Three-Particle Scattering—A Relativistic Theory*

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We give relativistic extensions of the Faddeev equations for three-particle scattering. They are linear integral equations for six amplitudes, one of the particles being off the mass shell. They satisfy exactly three-particle unitarity. These equations are obtained by applying the techniques introduced by Blankenbecler and Sugar to multiladder diagrams. Accordingly, the two-particle scattering amplitudes, which appear in the kernel of the equations, depend on the energy of the third particle. We study the equations for these amplitudes. If one replaces these two-body amplitudes by phenomenological amplitudes satisfying unitarity, one gets phenomenological relativistic equations for the three-body problem. When the two-body amplitudes are approximated by the contributions of bound states and resonances, the equations can be reduced to a set of integral equations in one variable. As a by-product of this study, the following results have been obtained: (a) a new proof of unitarity for the Lippmann-Schwinger and Faddeev equations; (b) a proof of the analytic properties of a resonance form factor in the nonrelativistic theory; (c) a proof of the asymptotic behavior of these form factors, which uses functional-analysis techniques and is a rather general method for investigating the asymptotic behavior of solutions of Fredholm equations.

I. INTRODUCTION

SEVERAL sets of equations have been proposed recently for solving the nonrelativistic scattering three-body problem.[1–4] Each of them is a set of linear integral equations in momentum space which allow one in principle to compute the off-the-energy-shell three-body scattering amplitude once one knows the off-the-energy-shell two- body amplitude. They ensure unitarity in the three-particle channel.[2]

As they stand, these equations are not very useful because of the high number of variables involved in the integrations. However, a natural approximation consists in taking into account only the contributions of bound states, resonances, and virtual states to the two-body scattering amplitude, i.e., to the kernel of the Faddeev equations.[3] As a result, these equations separate into one-dimensional integral equations which are easily solved.[3,5]

Some of the most attractive three-body problems are concerned with resonances of three elementary particles. However, it is obvious that in that case, a relativistic version of the Faddeev equations must be found.

This paper is concerned with the search for such relativistic three-body equations. More precisely we have looked for equations having the following properties:

(a) They should be relativistically invariant, i.e., their form should not depend on the reference system.

(b) They should be linear integral equations for an off-the-mass-shell three-body scattering amplitude.

(c) They should exhibit explicitly the nonconnectedness of the three-body scattering amplitude.

(d) They should involve only some two-body scattering amplitudes as ingredients in the inhomogeneous terms and the kernel.

(e) They should be Fredholm-type integral equations. More technically, some power of their kernel should be of the Hilbert-Schmidt type.

(f) They should satisfy, automatically, unitarity in the three-particle channel.

(g) If possible, they should be simple enough to be solved on a computer, by use of reasonable approximations.

Most of these conditions have been inspired by an analogy with the Faddeev equations. They have no fundamental meaning. For instance, condition (d) is rather a limitation than a desirable feature, and crossing is not mentioned because it is not compatible with these properties. However, such equations would help us to extend our understanding of relativistic three-body systems.

In Sec. 2, we recall the method used by Blankenbecler and Sugar to write an equation for a relativistic off-the-mass-shell two-body scattering amplitude satisfying exactly two-body unitarity. Some ambiguity is exhibited in this procedure.[6] In Sec. 3, we introduce for comparison an off-the-energy-shell, on-the-mass-shell nonrelativistic two-body amplitude satisfying an equation analogous to the Lippmann-Schwinger[7] and to the Blankenbecler-Sugar equation. In Sec. 4, we give a proof of unitarity for the ordinary Lippmann-Schwinger equation. This proof is new and essentially based on the fact that the Lippmann-Schwinger equation is of the Fredholm type. It is easily extended to the Blankenbecler-Sugar types of equations as well as to the equa-

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[1] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)]; Dokl. Akad. Nauk SSSR **138**, 565 (1961) and **145**, 301 (1962) [English transls.: Soviet Phys.—Doklady **6**, 384 (1961) and **7**, 600 (1962)].

[2] C. Lovelace, in *Lectures at the 1963 Edinburgh Summer School*, edited by R. G. Moorhouse (Oliver and Boyd, London, 1964).

[3] C. Lovelace, Phys. Rev. **135**, B1225 (1964).

[4] S. Weinberg, Phys. Rev. **133**, B232 (1964).

[5] J. L. Basdevant, Phys. Rev. **138**, B892 (1965).

[6] R. Blankenbecler and R. Sugar (unpublished).

[7] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1960).

tions introduced in Sec. 3. In Sec. 5, we introduce a three-particle propagator which is the function E_3 used by Blankenbecler and Sugar. Some ambiguities are unavoidable in this definition and are exhibited.

In Sec. 6, we write linear equations for two-body scattering amplitudes constructed with a three-body propagator. In Sec. 7, these equations are shown to be of the Fredholm type and to ensure two-particle unitarity. It is shown also that if there is a two-particle bound state or a resonance, these amplitudes have a pole when the total energy is equal to the mass of the bound state or the resonance. The residue of the amplitude then factorizes in the product of two form factors. In Sec. 8, we prove the analyticity properties of the form factors. The proof is made for the ordinary Lippmann-Schwinger equation, but it is valid for all the equations used in this paper. This proof is valid for the form factors of resonances and virtual states as well as for bound states. In Sec. 9, we prove the asymptotic properties of the form factors when the energy tends to infinity. While it is easy to guess their asymptotic behavior, it is less easy to prove it. Our proof makes use of a new functional-analysis technique for finding the asymptotic behavior of a Fredholm-type equation. It is valid for a bound state or a resonance. In Sec. 10, we describe the three-particle scattering amplitudes that we are going to use. They can be considered as defined by the sum of a set of ladder-type diagrams. However, the rules of correspondence between graphs and matrix elements are not the Feynman rules and are given explicitly. These rules have been found as the result of a trial-and error process. We believe they are unique but we have no proof for that belief.

In Sec. 11, we write linear integral equations for these three-particle amplitudes. They satisfy conditions (a), (b), (c), and (d). In Sec. 12, we show that these equations are of the Fredholm type and we give the essential steps of a proof of unitarity. (This proof is extended in the Appendix.) In Sec. 13, we show how these equations reduce when one makes use of the conservation of angular momentum and parity. It has also been checked that they satisfy the Utopian condition (g). That is, if the two-body scattering amplitudes in the kernel are replaced by the sums of their bound states and resonances poles, then the equations reduce to a set of coupled integral equations in one variable. In Sec. 14, we give a three-body version of Sec. 3. It gives a set of equations for on-the-mass-shell, off-the-energy shell three-body scattering amplitudes. These equations are written in the total center-of-mass system of the three particles and are very similar to the Faddeev equations. They are not covariant although they use a correct relativistic kinematics and satisfy the relativistic form of three-body unitarity. When the customary pole approximation is made on the two-body scattering amplitudes in the kernel, these equations yield a set of coupled integral equations in one variable. Section 15 contains the conclusions.

An Appendix gives an explicit proof of unitarity for the ordinary Faddeev equations and for the equations of this paper. These proofs are new. Sections 4, 8, 9, and 12, which contain important but rather technical points, can be skipped in a first reading.

