the j plane (from region of large values of j where $\Delta_4 f_j$ has no singularities) such that the two solutions $t_2^{(+)}$ and $t_2^{(-)}$ actually pinch the contour C_2 in the t_2 plane as j tends to the value (39).

When $t=16\mu^2$, all the three singularities (35), (36), and (39) of the function $\Delta_4 f_j(t)$ coincide. As t is decreased and the point $t=16\mu^2$ is crossed, singularities (36) and (39) go around each other as they move in the j plane. In Appendix B we show that the function $f_j(t)$ has no singularities (35) and (36) at $t<16\mu^2$ if the cut from singularity (39) in the j plane is made to the left along the real axis [singularities (35) and (36) come onto another sheet connected with this cut]. The movement of the singularities (35), (36), and (2) as j is varied was considered by Simonov.[25]

Let us calculate the discontinuity $\delta_jf_j(t)$ of the function $f_j(t)$ across the cut of interest to us at singularity (39) in the j plane. Since the singularities of $f_j^{(4)}(t)$ do not coincide with those of $f_j(t)$ we have

$$\delta_j\Delta_4 f_j(t)\equiv\delta_j(1/2i)[f_j(t)-f_j^{(4)}(t)]=(1/2i)\delta_jf_j(t)\,.$$

Therefore it is sufficient to determine the discontinuity of integral (31)

$$\Delta_4' f_j(t)=\frac{1}{2i}\int_{C_2}\phi_j(t,t_2)dt_2\,. \quad (40)$$

The latter results, since the two singularities $t_2=t_2^{(+)}$ and $t_2=t_2^{(-)}$ of the function

$$\phi_j(t,t_2)=\frac{1}{4i}\int_{C_1}\frac{N_{j\alpha_1\alpha_2}N_{j\alpha_1\alpha_2}^{(4)}}{j+1-\alpha(t_1)-\alpha(t_2)}\frac{2p(t;\,t_1,t_2)}{t^{1/2}}dt_1 \quad (41)$$

approaching in the plane t_2 the contour C_2 of Fig. 2 deform it in a different way depending on the method of

[26] Besides these points, quantity (31) has singularities at complex-conjugate points.

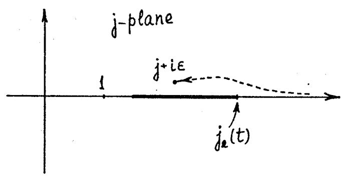

FIG. 13. The path of enclosure in j plane for the singular point $j=j_2(t)$, for which the point j proves to be on the upper edge of the cut.

enclosure of the point $j=j_2(t)$ in the j plane of Fig. 13 [for the time being we neglect the terms in $\delta_j f_j(t)$ resulting from dependence on j of the amplitudes $N_{j\alpha_1\alpha_2}$ and $N_{j\alpha_1\alpha_2}{}^{(4)}$ themselves; see the following section].

Calculating the discontinuity of the integral (40) we obtain for $\delta_j f_j(t)$:

$$\delta_j f_j(t)=\int_{t_2^{(+)}}^{t_2^{(-)}} \Delta_{t_2}\phi_j(t,t_2)dt_2\,, \qquad (42)$$

where $t_2^{(+)}$ is that one of the two singularities of the function (41) which has a positive imaginary part at $j<j_2$ (and at j close to j_2). Here $\Delta_{t_2}\phi_j(t,t_2)$ is a discontinuity of integral (41) on the contour drawn between the points $t_2^{(+)}$ and $t_2^{(-)}$ (i.e., the difference of its values on both sides of the contour divided by $2i$).

This discontinuity equals, accurately to the factor $2i$, the integral (41) taken over the contour $C_2{}^0$ enclosing in the negative direction (as indicated in Fig. 3) the point $t_1=t_1{}^0$ in which the nominator in Eq. (41) vanishes:

$$2i\Delta_{t_2}\phi_j(t,t_2)=\frac{1}{4i}\int_{C_2{}^0}\frac{N_{j\alpha_1\alpha_2}N_{j\alpha_1\alpha_2}{}^{(4)}}{j+1-\alpha(t_1)-\alpha(t_2)}\frac{2p(t;\,t_1,t_2)}{t^{1/2}}dt_1 =\frac{\pi}{2\alpha'(t_1{}^0)}N_{j\alpha_1\alpha_2{}^0}N_{j\alpha_1\alpha_2{}^0}{}^{(4)}\frac{2p(t,t_2,t_1{}^0)}{t^{1/2}}\,. \qquad (43)$$

This value can conveniently be written symbolically

$$\Delta_{t_2}\phi_j=\frac{1}{2}\frac{\pi}{2i}\int N_{j\alpha_1\alpha_2}N_{j\alpha_1\alpha_2}{}^{(4)} \times\frac{2p(t;\,t_1,t_2)}{t^{1/2}}\delta(j+1-\alpha(t_1)-\alpha(t_2))dt_1\,,$$

assuming that the contour of integration in the complex t_1 plane is chosen so that the argument of the δ function runs through zero taking real values. This form of Eq. (43) is convenient (though not necessary) for the following.

Thus the discontinuity $f_j(t)$ at a two-Regge-pole singularity can be represented as

$$\delta_j f_j(t)=\frac{\pi}{2}\int_{t_2^{(+)}}^{t_2^{(-)}}\frac{dt_2}{2i}\int dt_1 N_{j\alpha_1\alpha_2}N_{j\alpha_1\alpha_2}{}^{(4)} \times\frac{2p(t;\,t_1,t_2)}{t^{1/2}}\delta(j+1-\alpha(t_1)-\alpha(t_2))\,, \qquad (44)$$

where $t_2^{(+)}$ and $t_2^{(-)}$ are, as indicated above, the roots of Eq. (34) which can be regarded as a consequence of the two conditions

$$j+1-\alpha(t_1)-\alpha(t_2)=0\,,$$
$$p(t,t_1,t_2)=0\,,$$

or more accurately the equation $t^{1/2}=t_1{}^{1/2}+t_2{}^{1/2}$.

Let us determine the behavior of $\delta_j f_j(t)$ at j close to $j_2(t)$, when values of t_1 and t_2 close to $t/4$ are required in Eq. (44).

Under these conditions the amplitude N_{j,α_1,α_2} cannot be considered as constant (in spite of the fact that at $j\to j_2$ the regions of integration over t_1 and t_2 tend to zero) because at $p=p(t,t_1,t_2)\to 0$, $N_{j,\alpha_1\alpha_2}$ has a threshold singularity of the type $N_{j,\alpha_1\alpha_2}=C_j(2p)^L$. Here $L=j-\alpha_1-\alpha_2$ is the minimal value of the orbital angular momentum of relative motion of two Regge poles. In the integral (45) $L=-1$ and, consequently,

$$N_{j,\alpha_1\alpha_2}=C_j(t)/p\,, \qquad (45)$$

i.e., it tends to infinity at $p\to 0$.

It should be noted that owing to the unitarity condition the production amplitude of two particles $N_{j,l_1l_2}(t,m_1{}^2,m_2{}^2)$ (but not of two Regge poles) cannot tend to infinity as $1/p$. Its true threshold behavior is determined[8] by the formula

$$N_{j,l_1l_2}(t,m_1{}^2,m_2{}^2)=k^L\bigg/\left(\Lambda+\frac{ik^{2L+1}}{j_1n\pi L}e^{i\pi L}\right),$$

where $k=p(t,m_1{}^2,m_2{}^2)$, m_1 and m_2 being the masses of two real particles. At $L=j-l_1-l_2=-1$ and at $k\to 0$ the amplitude $N_{j,l_1l_2}(t,m_1{}^2,m_2{}^2)$ tends to a constant but not to $1/k$. The compensating factor $k^{2L+1}=1/k$ (at $L=-1$) in the denominator results from summing up the diagrams with two particles (with the masses m_1 and m_2) in the intermediate state. Each of these diagrams has a singularity at $k=0$, i.e., at $t^{1/2}=m_1+m_2$.