2. TWO-PARTICLE SCATTERING: ORIENTATION

It has been shown by Blankenbecler and Sugar how to construct an equation for a relativistic two-body scattering amplitude satisfying exactly two-body unitarity for any value of the total energy.[6] As their considerations will be used and extended here, we recall their main results.

One starts from the Bethe-Salpeter equation,[8]

$$t(p_1,p_2;\,p_1',p_2') = V(p_1-p_1')+\int V(p_1-q_1)\delta(p_1+p_2-q_1-q_2)dq_1dq_2 \times G(q_1,q_2)t(q_1,q_2;\,p_1'p_2'),\quad (2.1)$$

where

$$V(p)=g^2[p^2-\mu^2]^{-1},\quad (2.2)$$

$$G(q_1,q_2)=G_0(q_1;\,q_2)\equiv -i[(q_1^2-m^2)(q_2^2-m^2)]^{-1}.\quad (2.3)$$

In place of this propagator $G(q_1,q_2)$ one introduces the propagator

$$G_1(q_1,q_2)=E_2(Q,q),\quad (2.4)$$

where

$$Q=q_1+q_2;\quad s=Q^2,\quad q=\tfrac{1}{2}(q_1-q_2),\quad (2.5)$$

and

$$E_2(Q,q)=2\pi\int_{4m^2}^{\infty}\frac{ds'}{s'-s}\delta^+\left\{\left[\left(\frac{s'}{s}\right)^{1/2}\frac{Q}{2}+q\right]^2-m^2\right\} \times\delta^+\left\{\left[\left(\frac{s'}{s}\right)^{1/2}\frac{Q}{2}-q\right]^2-m^2\right\},\quad (2.6)$$

where

$$\delta^+(p^2-m^2)=\delta(p^2-m^2)O(p_0).\quad (2.7)$$

This function is easily computed if one uses the elementary identity

$$\delta(a+b)\delta(a-b)=\tfrac{1}{2}\delta(a)\delta(b),\quad (2.8)$$

and is given by

$$E_2(Q,q)=4\pi\delta[Q.q][s/(m^2-q^2)]^{1/2}[4(\mu^2-q^2)-s]^{-1}.\quad (2.9)$$

Equation (2.1) with $G=G_1$ is a linear equation for $t(p_1,p_2;\,p_1',p_2')$, which is now an off-the-mass-shell ($q_1^2\neq m^2;\,q_2^2\neq 2$), on-the-energy-shell ($Q=P=P'$) amplitude. It satisfies two-body unitarity for any value of s as a comparison of Eq. (2.6) and the Cutkosky rule suggests.[9] This point will be explicitly verified in Sec. 4. The δ function in Eq. (2.9) ensures the three-dimensional character of the equation.

[8] E. E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951).
[9] R. Cutkosky, J. Math. Phys. 1, 431 (1960).

It is important to realize that there is an ambiguity in all this procedure and that the propagator is not completely specified by the requirement of unitarity. For instance, one could introduce

$$G_2(q_1,q_2)=2\pi\delta^+(q_1^2-m^2)\int_{4m^2}^{\infty}\frac{ds'}{s'-s}\times\delta^+\left\{\left[\left(\frac{s'}{s}\right)^{1/2}Q-q_1\right]^2-m^2\right\}. \quad (2.10)$$

Then Eq. (2.1) would define an amplitude for q_1 on the mass shell, and this new amplitude would also be unitary. The prescription (2.6) is the only one which is symmetric with respect to both particles.

3. LIPPMANN-SCHWINGER EQUATION

One can put both particles on the mass shell by using another propagator,

$$G_3(q_1,q_2)=\delta^+(q_1^2-m_1^2)\delta^+(q_2^2-m^2)\int_{4m^2}^{\infty}\frac{\delta[\sqrt{s'}-Q^0]ds'}{s'-s}$$
$$=\delta^+(q_1^2-m^2)\delta^+(q_2^2-m^2)/s^{-1/2}(Q_0^2-s). \quad (3.1)$$

In order to restore the balance of conserved quantities and homogeneity, it is necessary to suppress the δ function of energy conservation in Eq. (2.1), which now reads

$$t(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_1',\mathbf{p}_2')=\int V(p_1-q_1)\delta(\mathbf{P}-\mathbf{Q})d^3\mathbf{q}_1d^3\mathbf{q}_2[Q_0^2-s]^{-1}\times t(\mathbf{q}_1,\mathbf{q}_2;\mathbf{q}_1',\mathbf{q}_2')(q_1^0q_2^0)^{-1} \quad (3.2)$$

with

$$q_1^0=(\mathbf{q}_1^2+m^2)^{1/2};\quad p_1^0=(\mathbf{p}_1^2+m^2)^{1/2};\quad q_2^0=(\mathbf{q}_2^2+m^2)^{1/2}. \quad (3.3)$$

The Lippmann-Schwinger equation (3.2) is very similar to Eq. (2.1) with the prescription (2.4), although not identical. Its unitarity will also follow immediately from the proof of the next section. It is obvious, but important, that the prescriptions which lead to Eq. (3.2), as well as the equation itself, are not covariant.

4. EQUATIONS OF THE LIPPMANN-SCHWINGER TYPE AND UNITARITY

We shall prove the unitarity property for the solution of the ordinary Lippmann-Schwinger equation

$$t(s)=V+V(H_0-s)^{-1}t(s), \quad (4.1)$$

where

$$H_0=\mathbf{p}_1^2/2m_1+\mathbf{p}_2^2/2m_2. \quad (4.2)$$

The proof will be extendable trivially to the case of the equations introduced in Secs. 2 and 3. Let us suppose that V is a converging superposition of Yukawa potentials. It has been shown by several authors that in that case the kernel in Eq. (4.1) is of the Fredholm type, even when s tends to the positive real axis.[10] The proof by Lovelace, for instance, can be extended immediately to the equations of Secs. 2 and 3.

In order to prove the unitarity property of the solution of Eq. (4.1) let us proceed through the following steps:

(a) Write the symmetric Lippmann-Schwinger equation

$$t(s)=V+t(s)(H_0-s)^{-1}V. \quad (4.3)$$

It is of essential importance for the proof that Eqs. (4.1) and (4.3) have the same solution.

(b) Take the adjoint of Eq. (4.3), namely, write

$$t^\dagger(s)=V+V(H_0-\bar{s})^{-1}t^\dagger(s). \quad (4.4)$$

(c) Introduce the difference

$$\Delta t(s)=t(s)-t^\dagger(s). \quad (4.5)$$

Use Eqs. (4.3) and (4.4). Then, we get

$$\Delta t(s)=2\pi iV\delta(H_0-s)t^\dagger(s)+V(H_0-s)^{-1}\Delta t(s). \quad (4.6)$$