In the case of the amplitude of two-Regge-pole production the factor $p^L(t,t_1,t_2)$ could be correspondingly compensated only by the contribution of the multiparticle intermediate states. However, this contribution has the form of the integral over the energies t_1', t_2' of the groups of particles produced in the intermediate state and, consequently, has a singularity at the real thresholds ($t=16m^2$, $36m^2$, etc.). Therefore they cannot compensate the singularity at $t^{1/2}=t_1{}^{1/2}+t_2{}^{1/2}$, where t_1 and t_2 are arbitrary energies of particles in the final state.

Let us calculate the integral (44) over t_1 and t_2 at $j\to j_2$ using the form (45) of $N_{j,\alpha_1\alpha_2}$. To this end, assuming $t_1=t/4+x_1$, $t_2=t/4+x_2$, we expand $\alpha(t_1)$ and $\alpha(t_2)$ into series

$$\alpha(t_i)=\alpha(t/4)+\alpha'x_i+\tfrac{1}{2}\alpha''x_i{}^2\,,\quad i=1,\,2$$

and substitute the value x_1 from the condition

$$j-j_2(t)=\alpha'(x_1+x_2)+\tfrac{1}{2}\alpha''(x_1{}^2+x_2{}^2)$$

into

$$p(t,t_1,t_2)=1/2t^{1/2}[(x_1-x_2)^2-2t(x_1+x_2)]^{1/2}.$$

We obtain from (44)

$$\delta_j f_j(t)=\pi C_j C_j^{(4)}\cdot B_2, \qquad (46)$$

where $B_2=-\frac{1}{8}\pi[\alpha_0'(\alpha_0'+\frac{1}{2}\alpha_0''t)]^{-1/2}$ does not depend on j at $j\to j_2(t)$.

If C_j at $j\to j_2$ had no singularities (as was the case we supposed so far) then it would follow from (46) that $f_j(t)$ at $j\to j_2(t)$ has a logarithmic singularity of the form

$$f_j(t)\simeq A+B_2C_jC_j^{(4)}\ln[j-j_2(t)]. \qquad (47)$$

Because of the fact that at $j=j_2(t)$, Eq. (47) tends to infinity, the singularity of C_j changes essentially the character of the singularity of $f_j(t)$. In the following this question will be considered in detail.

The presence of this (and other similar singularities) changes radically the analytical properties of $f_j(t)$ in the t plane. Besides the cuts indicated in Fig. 1 and connected with the thresholds of production of usual particles, in the t plane there must appear, obviously, a logarithmic branch point at $t=t_2(j)$ where $t_2(j)$ is the solution of the equation

$$j=j_2(t).$$

Therefore, to determine $f_j(t)$ unambiguously in the t plane it is necessary, besides the cuts indicated in Fig. 1, to make a cut indicated in Fig. 14 from the point $t=t_2(j)$. It can readily be noticed that the discontinuity $\delta_t^{(2)}f_j(t)$ across this cut differs from $\delta_j f_j(t)$ only in sign:

$$\delta_t^{(2)}f_j(t)=-\delta_j f_j(t). \qquad (48)$$

Therefore, if we know the magnitude of $\delta_j f_j$ we can restore the amplitude $f_j(t)$ with the aid of the dispersion integral

$$f_j(t)=\frac{1}{\pi}\int_{t_2(j)}^{\infty}\frac{\delta_{t'}^{(2)}f_j(t')dt'}{t'-t}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\Delta_2 f_j(t')dt'}{t'-t}+\frac{1}{\pi}\int_{16\mu^2}^{\infty}\frac{\Delta_4 f_j(t')dt'}{t'-t}. \qquad (49)$$

To take into account in the t plane not only two-Regge-pole singularities but all multi-Regge-pole (three-, four-, etc., Regge-pole) singularities, the additional terms[27]

$$\frac{1}{\pi}\int_{t_n(j)}^{\infty}\frac{\delta_j^{(n)}f_j(t')dt'}{t'-t}$$

should be included into Eq. (49).

[27] From Eq. (2) it follows that at real j the singularity $t=t_n(j)$ emerges on the physical sheet of Fig. 1 in the t plane only for such n for which $j_n^{(0)}>j$, where $j_n^{(0)}=n[\alpha(4\mu^2)-1]+1$, while $1\leqslant\alpha(4\mu^2)\leqslant 2$.

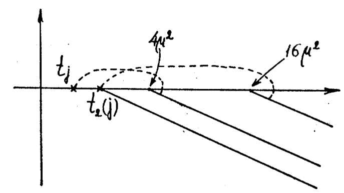

FIG. 14. The cut in t plane corresponding to the branch point $t=t_2(j)$; broken lines indicate the trajectories of the branch point $t=t_2(j)$ and the movements of the pole $t=t_j$ with the decreasing of $j\to j+i\epsilon$.

VI. UNITARITY CONDITION FOR TWO-REGGE-POLE SINGULARITY

While investigating the singularities of $f_j(t)$ we have so far neglected the fact that the amplitudes $N_{j\alpha}$ and $N_{j\alpha_1\alpha_2}$ entering into $\Delta_3 f_j$ or $\Delta_4 f_j$ may themselves have singularities at the same points as $f_j(t)$. Actually, by unitarity, these singularities must be present in all the amplitudes.

Let us show that taking this into account changes the value $\delta_j f_j(t)=-\delta_t^{(2)}f_j(t)$ obtained earlier and results in the fact that the accurate unitarity condition, giving the magnitude of the discontinuity $\delta_t f_j(t)$ of the amplitude $f_j(t)$ at the two-Regge-pole singularity $t=t_2(j)$ in the t plane, has the form

$$\delta_j^{(2)}f_j(t)=\frac{\pi}{2!}\int_{t_2^{(-)}}^{t_2^{(+)}}\frac{dt_2}{2i}\int dt_1 N_{j^+,\alpha_1\alpha_2}\times N_{j^-,\alpha_1\alpha_2}\frac{2p(t;t_1,t_2)}{t^{1/2}}\delta(j+1-\alpha(t_1)-\alpha(t_2)), \qquad (50)$$

where

$$N_{j^\pm,\alpha_1\alpha_2}=N_{j\pm i\epsilon,\alpha_1\alpha_2}=N_j(t\mp i\epsilon,t_1,t_2)$$

are the amplitudes of the production of two Regge poles on both banks of the cut $t=t_2$ in Fig. 15 at $t>t_2(j)$. It should be emphasized that both quantities $N_{j^+,\alpha_1\alpha_2}$ and $N_{j^-,\alpha_1\alpha_2}$ entering into this formula are determined on the same physical sheet of the t plane (with respect to the thresholds of production of usual particles) in contrast to Eq. (44) containing the amplitude $N_{j\alpha_1\alpha_2}^{(4)}$ on an unphysical sheet.

Equation (50) can be interpreted as a two-Regge-pole unitarity condition in this sense. The branch point $t=t_2(j)$ comes onto the physical sheet of the t plane of Fig. 14 from under the cut running from $t=16\mu^2$, and moves as indicated by a broken line in Fig. 14 as j is decreased [along the real axis from large values for which $f_j(t)$ has no singularity $t=t_2(j)$ on a physical sheet]. Since the singularity at $t=t_2(j)$ can be regarded as a threshold singularity corresponding to the production of two Regge poles, Eq. (50), the right-hand side of which contains the amplitudes $N_{j^\pm,\alpha_1\alpha_2}\equiv N_{j\alpha_1\alpha_2}(t\mp i\epsilon,t_1,t_2)$ of the production of two Regge poles on both banks of the cut connected with the same singularity, is perfectly similar to the conventional unitarity condition.