(d) Compare Eqs. (4.1) and (4.6). They are integral equations for $t(s)$ and $\Delta t(s)$, respectively. The kernels are the same. The inhomogeneous term of Eq. (4.6) is obtained by letting the operator $\delta(H_0-s)t^\dagger(s)$ act on the variables of the final state, which are but dummy indexes for the integral equation. Since the kernel is of the Fredholm type, the solutions are related by the same linear relation as their inhomogeneous terms, i.e.,

$$\Delta t(s)=2\pi it(s)\delta(H_0-s)t^\dagger(s). \quad (4.7)$$

The same proof holds for the equations of Secs. 2 and 3. Here the only delicate step is step (a), i.e., to show that the symmetric equation has the same solution. The simplest, if not rigorous, proof is by perturbation expansion. A correct proof is via a Fredholm expansion. The results are, for Eqs. (2.1), (2.4), and (2.6),

$$\Delta t(p_1,p_2;p_1',p_2')=2\pi i\int t(p_1,p_2;q_1,q_2)\delta^+(q_1^2-m^2)\delta^+(q_2^2-m^2)\times\delta(Q^2-s)s^{-1/2}d^4q_1d^4q_2t^\dagger(q_1,q_2;p_1',p_2'), \quad (4.8)$$

with p_1, p_2, p_1', p_2' on the mass shell. For Eq. (3.2),

$$\Delta t(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_1',\mathbf{p}_2')=2\pi i\int t(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_1',\mathbf{p}_2')\times\delta(Q_0-s^{1/2})d^3\mathbf{q}_1d^3\mathbf{q}_2(q_1^0q_2^0)^{-1}t(\mathbf{q}_1,\mathbf{q}_2;\mathbf{p}_1',\mathbf{p}_2'). \quad (4.9)$$

[10] L. D. Faddeev, Stoklov Mathematical Institute Report No. 69, 1963 (unpublished); L. Brown, D. I. Fivel, B. W. Lee, and R. F. Sawyer, Ann. Phys. (N. Y.) **23**, 187 (1963); C. Lovelace, Ref. 3; and S. Weinberg, M. Scadron, and J. Wright, Phys. Rev. **135**, B202 (1964).

5. THREE-PARTICLE PROPAGATORS

One can define a propagator E_3 for three particles:

$$E_3=4\pi^2\int_{9m^2}^{\infty}\frac{ds'}{s'-s}\times\delta^+(q_1^2-m^2)\delta^+(q_2^2-m^2)\delta^+(q_3^2-m^2)\,, \quad (5.1)$$

where

$$s'=(q_1+q_2+q_3)^2.$$

This function has also been introduced by Blankenbecler and Sugar.[6] In order to define it precisely, it is necessary to introduce a basis of three vectors dependent on q_1, q_2, q_3 including $Q=q_1+q_2+q_3$ as a basis vector in place of the set q_1, q_2, q_3. Let us call (Q,p_1,p_2) such a basis. E_3 will then be defined as

$$E_3(Q,p_1,p_2)=4\pi^2\int_{9m^2}^{\infty}\frac{ds'}{s'-s}D\left[\left(\frac{s'}{s}\right)^{1/2}Q,p_1,p_2\right], \quad (5.2)$$

where

$$D(Q,p_1p_2)=\delta^+(q_1^2-m^2)\delta^+(q_2^2-m^2)\delta^+(q_3^2-m^2)$$
$$s=Q^2. \quad (5.3)$$

It is easily checked that the function $E_3(Q,p_1',p_2')$, where (p_1',p_2') is different from (p_1,p_2), will generally be different from $E_3(Q,p_1,p_2)$.

Contrary to the two-particle case, there is no basis symmetric in all momenta for more than two particles, so that there is not a single "natural" function E_3. As we shall see in the following, this ambiguity will be in fact a useful tool for finding three-body equations.

We shall use the basis

$$Q=q_1+q_2+q_3;\quad p_1=q_\beta;\quad p_2=q_\gamma\,, \quad (5.4)$$

where

$$(\alpha,\beta,\gamma)=(1,2,3)\,,$$

and we shall hence define the three-body propagator as

$$E^{(\alpha)}(s)=4\pi^2\delta^+(q_\beta^2-m^2)\delta^+(q_\gamma^2-m^2)\times\int_{9m^2}^{\infty}\frac{ds'}{s'-s}\delta^+\left\{\left[\left(\frac{s'}{s}\right)^{1/2}Q-q_\beta-q_\gamma\right]^2-m^2\right\}, \quad (5.5)$$

or, in the total center-of-mass system,

$$E^{(\alpha)}(s)=\frac{\pi^2}{\omega_1\omega_2\omega_3}\frac{\omega_1+\omega_2+\omega_3}{(\omega_1+\omega_2+\omega_3)^2-s}\times\delta(q_\beta^0-\omega_\beta)\delta(q_\gamma^0-\omega_\gamma)\,, \quad (5.6)$$

$$\omega_\alpha=(\mathbf{q}_\alpha^2+m^2)^{1/2}. \quad (5.7)$$

We have dropped the index 3 in $E^{(\alpha)}$ since it will be the only propagator used from now on.

6. TWO-BODY AMPLITUDE WITH THREE-BODY PROPAGATOR

Let us introduce a new two-body scattering amplitude $t_1^{(2)}(p_2,p_3;\,p_2',p_3')$. It describes the scattering of particles 2 and 3 under the influence of a Yukawa potential, a third particle (1) being present and not taking place in the scattering.

This amplitude will be chosen to satisfy, for instance,

$$\langle\mathbf{p}_1p_2p_3|t_1^{(2)}(s)|\mathbf{p}_1'p_2'p_3'\rangle=\langle\mathbf{p}_1p_2p_3|V_1|\mathbf{p}_1'p_2'p_3\rangle+\int d^4q_1d^4q_2d^4q_3\times\langle\mathbf{p}_1p_2p_3|V_1|\mathbf{q}_1q_2q_3\rangle E^{(3)}(q_1,q_2,q_3)\times\langle\mathbf{q}_1q_2q_3|t_1^{(2)}(s)|\mathbf{p}_1'p_2'p_3'\rangle\,, \quad (6.1)$$

where

$$\langle p_1p_2p_3|V_1|p_1'p_2'p_3'\rangle=\omega_1\delta(\mathbf{p}_1-\mathbf{p}_1')\times\delta(p_2+p_3-p_2'-p_3')[(p_2-p_2')^2+\mu^2]^{-1}. \quad (6.2)$$

We have taken the spectator particle (1) as being explicitly on the mass shell. The factor ω_1 comes from the invariant normalization of momentum eigenstates. Then, we can factorize $\langle t_1^{(2)}(s)\rangle$ as follows,

$$\langle p_1p_2p_3|t_1^{(2)}(s)|p_1'p_2'p_3'\rangle=\omega_1\delta(\mathbf{p}_1'-\mathbf{p}_1)\times\delta(p_1'+p_2'-p_1-p_2)t_1^{(2)}(p_2,p_3;\,p_2'p_3')\,, \quad (6.3)$$

so that Eq. (6.1) will become, written, for instance, in the total c.m. system of the three particles,

$$t_1^{(2)}(\mathbf{p}_2,p_3;\,p_2',p_3')=V_1(p_2-p_2')+\int V_1(p_2-p_2)\delta(q_2+q_3-p_2-p_3)d^3\mathbf{q}_2d^4q_3\times\frac{\pi^2(\omega_1+\omega_2+\omega_3)}{\omega_2\omega_3[(\omega_1+\omega_2+\omega_3)^2-s]}t_1^{(2)}(\mathbf{q}_2,q_3;\,p_2',p_3')\,. \quad (6.4)$$

We have written $\mathbf{p}_2$ to insist on the fact that this vector is on the mass shell.