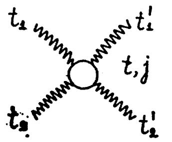

FIG. 15. The amplitude of the transition of two Regge poles with masses $t_1^{1/2}$ and $t_2^{1/2}$ into two Regge poles with masses $t_1'^{1/2}$ and $t_2'^{1/2}$.

To avoid a tiresome number of subscripts, we derive Eq. (50) in the symbolic-operator form. The state of four particles will be characterized by the quantum number n including the angular momenta and helicities l_1, m_1 and l_2, m_2 of two pairs of particles and their energies $t_1^{1/2}$ and $t_2^{1/2}$. We shall need the unitarity conditions for the amplitudes of the transition of two particles into two and into four: $f_j(t)$ and $f_n{}^j \equiv f^j{}_{l_1m_1l_2m_2}(t;t_1,t_2)$, and for the amplitudes of the transition of four particles into four $f_{nn'}{}^j$. The amplitudes $f_n{}^j$ and $f_{nn'}{}^j$ are written in a form similar to Eqs. (10) and (11)

$$f_n{}^j=\frac{G_{j;n}}{D_{l_1}(t_1)D_{l_2}(t_2)};\qquad f_{nn'}{}^j=\frac{H_{j;nn'}}{D_{l_1}(t_1)D_{l_2}(t_2)D_{l_1'}(t_1')D_{l_2'}(t_2')}\,. \tag{51}$$

Here $H_{j;nn'}$ (just as $G_{j;n}$) has no singularities at the two-particle thresholds in the variables t_1, t_2 and t_1', t_2', while the quantity

$$M_{j;n_0n_0'}=g_1g_2g_1'g_2'H_{j;n_0n_0'}\Lambda_j\Lambda_j',\quad \Lambda_j=\Lambda(j,\alpha_1,\alpha_2),\quad \Lambda_j'=\Lambda(j,\alpha_1',\alpha_2');\quad g_i=g(t_i),\quad g_i'=g(t_i') \tag{52}$$

[where n_0 is the state with $l_i=m_i=\alpha(t_i)$, $l_i'=m_i'=\alpha(t_i')$] has the meaning of the amplitude of the transition of two Regge poles with masses $t_1^{1/2}$ and $t_2^{1/2}$ into two Regge poles with masses $t_1^{1/2}$ and $t_2'^{1/2}$ (Fig. 15).

In these notations the four-particle unitarity conditions for the amplitudes f_j, $f_n{}^j$, and $f_{nn'}{}^j$ can be written as

$$(1/2i)[f_j{}^+-f_j{}^{(4)}]=(G_j{}^+\Gamma_jG_j{}^{(4)})\,, \tag{53a}$$

$$(1/2i)[G_j{}^+-G_j{}^{(4)}]=(G_j{}^+\Gamma_j{}^+H_j{}^{(4)})=(G_j{}^{(4)}\Gamma_j{}^+H_j{}^+)\,, \tag{53b}$$

$$(1/2i)[H_j{}^+-H_j{}^{(4)}]=(H_j{}^+\Gamma_j{}^+H_j{}^{(4)})\,. \tag{53c}$$

Here $(G_j\Gamma_jG_j{}^{(4)})=(G_{jn'}\Gamma_{jn'}G_{jn'}{}^{(4)})$ symbolizes the right-hand side of Eqs. (31) and (32). Similarly, $(G_j\Gamma_jH_j{}^{(4)})\equiv(G_{jn'}\Gamma_{jn'}H_{jn'n}{}^{(4)})$ and $(H_j\Gamma_jH_j{}^{(4)})$. The quantity Γ_{jn} contains all the singular factors entering into Eq. (31).

Calculating the discontinuity from both parts of Eq. (53) in j at the singularity $j=j_2$, we obtain

$$(1/2i)\delta_jf_j=(G_j{}^-\delta\Gamma_jG_j{}^{(4)})+(\delta G_j\Gamma_j{}^+G_j{}^{(4)})\,, \tag{54a}$$

$$(1/2i)\delta_jG_j=(G_j{}^-\delta\Gamma_jH_j{}^{(4)})+(\delta G_j\Gamma_j{}^+H_j{}^{(4)})\,, \tag{54b}$$

$$(1/2i)\delta_jH_j=(H_j{}^-\delta\Gamma_jH_j{}^{(4)})+(\delta H_j\Gamma_j{}^+H_j{}^{(4)})\,, \tag{54c}$$

where $j^\pm=j\pm i\tau$ means that the quantities on the right-hand side are taken on the upper or lower bank of the cut of Fig. 12 corresponding to the singularity $j=j_2(t)$. The first terms in the right-hand sides of these equations are precisely those discontinuities of integrals like (31), one of which [the first in the right-hand side of Eq. (54a) was calculated above in Eq. (44)]. Comparing the first term in the right-hand side of Eq. (54a) with Eqs. (44) and (32), we notice that[28]

$$\delta\Gamma_{jn}\sim\frac{\pi\Lambda^2}{(2i)^2}g_1{}^2g_2{}^2\frac{2p(t,t_1,t_2)}{t^{1/2}}\delta(j+1-\alpha(t_1)-\alpha(t_2))\times\delta(l_1-\alpha_1)\delta(m_1-\alpha_1)\delta(l_2-\alpha_2)\delta(m_2-\alpha_2)\,.$$

Equation (54c) is an equation with respect to δH_j; comparing it with Eq. (53c) regarded as an equation with respect to $H_j{}^+$

$$(1/2i)H_j{}^+=(1/2i)H_j{}^{(4)}+(H_j{}^+\Gamma_j{}^+H_j{}^{(4)})\,, \tag{55}$$

we notice that the solution of Eq. (54c) is

$$(1/2i)\delta_jH_j=(H_j{}^-\delta\Gamma_jH_j{}^+) \tag{56}$$

[since on the application of the operator $H_j{}^-\delta\Gamma_j$ to both sides of Eq. (55) the equation obtained identically coincides with Eq. (54c) under the condition (56)].

Equation (56) is the unitarity condition for the amplitude (52) of the transition of two Regge poles into two Regge poles. Similarly, comparing Eqs. (55) and (54b), we notice that

$$(1/2i)\delta_jG_j=(G_j{}^-\delta\Gamma_jH_j{}^+)\,. \tag{57}$$

Substituting this value into Eq. (54a) and taking into account the second equation (53b), we obtain

$$(1/2i)\delta_jf_j(t)=(G_j{}^+\delta\Gamma_jG_j{}^-)\,. \tag{58}$$

This equation is, according to Eq. (44), that unitarity condition (50) which has to be demonstrated.