We shall also use the amplitude $t_1^{(3)}(p_2,\mathbf{p}_3;\,p_2',p_3')$ whose $\mathbf{p}_3$ is maintained on the mass shell. $t_1^{(2)}$ and $t_1^{(3)}$ are on-the-energy-shell covariant scattering amplitudes, one particle being kept off the mass shell. They are dependent, however, on the spectator particle 1. Note that, after integration upon d^4q_3, the scattering amplitude $t_1^{(2)}$ can be thought of as an on-the-mass-shell, off-the-energy-shell amplitude $t_1^{(2)}(\mathbf{p}_2,\mathbf{p}_3;\,\mathbf{p}_2',\mathbf{p}_3';\,s^{1/2})$, with $s\neq\omega_1+\omega_2+\omega_3$.

7. PROPERTIES OF $t_1^{(2)}$

If one integrates over dq_3^0 in Eq. (6.4), using $\delta(q_2^0+q_3^0-p_2^0-p_3^0)$, one is left with an equation very similar, once again, to the Lippmann-Schwinger equation. Accordingly, we shall mention without detailed proofs the following results which are obtained easily by simple modifications of proofs valid for the Lippmann-Schwinger equation.

(a) Equation (6.4) is of the Fredholm type. The proofs of Faddeev or Lovelace can be used for this result.[10] In particular, the proof by Faddeev ensures that the kernel in Eq. (6.4) is compact in a Banach space B. This space is the space of functions $f(\mathbf{q}_2)$ with the norm

$$\sup_{\mathbf{q}_2,\mathbf{q}_2'}(1+\mathbf{q}_2^2)\left[|f(\mathbf{q}_2)|+\frac{|f(\mathbf{q}_2)-f(\mathbf{q}_2')|^\mu}{|\mathbf{q}_2-\mathbf{q}_2'|^\mu}\right], \quad (7.1)$$

where $0<\mu\leq 1$.

(b) The proof of unitarity of Sec. 4 can be easily extended to $t_1^{(2)}$ to show that it satisfies relativistic two-body unitarity as given in Eq. (6.8).

It will be useful to have the unitarity of $t_1^{(2)}$ expressed as three-body unitarity. If one introduces the discontinuity of $E^{(\alpha)}(q_1,q_2,q_3)$; $2\pi i\Delta_3(s)$, where

$$\Delta_3(s)=\Delta_3(q_1,q_2,q_3)$$
$$=\delta^+(q_1^2-m^2)\delta^+(q_2^2-m^2)\delta\dagger(q_3^3-m^3), \quad (7.2)$$

Eq. (6.1) can be shown to give

$$\Delta t_1^{(2)}(s)=2\pi i t_1^{(2)}(s)\Delta_3(s)t_1^{(2)\dagger}(s). \quad (7.3)$$

In this equation, the operator multiplication is understood as an integration over $d^4q_1d^4q_2d^4q_3$, so that Eq. (4.5) obviously is relativistic unitarity.

(c) The method of Lovelace can be used to show that, as a function of s, the scattering amplitude $t_1^{(2)}$ is a meromorphic function defined in two Riemann sheets connected along the real axis cut from $(\omega_1+2m)^2$ to infinity. When some of the eigenvalues of the kernel are equal to 1 for $s=s_0$ in that domain, $t_1^{(2)}(s)$ has a pole at $s=s_0$, i.e., in the neighborhood of s_0,[11]

$$t_1^{(2)}(\mathbf{p}_2,p_3;\,\mathbf{p}_2'p_3',s)\simeq\frac{R(\mathbf{p}_2,p_3;\,\mathbf{p}_2',p_3')}{s-s_0}. \quad (7.4)$$

Inserting this expression into Eq. (6.4), one finds that the residue factorizes into

$$R(\mathbf{p}_2,p_3;\,\mathbf{p}_2',p_3')=g(\mathbf{p}_2,p_3)g^*(\mathbf{p}_2'p_3'). \quad (7.5)$$

The function $g(p_1,p_2)$ is called the form factor. It satisfies the homogeneous equation

$$g(\mathbf{p}_2,p_3)=\int V_1(p_2-q_2)\delta(p_2+p_3-q_2-q_3)d^3\mathbf{q}_2d^4q_3$$
$$\times\pi^2(\omega_1+\omega_2+\omega_3)(\omega_2\omega_3)^{-1}$$
$$\times[(\omega_1+\omega_2+\omega_3)^2-s]^{-1}g(\mathbf{q}_2,q_3). \quad (7.6)$$

These properties are well known in the case of the ordinary Lippmann-Schwinger equation. A convenient approximation which has been discussed by Lovelace[3] and Basdevant[5] consists in retaining only in $t_1^{(2)}$ the expressions of Eq. (5.6), i.e., to write a sum over bound states, resonances, and virtual states,

$$t_1^{(2)}(\mathbf{p}_2,p_3;\,\mathbf{p}_2'p_3')=\sum_n\frac{g_m(\mathbf{p}_2,p_3)g_m(\mathbf{p}_2',p_3')}{\sqrt{s}-\sqrt{s_n}}. \quad (7.7)$$

We have replaced s by $s^{1/2}$ in order to reduce the dependence upon ω_1.

[11] There is a dependence of all these expressions upon ω_1, which we do not write here explicitly.

7. ANALYTIC PROPERTIES OF FORM FACTORS

Because of rotational invariance, the form factors will behave as a member of an irreducible representation of the rotation group in the total c.m. system[12]:

$$g(\mathbf{p}_2,p_3)=\delta(P-p_2-p_3)\delta(p_2^0-\omega_2)g(p_2)Y_l^m(\hat{p}_2), \quad (8.1)$$

where

$$g(p)=\int V_l(p,q)\frac{q^2dq\pi^2(\sum\omega)}{[(\sum\omega)^2-s]\omega_2\omega_3}g(q) \quad (8.2)$$

and

$$V_l(p,q)=\frac{g^2}{pq}Q_l\left[\frac{p^2+q^2+\mu^2+(\omega_2-q_2^0)^2}{2pq}\right], \quad (8.3)$$

where

$$\omega_2=(\mathbf{p}^2+m^2)^{1/2},\quad q_2^0=(\mathbf{q}^2+m^2)^{1/2} \quad (8.4)$$

and

$$\sum\omega=\omega_1+\omega_2+\omega_3.$$

When s is in the second Riemann sheet (i.e., in the case of a resonance or a virtual state), Eq. (8.2) should be replaced by

$$g(p)=2\pi^3 iV_l(p,p_0)\frac{p_0}{\omega_3}g(p_0)$$
$$+\int V_l(p,q)\frac{q^2dq\pi^2(\sum\omega)}{[(\sum\omega)^2-s]\omega_0\omega_3}g(q), \quad (8.5)$$

where p^0 is defined as the value of $|\mathbf{p}_2|$ which satisfies

$$\omega_1+\omega_2+\omega_3=s. \quad (8.6)$$

It is important to know the analytic properties of $g(p)$ as a function of p in order to reduce the possible functional forms by which it will be approximated in practice. In the case of the ordinary Lippmann-Schwinger equation this study has been made for the form factors of bound states.[13] Lovelace has also given a study of the case of resonances, but the domain he obtains is not large enough for practical applications.[2]

We shall now sketch another method which will be exemplified in the case of the ordinary Lippmann-Schwinger equation and of a bound state. It is trivial to

[12] This procedure is not relativistically invariant, because of the dependence of the form factors upon ω_1.