Multiplying both sides of Eq. (57) by the quantity $\Lambda(j,\alpha_1,\alpha_2)g(t_1)g(t_2)$ and both sides of Eq. (56) by $(\Lambda g_1g_2)(\Lambda'g_1'g_2')$ and using our notations, we notice that these equations are the unitarity conditions for the amplitudes (32) and (52) of the transition of two particles into two Regge poles and of two Regge poles into two Regge poles. Quite similarly to Eq. (50) they can be represented as

$$\delta_jN_{\alpha_1\alpha_2}{}^j=\frac{\pi}{2i}\int_{t_2^{(-)}}^{t_2^{(+)}}dt_2'\times\int dt_1'N_{\alpha_1'\alpha_2'}{}^{j+}M_{\alpha_1'\alpha_2',\alpha_1\alpha_2}{}^{j-}\times\frac{2p(t,t_2',t_2')}{t^{1/2}}\delta(j+1-\alpha_1'-\alpha_2')\,, \tag{59}$$

[28] Here the δ functions denote symbolically that instead of the sum over l_1m_1 and l_2m_2 of the form (14) we must only consider one term with $l_1=m_1=\alpha(t_1)$ and $l_2=m_2=\alpha(t_2)$, and instead of the factor $C_j(\lambda_4)$ entering into Eq. (14) substitute unity.

$$\delta_j M_{\alpha_1'\alpha_2',\alpha_1\alpha_2}{}^j=\frac{\pi}{2i}\int_{t_2^{(-)}}^{t_2^{(+)}}dt_2'\times\int dt_1' M_{\alpha_1'\alpha_2',\alpha_1''\alpha_2''}{}^{j+}M_{\alpha_1''\alpha_2'',\alpha_1\alpha_2}{}^{j-}\times\frac{2p(t,t_2'',t_2'')}{t^{1/2}}\delta(j+1-\alpha_1''-\alpha_2''). \quad (60)$$

Formulas (50), (59), and (60) make it possible to determine the character of singularities of the amplitudes $f_j(t)$, $N_{\alpha_1\alpha_2}{}^j$, and $M_{\alpha_1\alpha_2,\alpha_1'\alpha_2'}{}^j$ at $j\to j_2(t)$. Substituting into the right-hand parts of these formulas at $t_1\to t/4$ and $t_2\to t/4$ the amplitude $N_{\alpha_1\alpha_2}{}^j$ in the form

$$N_{\alpha_1\alpha_2}{}^j=C_j/p(t,t_1,t_2)$$

[see (45)] and similarly (at $j-\alpha_1-\alpha_2=-1$ and $j-\alpha_1'-\alpha_2'=-1$)

$$M_{\alpha_1'\alpha_2',\alpha_1\alpha_2}{}^j=d_j/p(t,t_1',t_2')\cdot p(t,t_1,t_2),$$

we obtain

$$\delta_j^{(2)}d_j=\pi B_2 d_j d_j^+,$$
$$\delta_j^{(2)}C_j=\pi B_2 C_j d_j^+,$$
$$\delta_j^{(2)}f_j=\pi B_2 C_j C_j^+.$$

It follows from the first equality that

$$\delta_j^{(2)}(1/d_j)=-\pi B_2.$$

Hence

$$d_j=1/[A-B_2\ln(j-j_2)].$$

From the second and third equalities we obtain

$$C_j=\nu/[A-B_2\ln(j-j_2)],$$
$$f_j=\nu^2/[A-B_2\ln(j-j_2)]+f_0,$$

where A, ν and f_0 have no singularities at $j\to j_2(t)$.

VII. THREE- AND MULTI-REGGE-POLE STATES

The Regge poles of the amplitude $f_j(\lambda_4)$ of Fig. 11 of the transition of two particles into four, in the angular momenta l_1 and l_2 of the pairs of the particles produced, result from the interaction of the particles within these pairs. As shown in the previous section, the same interaction results in the angular-momentum plane j in branch points of the type (47). It can readily be noticed that taking into account the new singularities of the amplitude $f_{j;\lambda_n}=f_{j;l_1m_1,l_2m_2}$ in the variables l_1 and l_2 in integrals of the type (30) leads to a series of branch points at $j=j_n(t)$, where $j_n(t)$ is given in Eq. (2). From this formula it follows that the nth branch point becomes complex at $t=(2\mu n)^2$. Therefore, it should be expected that the nth point of this series is connected by the $2n$-particle unitarity condition, or more accurately, that it is to come into the tth plane (as j decreases from large values along the real axis) from the unphysical sheet of Fig. 1 resulting from the production of $2n$ particles.

Therefore, when studying the nth singularity (2), it is natural to consider the $2n$-particle unitarity condition (18). However, before proceeding to the general case we shall have to deal briefly with the six-particle (three-Regge-pole) unitarity condition (15)–(16). Let us consider its analytical continuation to complex j. If this continuation is written in the form of contour integrals of the type (30) over l_1m_1, l_2m_2, $l_{12}m_{12}$, l_3m_3 and the Regge poles in l_1, l_2, and l_3 are taken into account, in the calculation of the singular part of the integral there will arise only the following difference from the two-Regge-pole case considered in Sec. V.

Two Γ functions entering into $C_j(l_{12},m_{12},l_3m_3)$ and $C_{l_{12}}(l_1m_1,l_2m_2)$ in Eq. (15)

$$\Gamma(j+1-m_{12}-m_3)\cdot\Gamma(l_{12}+1-m_1-m_2)$$

are essential for the emergence of a singularity. The singularity of $\Delta_6 f_j(t)$ results from the following points in the contour integrals over l_i, m_i, l_{12}, m_{12} of type (30)

$$l_1=m_1=\alpha(t_1),\quad l_2=m_2=\alpha(t_2),\quad l_3=m_3=\alpha(t_3)$$

and

$$l_{12}=m_{12}=\alpha(t_1)+\alpha(t_2)-1.$$

For the singular part of the integral we obtain an expression analogous to Eq. (31):

$$\Delta_6 f_j(t)=\frac{1}{2!}\int\frac{N_{j;12,\alpha_a}N_{j;12,\alpha_a}}{j+2-\alpha(t_1)-\alpha(t_2)-\alpha(t_3)}\times\frac{2p(t;t_{12},t_3)}{t^{1/2}}\frac{2p(t_{12},t_1,t_2)}{t^{1/2}}dt_{12}\frac{dt_1dt_2dt_3}{(2i)^3},\quad (61)$$

where $N_{j;12,\alpha_a}=N_{j;12,\alpha_1,\alpha_2,\alpha_3}$ is the amplitude of production of three Regge poles in a state with angular momentum l_{12} of one pair of them equal to $\alpha(t_1)+\alpha(t_2)-1$. The amplitude $N_{j;12,\alpha_a}$ is connected with the quantity $G_{j;l_{12},l_1,l_2l_3}(t;t_{12},t_1,t_2,t_3)$, given according to Eq. (12a), by

$$N_{j;12,\alpha_a}=\Lambda(j,l_{12},\alpha_3)\cdot\Lambda(l_{12},\alpha_1,\alpha_2)\cdot g_1\cdot g_2\cdot g_3 G_{j,l_{12},\alpha_1,\alpha_2\alpha_3}. \quad (61a)$$

With the normalization chosen in Eq. (16) there actually appears a factor $2^3/6!$ in front of the integral in (61). However, it should be borne in mind that in this case besides $6!/2^33!=15$ ways of distribution of six particles in three pairs with definite values of angular momenta there are three ways to group two of the three Regge poles in a pair with definite l_{12}. Therefore the factor $2^3/6!$ should be multiplied by $(6!/2^33!)\times 3$, which is what gives in Eq. (61) the factor $\frac{1}{2}$ corresponding to the identity of the two Regge poles forming a state with a given l_{12}.

The singularities of integral (61) result from the zeros of the denominator in Eq. (61) which is independent of t_{12}. Therefore we assume that integration over t_{12} has been performed [within $(t_1^{1/2}+t_2^{1/2})^2<t_{12}<(t^{1/2}-t_3^{1/2})^2$] and integration over t_a, $a=1, 2, 3$, is made over the contours C_a indicated in Eq. (16). A singularity of the integral arises on coincidence of a zero of the denominator with the boundaries of integration. The coincidence of a denominator zero with the lower limit $4\mu^2$ of one of the variables t_a gives rise to singularities of type (35) and (36) whose location depends on the particle masses. These singularities are of no interest to us since just as in the two-Regge-pole case they can be shown to occur, at a small t, on the unphysical sheets of the j plane.