[13] D. Fivel, Nuovo Cimento 22, 326 (1961); L. Bertocci, C. Ceolin, and M. Tonin, Nuovo Cimento 18, 770 (1960).

extend it to Eq. (8.2) or (8.5). The equation reads

$$g(p)=\int_0^\infty U(p,q)\frac{q^2dq}{q^2-z}g(q)\,, \tag{8.7}$$

$$U(p,q)=\frac{1}{pq}Q_l\left(\frac{p^2+q^2+\mu^2}{2pq}\right); \tag{8.8}$$

$U(p,q)$ has singularities at $q=0$ (but q^2U is not singular at that point) and at

$$p\pm iq=\pm i\mu\,. \tag{8.9}$$

The integral in Eq. (8.7) therefore converges into a strip S_μ: $|\mathrm{Im}p|<\mu$. We can then deform the contour of integration from 0 to ∞ into S_μ. Then the integral in Eq. (8.7) will converge in the strip $S_{2\mu}$ indented by the parts $(i\mu,2i\mu)$ and $(-i\mu_1,-2i\mu)$ of the imaginary axis. When z is complex the singularities in q of Eq. (8.9) can pinch the singularity of $q^2=z$. However, in that case one has a resonance as in Eq. (8.5), and this confluence of singularities can be taken care of by rewriting the equation in the form (8.2). The process of the strip extension can be iterated to show that $g(p)$ is analytic as a function of p into the complex plane cut from $i\mu$ to $i\infty$ and from $-i\mu$ to $-i\infty$. This proof is admittedly very simplified, but it is easy to make it into a rigorous proof as is rather obvious and as we have checked in detail.

9. ASYMPTOTIC BEHAVIOR OF THE FORM FACTORS

Our problem is now to investigate the behavior of $g(p)$ when p tends to infinity. Here again we shall work on the simple Lippmann-Schwinger equation (8.7) as an example of the techniques to be used.

When $p\to\infty$, one has, according to Eq. (8.8),

$$U(p,q)\simeq cq^lp^{-l-2}\,, \tag{9.1}$$

which suggests that $g(p)$ behaves like p^{-l-2} when $p\to\infty$. Indeed when this assumption is fed into Eq. (8.7) it is found to be consistent.

However, this does not constitute a proof, and it is found deceptively difficult to provide one. The proof that we are going to propose appears to be new and seems to be of a wide applicability to the solutions of Fredholm-type equations. It proceeds as follows:

(a) Introduce the Banach spaces B_α characterized by the norm

$$\sup_{\substack{q\in[0,\infty]\\ p\in[0,\infty]}}\left\{(1+p^2)^{\alpha/2}|f(p)|+\left|\frac{f(p)-f(q)}{p-q}\right|^\mu(1+p^2)\right\}. \tag{9.2}$$

Clearly $B_\alpha\subset B_\beta$ for $\alpha>\beta$ and, according to Sec. 7, $g\in B_\alpha$.

(b) The Lippmann-Schwinger kernel applies B_α to $B_{\alpha+1-\eta}\cup B_{l+2}$, η being as small as wanted. The proof of this lemma goes as follows: Introduce two fixed numbers $\epsilon\ll1$ and $A\gg1$ and suppose $f(q)\in B_\alpha$. Then

$$\left|\int_{Ap}^\infty U(p,q)\frac{q^2dq}{q^2-z}f(q)\right|<c\int_{Ap}^\infty\frac{1}{pq}\left[\frac{pq}{p^2+q^2+\mu^2}\right]^{l+1}\frac{dq}{q^\alpha}<c\int_{Ap}^\infty\frac{p^ldq}{q^{l+2+\alpha}}<\frac{c}{p^{\alpha+1}}\,, \tag{9.3}$$

$$\left|\int_{\epsilon p}^{Ap} U(p,q)\frac{q^2dq}{q^2-z}f(q)\right|<c\int_{\epsilon p}^{Ap}\frac{1}{p^2}\ln\frac{\mu^2}{p^2}\frac{dq}{p^\alpha}<c\frac{\ln p}{p^{\alpha+1}}<\frac{c}{p^{\alpha+1-\eta}}\,, \tag{9.4}$$

$$\left|\int_0^{\epsilon p} U(p,q)\frac{q^2dq}{q^2-z}f(q)\right|<c\int_0^{\epsilon p}\frac{1}{pq}\left[\frac{pq}{p^2+q^2+\mu^2}\right]^{l+1}\frac{q^2dq}{|q^2-z|}|f(q)|<c\int_0^\gamma\frac{q}{p^{l+2}}dq+c\int_\gamma^{\epsilon p}\frac{q}{p^{l+2}}dq\,q^{-\alpha}<\frac{c}{p^{l+2}}+\frac{c}{p^{\alpha+1}}\,. \tag{9.5}$$

Here $\gamma=\min\,(A,\epsilon p)$ and the c's are constants, generally different from each other. Putting these equations together, we get the above statement.

(c) Since $g\in B_2$, by iteration of lemma (b), one gets $g\in B_{l+2}$, i.e.,

$$g(p)p^{l+2}<c\quad\text{as}\quad p\to\infty\,.\quad\text{Q. E. D.} \tag{9.6}$$

The case in which p tends to zero can be treated in the same way to give

$$g(p)p^{-l}<c\quad p\to0\,. \tag{9.7}$$

10. THREE-BODY AMPLITUDE

In order to build up equations for the three-particle scattering amplitude, one could start from a sum of all types of ladder-type diagrams where three particles propagate. It is easy to write a set of linear integral equations for the amplitudes corresponding to the sums of these graphs.[14]

These equations are very similar both to the Bethe-Salpeter equation and to the Faddeev equations and have therefore the following properties:

(a) Their kernel depends on the off-the-mass-shell two-body scattering amplitude.

(b) They satisfy three-body unitarity only in a limited region of energy depending upon the range of the potential.

[14] D. Stojanov and A. N. Tavkhelidze, Phys. Letters **13**, 76 (1964); V. P. Shelest and D. Stojanov, *ibid.* **13**, 253 (1964).

From a practical standpoint they give rise to some difficulties:

(a) The usual difficulties of the Bethe-Salpeter equation with respect to the non-Euclidean metric of momentum space.

(b) Our ignorance of the properties of the solution of the two-body Bethe-Salpeter equation makes it difficult to discover whether the three-body equations are of the Fredholm type.

(c) They involve too many integrations to be of practical use on a computer. Therefore, these equations are not what we are looking for if we want to satisfy the conditions stated in the Introduction.

We are now going to extend the techniques of Blankenbecler and Sugar to a set of three-particle ladder diagrams. Consider for simplicity three distinguishable scalar particles of the same mass. In the ladder approximation, the three-body amplitude $T(p_1p_2p_3;p_1'p_2'p_3')$ will be given by the sum of all diagrams of the form shown in Fig. 1. We will call $T_\alpha(p_1p_2p_3|p_1'p_2'p_3)$ the sum of all Feynman diagrams in which the first interaction takes place between the pair (β,γ) $(\alpha\neq\beta\neq\gamma;$ $\alpha,\beta,\gamma=1,2,3)$. In an analogous way $T_\alpha'(p_1p_2p_3|p_1'p_2'p_3')$ will represent the sum of all diagrams in which the last interaction takes place between β' and γ'; and $T_{\alpha\beta}'(p_1p_2p_3|p_1'p_2'p_3')$ the corresponding sum of diagrams with particles $(\beta\gamma)$ interacting first and $(\alpha'\gamma')$ interacting last. In this way, it is obvious that

$$T(p_1p_2p_3;p_1'p_2'p_3') = \sum_{\alpha=1}^{3} T_a(p_1p_2p_3;p_1'p_2'p_3') = \sum_{\beta'=1}^{3} T_{\beta'}(p_i,p_i') = \sum_{\alpha,\beta'=1}^{3} T_{\alpha\beta'}(p_i,p_i'). \tag{10.1}$$

Needless to say, these amplitudes are considered off the mass shell. However, in our final equations we will be allowed to put two of the three initial or final particles on the mass shell. For example, if we are considering $T_\alpha(p_i,p_i')$ we will need it either when particles (α,β) or when particles (α,γ) are on the mass shell. Therefore, we shall use the following notation:

$$T_\alpha{}^\beta(p_i,p_i') = T_\alpha(p_i,p_i')\,|_{p_\alpha{}^2=m_\alpha{}^2,\,p_\beta{}^2=m_\beta{}^2}. \tag{10.2a}$$

The requirement of keeping only the elastic part of the amplitude means that our three-body amplitudes will have only the Landau singularities associated with the three propagators cut by lines B, C, D, $\cdots$ and the propagators cut by lines A, A'. The prescription we will follow for getting such an amplitude is to insert the function E_3 defined previously in the sections B, C, D of the diagram and also the same function in sections of the type A, A'. By using the Cutkosky rules for the discontinuity it is possible to convince oneself that this is the correct prescription. Anyway, we will come back to this point later.

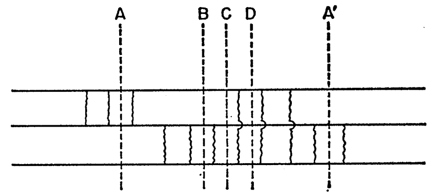

FIG. 1. Ladder-type diagrams considered in the derivation of the relativistic Faddeev equations.

Because of the ambiguities that are present in the calculation of E_3, we have to be more explicit in our prescription. If V_α represents an exchange between particle (β,γ), then following any V_α we will insert the function E^γ, with $\gamma\neq\alpha$, introduced in Eq. (5.6). This procedure is not symmetric starting from the right or from the left side of the diagram; however, the final explicit expression for a graph will be symmetric under the exchange of the initial and final variables. This result will be obtained by exploiting the ambiguity that we still have in the function $E_3{}^\gamma$; namely, given a certain V_α we can choose either E^β or E^γ; $a\neq\beta\neq\gamma$.

11. THE THREE-BODY EQUATIONS

Let us define the variable

$$s=(p_1+p_2+p_3)^2=(p_1'+p_2'+p_3')^2,$$

which is the total energy in the total center-of-mass system $\mathbf{P}=\mathbf{P}'=0$. It will be a fixed parameter in our equations. Following our prescription, it is easy to obtain the following equations for $T_\alpha(s)$ and $T_{\alpha'}(s)$:

$$T_\alpha(s)=t_\alpha(s)+\sum_\gamma t_\alpha(s)E^\beta(s)(1-\delta_{\alpha\gamma})T_\gamma(s), \tag{11.1}$$

$$T_{\alpha'}(s)=t_{\alpha'}(s)+\sum_{\gamma'} T_{\gamma'}(s)(1-\delta_{\alpha'\gamma'})E^{\beta'}(s)t_{\alpha'}(s), \tag{11.2}$$

$t_\alpha(s)$ being the solution of the two-body equation with the Green function E_3 studied in Sec. 6. Equation (11.1) is an equation in the initial variables p_i, the final variables p_i' being fixed parameters. The opposite is true for Eq. (11.2). We can also write equations for $T_{\alpha\alpha'}$ in the "right" or "left" variables:

$$T_{\alpha\alpha'}(s)=\delta_{\alpha\alpha'}t_\alpha(s)+\sum_\gamma t_\alpha(s)E^\beta(s)(1-\delta_{\alpha\gamma})T_{\gamma\alpha'}(s), \qquad \alpha\neq\beta\neq\gamma; \tag{11.3}$$

$$T_{\alpha\alpha'}(s)=\delta_{\alpha\alpha'}t_{\alpha'}(s)+\sum_{\gamma'} T_{\alpha\gamma'}(s)(1-\delta_{\alpha'\gamma'})E^{\beta'}(s)t_{\alpha'}(s), \qquad \alpha'\neq\beta'\neq\gamma'. \tag{11.4}$$

Notice that the Green's function E_3 that we use in these equations now is prescribed unambiguously. In the equation for $T_\alpha(s)$, for example, next to the amplitude $T_\gamma(s)$ we put the function $E^\beta(s)$, $\alpha\neq\beta\neq\gamma$.

Let us write Eqs. (11.1) more explicitly, in order to study them in more detail:

$$T_\alpha(\mathbf{p}_\alpha p_\beta p_\gamma,p_\alpha' p_\beta' p_\gamma')=\omega_\alpha\delta(\mathbf{p}_\alpha-\mathbf{p}_\alpha')t_\alpha(p_\beta p_\gamma,p_\beta' p_\gamma')+\int d^4q_\beta t_\alpha(p_\beta p_\gamma,q_\beta q_\gamma)\frac{\pi}{\omega_\beta\omega_\gamma}\frac{(\sum\omega)}{(\sum\omega)^2-s}\delta(q_{\beta_0}-\omega_\beta)T_\beta(\mathbf{p}_\alpha q_\beta q_\gamma,p_i')$$

$$+\int d^4q_\gamma t_\alpha(p_\beta p_\gamma,q_\beta q_\gamma)\frac{\pi^2}{\omega_\beta\omega_\gamma}\frac{(\sum\omega)}{(\sum\omega)^2-s}\delta(q_{\gamma_0}-\omega_\gamma)T_\gamma(\mathbf{p}_\alpha q_\beta q_\gamma;p_i'). \quad (11.5)$$

In this equation we have already separated the total δ functions of conservation of energy-momentum which appear in the two- and three-body scattering amplitudes. The integration variables are taken to be q_β and q_γ in the integrals which contain T_β and T_γ, respectively.

In Eq. (11.6) we have put the external momentum p_α on the mass shell. Similarly, if we write the equation for T_β and T_γ we see that we can put $p_\beta^2=m^2$ and $p_\gamma^2=m^2$, respectively. The three equations are still consistent with this restriction because when we integrate the functions T_β in (11.5) we can choose q_β and q_γ as variables, respectively, and still keep that restriction.