A singularity independent of particle masses results from coincidence of denominator zeros with the upper limit of integration over t_i given by the condition

$$\chi(t_1,t_2,t_3;\,t)=t_1^{1/2}+t_2^{1/2}+t_3^{1/2}-t^{1/2}=0. \qquad (62)$$

For integral (61) to really have a singularity in the j plane it is necessary that in each consecutive integration the singularities of the previous integral pinch the contour of integration.

Consideration of the integrals over t_1, t_2, and t_3 one after another suggests that for the above purpose it is necessary [just as in the case of a simpler integral (31)] that the denominator in Eq. (61),

$$\square(t_1,t_2,t_3;\,j)=j+2-\alpha(t_1)-\alpha(t_2)-\alpha(t_3), \qquad (63)$$

have an extremum under condition (62).

Therefore the location of the singularity can be determined from the absolute extremum condition for the function

$$\square'=j+2-\alpha(t_1)-\alpha(t_2)-\alpha(t_3)-\lambda\cdot\chi(t_1,t_2,t_3,t), \qquad (64)$$

the condition

$$\square(t_1,t_2,t_3;\,j)=0, \qquad (65)$$

and the condition (62). The extremum condition leads to

$$\alpha'(t_a)=\lambda/2t_a^{1/2},\quad a=1,2,3.$$

Let t_0 denote the solution of this equation; then from the requirement $\chi=0$ we get $t_0=t/9$, whence we obtain from the condition (65) for the location of the singularity $j=j_3(t)$ the value (2) with $n=3$.

Let us determine the discontinuity $\delta_j^{(3)}\Delta_3 f_j=(1/2i)\times\delta_j^{(3)}f_j(t)$ at this singularity [at $t>0$, $j<j_3(t)$] neglecting the singularities $N_{j,12,\alpha_a}$ for the time being.

Quite similarly to Eq. (42) we obtain

$$\delta_j^{(3)}f(t)=\int_{t_3^{(+)}}^{t_3^{(-)}}\Delta_{t_3}\phi_j^{(3)}(t,t_3)dt_3, \qquad (66)$$

where $\Delta_{t_3}\phi_j^{(3)}$ is the discontinuity of the integral

$$\phi_j^{(3)}(t,t_3)=\frac{1}{2}\cdot\frac{1}{(2i)^2}\times\int_{C_2}\int_{C_1}\frac{dt_2dt_1N_{j;12,\alpha_a}N_{j;12,\alpha_a}^{(6)}I_3(t_1,t_2,t_3;\,t)}{j+1-\alpha(t_1)-\alpha(t_2)-\alpha(t_3)} \qquad (67)$$

across the contour in the t_3 plane drawn between its two singularities $t_3^{(+)}$ and $t_3^{(-)}$ (the singularity $t_3^{(+)}$ is to the right when $j>j_3$; it goes into the upper half-plane if j goes round, as it decreases, the point $j=j_3$ from above). In Eq. (67), $I_3(t_1,t_2,t_3;\,t)$ denotes the total phase space of a system of three particles with masses $t_1^{1/2}$, $t_2^{1/2}$, $t_3^{1/2}$, and energy $t^{1/2}$,

$$I_3(t_1,t_2,t_3;\,t)=\int_{(t_1^{1/2}+t_2^{1/2})^2}^{(t^{1/2}-t_3^{1/2})^2}dt_{12}\frac{2p(t;\,t_{12},t_3)}{t^{1/2}}\cdot\frac{2p(t_{12};\,t_1,t_2)}{t_{12}^{1/2}},$$

whose nonrelativistic value (at $\chi\to 0$) is proportional to χ^2.

The location of the singularities $t_3^{(+)}$ and $t_3^{(-)}$ of integral (67) is given as the solution of Eq. (65) at those values t_1 and t_2 for which the quantity $\square(t_1,t_2,t_3,j)$ has an extremum at the upper limit of integration, i.e., at $\chi(t_1,t_2,t_3;\,t)=0$. In other words, the values $t_1=t_1(t,t_3)$ and $t_2=t_2(t,t_3)$ which must be substituted into Eq. (65), can be found from the equations

$$\frac{\partial\square}{\partial t_1}=\lambda\frac{\partial\chi}{\partial t_1},\quad \frac{\partial\square}{\partial t_2}=\lambda\frac{\partial\chi}{\partial t_2}, \qquad (68)$$

the Lagrange parameter λ being determined from the condition

$$\chi(t_1,t_2,t_3,t)=0.$$

The discontinuity of integral (67) across the contour between $t_3^{(+)}$ and $t_3^{(-)}$ can be calculated just as (Sec. V) the discontinuity of integral (51) was calculated earlier. Substituting its value into Eq. (67) we obtain for $\delta_t^{(3)}f_j(t)=-\delta_j^{(3)}f_j(t)$:

$$\delta_t^{(3)}f_j(t)=\frac{\pi}{2!}\int_{t_3^{(-)}}^{t_3^{(+)}}\frac{1}{2i}dt_3\int_{t_2^{(-)}}^{t_2^{(+)}}\frac{1}{2i}dt_2\int dt_1\times\int_{(t_1^{1/2}+t_2^{1/2})^2}^{(t^{1/2}-t_3^{1/2})^2}dt_{12}N_{j;12,\alpha_a}N_{j;12,\alpha_a}^{(6)}\times\frac{2p(t,t_{12},t_3)}{t^{1/2}}\frac{2p(t_{12},t_1,t_2)}{t_{12}^{1/2}}\delta(\square), \qquad (69)$$

where $t_2^{(-)}$ and $t_2^{(+)}$ are determined at a given t_3 [when $t>0$ and $j<j_2(t)$] as two solutions of the equations

$$\square(t_1,t_2,t_3;\,j)=0;$$
$$\chi(t_1,t_2,t_3,t)=0.$$

These equations also determine the value t_1 essential in Eq. (69).

After a treatment similar to that of the previous section we obtain, taking into account the singularity $N_{j;12,\alpha_a}$, the three-Regge-pole unitarity condition in this form

$$\delta_t^{(3)}f_j(t)=\frac{\pi}{2!}\int_{t_1^{(-)}}^{t_1^{(+)}}\frac{dt_1}{2i}\int_{t_2^{(-)}}^{t_2^{(+)}}\frac{dt_2}{2i}\int dt_3\times\int_{(t_1^{1/2}+t_2^{1/2})^2}^{(t^{1/2}-t_3^{1/2})^2}dt_{12}N_{j^-;12,\alpha_a}N_{j^+;12,\alpha_a}\times\frac{2p(t;\,t_{12},t_3)}{t^{1/2}}\frac{2p(t_{12};\,t_1,t_2)}{t_{12}^{1/2}}\delta(\square). \qquad (70)$$

To calculate the character of the singularity at $j \to j_3(t)$ let us take into account that in the region $t_1 \to t/9$, $t_2 \to t/9$, $t_3 \to t/9$, and $t_{12} \to 4t/9$ that is essential in the integral (70), we have $p(t_{12},t_1,t_2) \to 0$ and $p(t,t_{12},t_3) \to 0$. The threshold behavior of the amplitudes $N_{j;12,\alpha_i}$ has a form analogous to (45):

$$N_{j;12,\alpha_i} \simeq C_{j,3}/p(t_{12},t_1,t_2) \cdot p(t,t_{12},t_3).$$

Substituting $N_{j;12,\alpha_i}$ in this form into (70) and calculating the integral at $j \to j_3(t)$ we obtain

$$\delta_j^{(3)} f_j(t) = \pi C_{j,3} C_{j,3}^{+} \cdot B_3 \cdot (j - j_3),$$

where B_3 is a definite constant. Hence at $j \to j_3(t)$

$$f_j(t) = A_3 + B_3'(j - j_3) \ln(j - j_3),$$

where $B_3' = B_3 C_{j,3} C_{j,3}^{+}$. In this case the singularity C_j at $j = j_3$ should not be taken into account since the contribution from this singularity at $j \to j_3$ tends to zero [being proportional to $j - j_3$, as in the case of $f_j(t)$].