Therefore, after one performs the integrations over dq_{β_0} or dq_{β_0}, Eq. (11.5) reduces to

$$T_\alpha(\mathbf{p}_\alpha p_\beta p_\gamma,p_i')=\omega_\alpha\delta(\mathbf{p}_\alpha-\mathbf{p}_\alpha')t_\alpha(p_\beta p_\gamma,p_\beta' p_\gamma')+\int d\mathbf{q}_\beta t_\alpha(p_\beta p_\gamma,\mathbf{q}_\beta q_\gamma)\frac{\pi}{\omega_\beta\omega_\gamma}\frac{(\sum\omega)}{(\sum\omega)^2-s}T_\beta{}^{(\alpha)}(\mathbf{p}_\alpha\mathbf{q}_\beta q_\gamma,p_i')$$

$$+\int d\mathbf{q}_\gamma t_\alpha(p_\beta p_\gamma,q_\beta\mathbf{q}_\gamma)\frac{\pi^2}{\omega_\beta\omega_\gamma}\frac{(\sum\omega)}{(\sum\omega)^2-s}T_\gamma{}^{(\alpha)}(\mathbf{p}_\alpha q_\beta\mathbf{q}_\gamma,p_i'),\quad \alpha\neq\beta\neq\gamma;\quad \alpha,\beta,\gamma=1,2,3. \quad (11.6)$$

These equations can be simplified still further. In the integral terms of the equations for T_β and T_γ we will need only $T_\alpha{}^{(\beta)}$ or $T_\alpha{}^{(\gamma)}$, respectively. Then, we can put either p_β or p_γ on the mass shell in Eq. (11.6) and get equations for $T_\alpha{}^{(\beta)}$ and $T_\alpha{}^{(\gamma)}$. Notice that once we have done that, we are left with two- and three-body amplitudes in which only one of the particles is off the mass shell. In the total center-of-mass system we have

$$s^{1/2}=\sum_i p_{i0}=\sum_i q_{i0}=\sum_i p_{i0}'. \quad (11.7)$$

If two particles, say α and β, are fixed on the mass shell, then $p_{\gamma 0}=s^{1/2}-\omega_\alpha-\omega_\beta$ and we can write, for example,

$$T_\alpha{}^{(\beta)}(\mathbf{p}_\alpha,\mathbf{p}_\beta,\mathbf{p}_\gamma,p_\gamma{}^0;p_i')=T_\alpha{}^{(\beta)}(\mathbf{p}_\alpha,\mathbf{p}_\beta,\mathbf{p}_\gamma;s^{1/2};p_i'), \quad (11.8)$$

so we can forget about the $p_\gamma{}^0$ variable and look at this amplitude as being "off the energy shell," because $s^{1/2}\neq\sum\omega_i$. The correct mass-shell amplitude is got by putting s on the "energy shell."

We now introduce the notation:

$$t_\alpha(\mathbf{p}_\beta,p_\gamma;q_\beta,\mathbf{q}_\gamma)=t_\alpha{}^{\beta\gamma}(\mathbf{p}_\beta,\mathbf{p}_\gamma;\mathbf{q}_\beta,\mathbf{q}_\gamma;s^{1/2}). \quad (11.9)$$

These amplitudes can be obtained by solving Eq. (6.4) for $t_\alpha{}^\beta(\mathbf{p}_\beta p_\gamma,\mathbf{q}_\beta q_\gamma)$ for example, because in that equation q_β, q_γ are dummy variables which can be given any value.

The final set of three-body equations is

$$T_\alpha{}^\beta(\mathbf{p}_\alpha\mathbf{p}_\beta\mathbf{p}_\gamma,s;p_i')=\omega_\alpha\delta(\mathbf{p}_\alpha-\mathbf{p}_\alpha')t_\alpha{}^\beta(\mathbf{p}_\beta\mathbf{p}_\gamma,s,\mathbf{p}_\beta'\mathbf{p}_\gamma')+\int d\mathbf{q}_\beta t_\alpha{}^{\beta\beta}(\mathbf{p}_\beta\mathbf{p}_\gamma,s,\mathbf{q}_\beta\mathbf{q}_\gamma)\frac{\pi^2}{\omega_\beta\omega_\gamma}\frac{\sum\omega_i}{(\sum\omega_i)^2-s}T_\beta{}^\alpha(\mathbf{p}_\alpha\mathbf{p}_\beta\mathbf{p}_\gamma,s,p_i')$$

$$+\int d\mathbf{q}_\beta t_\alpha{}^{\beta\gamma}(\mathbf{p}_\beta\mathbf{p}_\gamma,s,\mathbf{q}_\beta\mathbf{q}_\gamma)\frac{\pi^2}{\omega_\beta\omega_\gamma}\frac{\sum\omega_i}{(\sum\omega_i)^2-s}T_\gamma{}^\alpha(\mathbf{p}_\alpha\mathbf{p}_\beta\mathbf{p}_\gamma,s,p_i'), \quad (11.10)$$

$\alpha\neq\beta\neq\gamma$; α, β, $\gamma=1, 2, 3$. In a more explicit form, they are

$$\begin{bmatrix}T_1{}^2(s)\\T_1{}^3(s)\\T_2{}^1(s)\\T_2{}^3(s)\\T_3{}^1(s)\\T_3{}^2(s)\end{bmatrix}=\begin{bmatrix}t_1{}^2(s)\\t_1{}^3(s)\\t_2{}^1(s)\\t_2{}^3(s)\\t_3{}^1(s)\\t_3{}^2(s)\end{bmatrix}+\begin{bmatrix}0&0&t_1{}^{22}(s)&0&t_1{}^{23}(s)&0\\0&0&t_1{}^{32}(s)&0&t_1{}^{33}(s)&0\\t_2{}^{11}(s)&0&0&0&0&t_2{}^{13}(s)\\t_2{}^{31}(s)&0&0&0&0&t_2{}^{33}(s)\\0&t_3{}^{11}(s)&0&t_3{}^{12}(s)&0&0\\0&t_3{}^{21}(s)&0&t_3{}^{22}(s)&0&0\end{bmatrix}G_0(s)\begin{bmatrix}T_1{}^2(s)\\T_1{}^3(s)\\T_2{}^1(s)\\T_2{}^3(s)\\T_3{}^1(s)\\T_3{}^2(s)\end{bmatrix}, \quad (11.11)$$

$G_0(s)$ being an operator diagonal in momentum space whose matrix element is

$$\frac{\pi^2}{\omega_1\omega_2\omega_3}\frac{(\sum\omega)_i}{(\sum\omega_i)^2-s}.$$

12. PROPERTIES OF THE EQUATIONS

The foregoing equations are relativistic generalizations of the Faddeev equations; they are three-dimensional and off-the-energy-shell. The main difference is that, when simplifying the equations we started with, we have doubled the number of amplitudes. As $T_\alpha{}^\beta(s)$ and $T_\alpha{}^\gamma(s)$ are different restrictions of the same function, the on-the-mass-shell three-body amplitude is one-half the sum of the six amplitudes $T_\alpha{}^\beta(s)$ taken on the energy shell.

As the two-body t matrices which appear in the kernel have essentially the same properties as the solutions of the Lippmann-Schwinger equations, the proof by Faddeev of the compactness of the square of the kernel of the nonrelativistic equations can be applied as well to the set of Eqs. (11.11) that we propose.

The proof of unitarity given in Sec. 4 can be applied also to the nonrelativistic Faddeev equations and to the relativistic equations of the previous sections. These proofs are carried out in detail in the Appendix. Using Eqs. (11.1)–(11.4) we prove that $T(s)$ satisfies the usual three-body elastic unitarity condition

$$T(s)-T^\dagger(s)=+2\pi iT(s)\Delta_3(s)T^\dagger(s). \qquad (12.1)$$

This is also a proof of the consistency of our prescriptions for obtaining those equations.