Let us now turn to the general n-Regge-pole case.

If there are more than three Regge poles, it is necessary to bear in mind the following. As indicated above, in order to obtain the contribution from a singularity it is necessary to consider all possible ways of grouping the particles (and groups of particles) into states with definite angular momenta and then add up the results obtained. So far, concerned with identical particles, we have been obtaining identical contributions from all such configurations and therefore the procedure has reduced to the multiplication by a certain number of the right-hand sides of the unitarity conditions.

From the four-Regge-pole case upward there appear groups of Regge poles of different types making different contributions.

In the four-Regge poles case there may be two different groups of Regge poles (see Sec. II) leading to two different terms in the Regge-pole unitarity condition. One of them contains the amplitude $N(jl_{12}l_{34}; l_{12}\alpha_1\alpha_2; l_{34}\alpha_3\alpha_4)$ of the production of four Regge poles in a state with definite momenta of two pairs made of them $l_{12} = \alpha(t_1) + \alpha(t_2) - 1$, $l_{34} = \alpha(t_3) + \alpha(t_4) - 1$, and the other contains the amplitude $N(jl_{123}\alpha_4, l_{123}l_{12}\alpha_3, l_{12}\alpha_1\alpha_2)$ of the production of four Regge poles in a state with a definite angular momentum of a pair $l_{12} = \alpha(t_1) + \alpha(t_2) - 1$ and that of a trio of Regge poles $l_{123} = \alpha(t_1) + \alpha(t_2) + \alpha(t_3) - 1$. The number of different configurations increases with increasing number of Regge poles.

The unitarity condition for the discontinuity $f_j(t)$ at the n-Regge-pole singularity $t = t_n(j)$ has the form

$$\delta_t^{(n)} f_j(t) = \sum_k \delta_t^{(n,k)} f_j(t) = \sum_k \frac{\pi}{2^{\nu_k}} \int N_{j^+;n,k}(t) N_{j^-;n,k}(t) \times \delta(\Box_n) \prod_{\alpha\beta} \frac{p_{\alpha\beta}(t_{\alpha\beta},t_\alpha,t_\beta)}{t_{\alpha\beta}^{1/2}} dt_\alpha dt_\beta \frac{1}{(2i)^{n-1}}, \quad (71)$$

where $N_{j;n,k}(t)$ denotes the kth type of the n-Regge-pole production amplitude corresponding to a certain grouping of them into a state with given angular momenta l_α, l_β ($t_\alpha^{1/2}$ and $t_\beta^{1/2}$ are the energies of these states). The total discontinuity equals the sum $\delta_t^{(n,k)} f_j(t)$ over all types k of such groups. The factor $1/2^{\nu_k}$ results from the multiplication of the initial factor $2^n/(2n)!$ in Eq. (18) by the number of ways of distribution of particles and Regge poles, $((2n)!/2^n \cdot n!) \cdot (n!/2^{\nu_k})$, which can bring about a given configuration. Here $2n!/2^n n!$ is the number of ways of distribution of $2n$ particles in n pairs and $n!/2^{\nu_k}$ the number of ways of grouping of n Regge poles into a given configuration.

The quantity ν_k depends on the form of a configuration and can readily be found (2^{ν_k} is the number of permutations of Regge poles as a result of which the form of the configuration does not change). For example, in the four-Regge-pole case we have for the first of the above configurations $\nu_1 = 3$ and for the second $\nu_2 = 1$.

By $\Box_n$ we denote a quantity analogous to (63)

$$\Box_n = j + n - 1 - \sum_{k=1}^{n} \alpha(t_k).$$

The location of the n-Regge-pole singularity in the plane is determined similarly to the two- and three-Regge-pole cases from the extremum condition for the function

$$\Box_n' = \Box_n - \lambda \cdot \chi_n,$$

where

$$\chi_n = \sum_{\alpha=1}^{n} t_\alpha^{1/2} - t^{1/2}$$

under the condition

$$\chi = 0 \quad \text{and} \quad \Box_n = 0.$$

The solution of these equations can readily be shown to be

$$t_\alpha = t/n^2, \quad \alpha = 1, 2, 3, \cdots, n,$$

and for the location of the nth singularity there follows the value (2)

$$j_n(t) = n\alpha(t/n^2) - n + 1.$$

The function $\delta(\Box_n)$ in Eq. (71) has a symbolic meaning just as in the three- and two-Regge-pole cases [like in Eq. (70) or in Eq. (50)] since the integration in Eq. (71) is performed over complex contours. The first integration in Eq. (71) over t_1 at fixed energies $t_2, t_3, \cdots, t_n$ of all the other Regge poles leads to substitution of the value of the integrand function at the point at which

$$\Box_n(t_1, t_2, \cdots, t_n; j) = 0. \quad (72)$$

The limits $t_2^{(-)}$ and $t_2^{(+)}$ of the second integration (over t_2 at fixed $t_3, t_4, \cdots, t_n$) are the two solutions of Eq.

(72) into which we have substituted the value t_1 from

$$\chi_n(t_1,t_2,\cdots,t_n;\,j)=0. \qquad (73)$$

The limits of subsequent integrations in Eq. (71) over t_3, $t_4,\cdots$, t_n, e.g., over t_i at fixed t_{i+1}, $t_{i+2},\cdots$, t_n are determined as a pair of solutions $t_i^{(\pm)}=f^{(\pm)}(t_{i+1},t_{i+2},\cdots,\times t_n,j,t)$ of Eq. (72) in t_i into which we substitute the values t_1, $t_2,\cdots$, t_{i-1} from

$$\frac{\partial\Box_n}{\partial t_k}=\lambda\frac{\partial\chi_n}{\partial t_k},\quad k=1,2,\cdots,i-1 \qquad (74)$$

and determine the parameter λ from (73). It can readily be checked that in the region $j\to j_n$ Eq. (72) actually has a pair of complex-conjugate roots $t_i^{(\pm)}$.

Using these rules we can readily determine the dependence of $\delta_j^{(n)}f_j(t)$ on j in the region $j<j_n$, $j\to j_n$ (i.e., to determine the character of the singularity j_n).

Near the singularity the limits of integration over all t_α, $t_{\alpha\beta}$, $t_{\alpha\beta\gamma},\cdots$ energies of Regge poles and Regge-pole groups contract (coinciding at $j=j_n$) and all the momenta of relative motion $p(t_{\alpha\beta},t_\alpha,t_\beta)$ tend to zero. The threshold behavior of the amplitudes $N_{j;n,k}$ in these conditions has the form

$$N_{j;n,k}=C_{j,n,k}/\textstyle\prod_{\alpha,\beta}p(t_{\alpha\beta},t_\alpha,t_\beta).$$

Taking $C_{j;n,k}$ out of the integral we obtain

$$\delta_t^{(n,k)}f_j(t)=\frac{\pi}{2^{\nu_k}}C_{j;n,k}C_{j;n,k}{}^{+}\frac{1}{(2i)^{n-1}}\times\int\delta(\Box_n)\prod_{\alpha,\beta}\frac{2dt_\alpha dt_\beta}{(t_{\alpha\beta})^{1/2}p(t_{\alpha\beta},t_\alpha,t_\beta)}.$$

The integral over the energies $t_{\alpha\beta}$, $t_{\alpha\beta\gamma},\cdots$ of the pairs and groups of Regge poles at fixed values of Regge-pole masses t_1, $t_2,\cdots$, t_n can be easily calculated. It contains $n-2$ integrations over these energies (e.g., one integration in the three-Regge-pole case, two integrations in the four-Regge-pole case, etc.) and $n-1$ factors $1/p(t_{\alpha\beta},t_\alpha,t_\beta)$. The regions of integration over all energies $t_{\alpha\beta}$, $t_{\alpha\beta\gamma},\cdots$, tend to zero at $t^{1/2}\to\sum_{\alpha=1}^n t_\alpha{}^{1/2}$ and each of them is of the order of $(t^{1/2}-\sum_{\alpha=1}^n t_\alpha{}^{1/2})$. Each momentum is of the order of $(t^{1/2}-\sum_{\alpha=1}^n t_\alpha{}^{1/2})^{1/2}$. Therefore, the integration over the relative energies gives

$$a_n(t^{1/2}-\sum_{\alpha=1}^n t_\alpha{}^{1/2})^{(n-3)/2},$$

where a_n has no singularities at $t_\alpha=t/n^2$. In this case Eq. (71) can be written as

$$\delta_j^{(n)}f_j=\textstyle\sum_k\delta_j^{(n,k)}f_j=A_n\displaystyle\int\left(\frac{n^2}{4t}\sum_{\alpha=1}^n x_\alpha{}^2-\sum_{\alpha=1}^n x_\alpha\right)^{(n-3)/2}\times\delta\left(j_n-j+\alpha'\cdot\sum_{\alpha=1}^n x_\alpha+\frac{\alpha''}{2}\sum_{\alpha=1}^n x_\alpha{}^2\right)(dx_\alpha)^n,$$

where account is taken of the fact that the energies of all Regge poles vary at $j\to j_n$ in a small region near $t_\alpha{}^0=t/n^2$ and therefore the quantities

$$x_\alpha=t_\alpha-t/n^2$$

are small as compared with t_α. By A_n we denote a certain constant whose value is inessential for the following. It can readily be seen from the latter value of $\delta_j^{(n)}f_j(t)$ that in the region near the singularity the quantities $x_\alpha{}^2$ and $\sum_{\alpha=1}^n x_\alpha$ are of the same order of smallness: of the order j_n-j [the discontinuity $\delta_j^{(n)}f_j(t)$ is not zero only in the region $j_n>j$]. Taking this into account, neglecting at $j\to j_n$ terms of a higher order in x_α and calculating the integral over x_1, we obtain

$$\delta_j^{(n)}f_j(t)=\frac{\pi A_n}{\alpha'(n-3)/2}\times\int\left[\frac{\gamma}{2}\sum_{\alpha=1}^n x_\alpha{}^2+(j_n-j)\right]^{(n-3)/2}\prod_{\alpha=2}^n\left(\frac{dx_\alpha}{i}\right),$$

where

$$\gamma=(n^2/\alpha't)(\alpha'+(2\alpha''/n)t).$$

Taking the above into account and substituting $x_1\simeq-\sum_{\alpha=2}^n x_\alpha$ we write $\sum_{\alpha=1}^n x_\alpha{}^2$ in the form

$$\sum_{\alpha=1}^n x_\alpha{}^2=(\sum_{\alpha=2}^n x_\alpha)^2+\sum_{\alpha=2}^n x_\alpha{}^2=2\sum_{\alpha\geqslant\alpha'\geqslant2}^n x_\alpha x_{\alpha'}.$$

Carrying out a linear transformation of the variables

$$\rho_\alpha=\sum_{\alpha'=2}^n\lambda_{\alpha\alpha'}x_{\alpha'},$$

we can readily select the coefficients $\lambda_{\alpha\alpha'}$ so as to make the form $\sum_{\alpha\geqslant\alpha'\geqslant2}^n x_\alpha x_{\alpha'}$ diagonal:

$$2\sum_{\alpha\geqslant\alpha'\geqslant2}^n x_\alpha x_{\alpha'}=\frac{2}{\gamma}\sum_{\alpha=2}^n\rho_\alpha{}^2=-\frac{2}{\gamma}\sum_{\alpha=2}^n\xi_\alpha{}^2,$$

where $i\xi_\alpha=\rho_\alpha$. Bearing in mind that

$$\prod_{\alpha=2}^n\left(\frac{dx_\alpha}{i}\right)=D_n\prod_{\alpha=2}^n\frac{d\rho_\alpha}{i}=D_n\prod_{\alpha=2}^n d\xi_\alpha,$$

where D_n is the determinant of the above linear transformation and writing accurately the limits of integration over ξ_α we obtain

$$\delta_t^{(n)}f_j(t)=\pi B_n'\int_{-(j_n-j)^{1/2}}^{(j_n-j)^{1/2}}d\xi_n\times\int_{-(j_n-j-\xi_n^2)^{1/2}}^{(j_n-j-\xi_n^2)^{1/2}}d\xi_{n-1}\cdots\int_{-(j_n-j-\sum_{\alpha=3}^n\xi_\alpha)^{1/2}}^{(j_n-j-\sum_{\alpha=3}^n\xi_\alpha^2)^{1/2}}d\xi_2\times[j_n-j-\sum_{\alpha=2}^n\xi_\alpha{}^2]^{(n-3)/2}=\pi B_n(j_n-j)^{n-2},$$

where B_n' and B_n are certain coefficients.[29]

Since $\delta_t^{(n)} f_j(t) = -\delta_j^{(n)} f_j(t)$, we have at $j \to j_n(t)$

$$\delta_j^{(n)} f_j(t) = \pi B_n (j - j_n)^{n-2}$$

and at $n \geqslant 3$

$$f_j(t) = A_n + B_n (j - j_n)^{n-2} \ln(j - j_n),$$

where A_n has no singularity at $j \to j_n$.

Thus, near the n-Regge-pole singularity, the smaller the singular part of $f_j(t)$, the larger is n. However, in the region of small t (at $\alpha_0' t \sim j - j_n \to 0$) the situation may be essentially different.

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APPENDIX A

We show that the second term in the sum (28), in which m_1 and m_2 have opposite signs, does not lead to singularities in $f_j(t)$ influencing the asymptotic behavior of the amplitude.

Let us write the second term as integrals (29) and (30) over contours L_1, L_2 and M_1, M_2 of the type indicated in Figs. 5, 10, and 11.

Its contribution differs from Eq. (30) by only the sign of m_2, i.e., it is given, after integration over l_1 and l_2, by the integral

$$\int_{M_1} \frac{dm_1}{\tan[(\pi/2)m_1]} \int_{M_2'} \frac{dm_2}{\tan[(\pi/2)m_2]} \times \frac{\Gamma(j+1-m_1+m_2)}{D_{m_1}(t_1) D_{m_2}(t_2)}, \quad \text{(A1)}$$

where m_2 denotes the value $|m_2|$. The contour M_2 in Fig. 10 encloses the poles of the Γ function so that their coincidence with the zeros of $\tan[(\pi/2)m_2]$ (at arbitrarily large j) does not lead to singularities of integral (30) in m_2. In the case of (A1) the poles of the Γ function are located at the points

$$m_2 = m_1 - j - 1 - n, \quad n = 0, 1, 2, \cdots, \quad \text{(A2)}$$

which shift, as j increases, not to the right as in Eq. (30) but to the left. Therefore, at a sufficiently large j they cannot coincide with the zeros of $\tan[(\pi/2)m_2]$ (at $m_2 = 0, 2, 4, \cdots$) and the contour M_2' must be drawn so that these poles (indicated by circles in Fig. 10) lie outside. It will be recalled that the singularity of integral (30) arises from coincidence in the plane of m_2 of a pole of the Γ function with a zero of $D_{l_2}(t_2)$ in Eq. (30) (at $l_2 = m_2$). If both poles of the Γ function and zeros of $D_{m_2}(t_2)$ lie outside the contour M_2' (integration over m_2 as in Fig. 10) this coincidence gives no rise to the singularity of integral (A1).

Therefore, unlike (30), integral (A1) may have singularities of another type, resulting from coincidence of poles of the Γ function and zeros of $\tan[(\pi/2)m_2]$. On a subsequent integration over t_1 and t_2 there arise, in this case, singularities of the form $j = \alpha\{(t^{1/2} - 2\mu)^2\} - 1 - n$, where $n = 0, 2, \cdots$, which do not affect the asymptotic behavior of the amplitude.

These singularities depend on the masses of particles, and at $t < 16\mu^2$ go through the cut associated with the singularity $j = j_2(t)$ onto the unphysical sheet [similar to the singularities of form (35) and (36) considered in the main text of the paper].

The authors are indebted to Ya. Azimov who has drawn their attention to the problem discussed above.

APPENDIX B

We show that at small t the contribution from the four-particle term of the unitarity condition in $f_j(t)$

$$f_j'(t) = \frac{1}{\pi} \int_{16\mu^2}^{\infty} \frac{\Delta_4 f_j(t') dt'}{t' - t} \quad \text{(B1)}$$

has no singularities (35), (36).

Substituting Eq. (31) into this integral and omitting for simplicity the factors $N_{j\alpha_1\alpha_2} N_j^{(4)}$ in Eq. (31) (near the singularity this factor can be taken outside the integral at $t_1 = t_2 = t/4$) we obtain, changing the order of integration,

$$f_j'(t) = \frac{1}{2!} \frac{1}{(2i)^2} \int_{C_2'} dt_2 \int_{C_1'} dt_1 \frac{F(t; t_1, t_2)}{j + 1 - \alpha(t_1) - \alpha(t_2)}, \quad \text{(B2)}$$

$$F(t; t_1, t_2) = \frac{1}{\pi} \int_{(t_1^{1/2} + t_2^{1/2})^2}^{\infty} \frac{2p(t', t_1, t_2)}{t'^{1/2}(t' - t)} dt', \quad \text{(B3)}$$

where the last relation must be understood in the sense that the required number of subtractions has been made

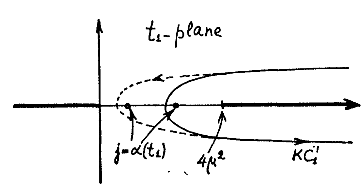

FIG. 16. The contour of integration in (B2) over t_1.

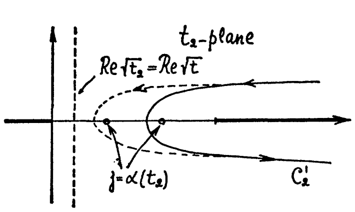

FIG. 17. The contour of integration in (B2) over t_2.

[29] They only differ by the numerical factor I_n:

$$I_n = \int_{-1}^{1} dy_n \int_{-(1-y_n^2)^{1/2}}^{(1-y_n^2)^{1/2}} dy_{n-1} \int_{-(1-y_n^2-y_{n-1}^2)^{1/2}}^{(1-y_n^2-y_{n-1}^2)^{1/2}} dy_{n-2} \cdots \times \int_{-(1-y_n^2-\cdots-y_3^2)^{1/2}}^{(1-y_n^2-y_{n-1}^2-\cdots-y_3^2)^{1/2}} dy_2 (1 - y_2^2 - y_3^2 - \cdots - y_n^2)^{(n-3)/2}.$$

in Eq. (B3). The contours C_1' and C_2' (Figs. 16 and 17) differ from those represented in Figs. 2 and 3 in that they are continued to ∞.

The function $F(t,t_1,t_2)$ has singularities at $t^{1/2}=t_1^{1/2}+t_2^{1/2}$, $t_1=0$, $t_2=0$. If $\mathrm{Re}t_2^{1/2}>\mathrm{Re}t^{1/2}$, the singularity $t_1^{1/2}=t^{1/2}-t_2^{1/2}$ of the function $F(t,t_1,t_2)$ is absent on the physical sheet of the plane t_1 represented in Fig. 16 since the point t_1 for which $\mathrm{Re}t_1^{1/2}<0$ lies below the cut made in Fig. 16 left of the singular point $t_1=0$.

Therefore at $\mathrm{Re}t^{1/2}<\mathrm{Re}t_2^{1/2}$ the singularity of the integral over t_1:

$$\phi_j(t,t_2)=\frac{1}{2}\frac{1}{2i}\int_{C_1'}\frac{F(t;t_1,t_2)}{j+1-\alpha(t_1)-\alpha(t_2)}dt_1$$

arises only for such j, t, and t_2 for which the zero of the denominator appearing across the cut $t_1>4\mu^2$ and deforming the contour C_1' of integration coincides with the points $t_1=0$. This singularity, given by the condition

$$j+1=\alpha(t_2)+\alpha(0)\,,$$

i.e., $j=\alpha(t_2)$, appears on the cut of the plane t_2, deforms the contour (as is indicated in Fig. 17) and reaching the line $\mathrm{Re}t_2^{1/2}=\mathrm{Re}t^{1/2}$ does not lead to the singularity of integral (B2)

$$f_j'(t)=\frac{1}{2i}\int_{C_2'}\phi_j(t,t_2)dt_2\,. \tag{B4}$$

This means that the singularity of this integral arises only from the region of small (or complex) t_2, $t_2\lesssim t$. Since t is small the quantity $t_1^{1/2}=t^{1/2}-t_2^{1/2}$ is also small (or if it is not, it is complex). In either case the particle masses cannot enter into the expression giving the location of the singularity. Actually the singularity of integral (B2), (B4) arises from the point $t_1=t_2=t/4$.

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Statistics of the Thermal Radiation Field*

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The characteristic functional is calculated for a system of bosons obeying linear field equations. The system is assumed to be in equilibrium, and the density matrix is taken to be of the form $\langle\{n\}|\rho|\{m\}\rangle=\Pi_\kappa\,\delta_{n_\kappa m_\kappa}(1-z_\kappa)z_\kappa^{n_\kappa}$, where κ labels the individual modes. From the characteristic functional, the moments and distribution functions of an arbitrary number of field components are derived. In addition, it is shown how to obtain the density matrix from the characteristic functional, and, for the system in question, the original density matrix is recovered. Explicit calculations are performed for the electromagnetic field in an unbounded domain and in a semi-infinite domain bounded by a perfectly conducting plane.

I. INTRODUCTION

USING the methods of quantum field theory, we shall compute the characteristic functional for an electromagnetic field in thermal equilibrium within an enclosure of arbitrary size and shape. From this functional, we shall compute the moments or correlation functions and the probability distributions for any number of field components at the same or different points in space-time.[1] We shall see that the probability distribution is a multivariate Gaussian function. Therefore, all correlation functions are expressible in terms of the two point correlation function. To exemplify the result, we shall explicitly calculate this correlation function for an unbounded domain and for a semi-infinite domain bounded by a perfectly conducting plane. For the unbounded domain our results agree with those of Sarfatt,[2] Bourret,[3] and Mehta and Wolf.[4] The correlation functions for a semi-infinite domain do not seem to have been calculated previously.

The deduction of the Gaussian distribution functions for black-body radiation in an unbounded domain has already been given by Glauber[5,6] and Holliday.[7] These distribution functions were used implicitly by Purcell[8] and explicitly by Mandel and Wolf[9] in order to analyze the intensity interferometry experiments of Hanbury-

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[1] Of course, the distribution functions are physically meaningful only when they refer to points at which the field components commute. For the electric and magnetic field components, this means that no two points lie on the same light cone.

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