It is important to realize that the proof of unitarity uses only the fact that the two-body amplitude satisfies unitarity and that the normalization of the state of the spectator particle is invariant. It is therefore possible to introduce into the equations any phenomenological form for the two-body amplitude satisfying unitarity without spoiling the unitarity of the three-body amplitude.

13. SEPARATION OF ANGULAR MOMENTUM

Owing to the complete analogy of our equations with the Faddeev equations, the separation of total angular momentum can be carried on in the same way.[15] If we work in the center-of-mass system the three momenta $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ form a closed triangle in a three-dimensional space, and the system can also be described by giving the three lengths $|\mathbf{p}_i|$ of the sides of the triangle, or else $\omega_i=(\mathbf{p}_i^2+m^2)^{1/2}$, and the orientation of the triangle in space, namely: the three Euler angles (ψ,θ,ϕ) which describe the rotation necessary to go from a space-fixed system of axes to a "body-fixed" system of axes linked to the triangle in a well-defined way. The set of quantum numbers canonically conjugated to these variables are the three subenergies ω_i; J, the total angular momentum; M, the projection of J over the body-fixed z axis; and M_z, its projection over the space-fixed z axis.

Of course, this procedure is not covariant; but it is the simplest from the practical point of view. It is straightforward to reproduce for this case the calculations carried out in Ref. 15. We shall not do it here; we only want to point out the only differences: Relativistic kinematics must be used, also the relativistic normalization of momentum eigenstates

$$\langle\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle=\prod_i \omega_i\delta^{(3)}(\mathbf{p}_i-\mathbf{p}_i'). \qquad (13.1)$$

The transformation kernel from the momentum to the angular-momentum representation will be

$$\delta(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{p}_3')\langle\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'|\mathbf{P},\omega_1\omega_2\omega_3,JMM_z\rangle =\left(\frac{2J+1}{8\pi^2}\right)^{1/2}\delta(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{p}_3')\delta(\mathbf{P}) \times\prod_i \delta(\omega_i-(p_i^2+m^2)^{1/2})D_{MM_z}{}^J(\psi,\theta,\phi). \qquad (13.2)$$

Let us relabel the six amplitudes $T_\alpha{}^\beta(s)$ which appear in Eq. (11.12) by $T^{(i)}(s)$, $i=1, 2, \cdots 6$. After separating angular momentum, we will get equations of the form

$$T_{MM'}{}^{(i)J}(\omega_1\omega_2\omega_3,\omega_1'\omega_2'\omega_3';s)=t_{MM'}{}^{(i)J}(\omega_1\omega_2\omega_3;\omega_1'\omega_2'\omega_3')+\sum_{j\neq i}\sum_{M''=-J}^{+J}\int\prod_i d\omega_i''K_{MM''}{}^{J(i,j)}(\omega_1\omega_2\omega_3,\omega_1''\omega_2''\omega_3'';s) \times T_{M''M'}{}^{(j)J}(\omega_1''\omega_2''\omega_3'';\omega_1'\omega_2'\omega_3';s),\quad i\neq j;\quad i,j=1,2,3, \qquad (13.3)$$

where i labels the rows and j the columns of the 6 by 6 matrices of Eq. (11.12). The reduction of parity can be also carried out,[15] the summation in (13.3) is reduced only to the even or odd value of M'', according to the quantum numbers J^p of the state one wants to look at.

In Eq. (13.3) there are only two variables of integration because the kernel contains a $\delta(\omega_\alpha-\omega_\alpha'')$ since there is always one particle which does not interact. In the corresponding nonrelativistic problem, Lovelace and Basdevant have shown that when one approximates the two-body t matrices by a bound state or resonance pole, because of the factorization of the residue in the initial and final variables the equations for $T_{MM'}{}^{(i)J}(\omega,\omega')$ can be transformed into a set of coupled linear integral equations in only one variable.[3,5] We have checked that the same is true for our relativistic equations. This fact makes them very appealing from the practical point of view, because the modified one-dimensional equations are very easy to handle in a computer.

[15] R. Omnes, Phys. Rev. 134, B1358 (1964).

14. ANOTHER SET OF EQUATIONS

Following the same method as in Sec. 3, we can define directly an off-the-energy-shell, on-the-mass-shell amplitude by considering the same set of ladder graphs as in Sec. 10, dropping all the δ functions which ensure energy conservation at each vertex, and introducing the propagator

$$G'(s)=(2\pi)^2\int \delta^+(q_1^2-m^2)\delta^+(q_2^2-m^2)\delta^+(q_3^2-m^2)\delta^+[\sqrt{s'}-(\omega_1+\omega_2+\omega_3)]\frac{ds'}{s'-s}. \quad (14.1)$$

The corresponding three-body amplitude can be written as

$$T=T_1+T_2+T_3, \quad (14.2)$$

where T_1, for instance, is the sum of all matrix elements whose graph has its leftmost interaction between particles 2 and 3.

The amplitudes satisfy

$$T_1(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3')=t_1(\mathbf{p}_1,\mathbf{p}_3;\mathbf{p}_2'\mathbf{p}_3')\delta(\mathbf{p}_1-\mathbf{p}_1')\cdot\omega_1 + \int t_1(\mathbf{p}_2,\mathbf{p}_3;\mathbf{p}_2'',\mathbf{p}_3'')\frac{4\pi^2}{8\omega_1''\omega_2''\omega_3''}\frac{(\omega_1''+\omega_2''+\omega_3'')}{(\omega_1''+\omega_2''+\omega_3'')^2-s}d\mathbf{p}_2''d\mathbf{p}_3''\delta(\mathbf{p}_2+\mathbf{p}_3-\mathbf{p}_2''-\mathbf{p}_3'')$$
$$\times[T_2(\mathbf{p}_1,\mathbf{p}_2'',\mathbf{p}_3'';\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3')+T_3(\mathbf{p}_1,\mathbf{p}_2'',\mathbf{p}_3'';\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')], \quad (14.3)$$

as well as two similar equations obtained by permuting the indices 1, 2, 3.

The amplitude $t_1(\mathbf{p}_2,\mathbf{p}_3;\mathbf{p}_2',\mathbf{p}_3')$ satisfies

$$t_1(\mathbf{p}_2,\mathbf{p}_3;\mathbf{p}_2'\mathbf{p}_3')=V(\mathbf{p}_2-\mathbf{p}_2')+\int V(\mathbf{p}_2-\mathbf{p}_2'')\frac{4\pi^2}{4\omega_2''\omega_3''}\frac{(\omega_1+\omega_2''+\omega_3'')}{(\omega_1+\omega_2''+\omega_3'')^2-s}$$
$$\times\delta(\mathbf{p}_2+\mathbf{p}_3-\mathbf{p}_2''-\mathbf{p}_3'')d\mathbf{p}_2''d\mathbf{p}_3''t_1(\mathbf{p}_2'',\mathbf{p}_3'';\mathbf{p}_2',\mathbf{p}_3') \quad (14.4)$$

with

$$V(\mathbf{p}-\mathbf{p}')=g^2[(\omega-\omega')^2-(\mathbf{p}-\mathbf{p}')^2-\mu^2]^{-1}. \quad (14.5)$$

(Note that the choice of the variable in the potential is somewhat arbitrary, owing to the nonconservation of energy.)

The proof of unitarity given in Sec. 4 for the Lippmann-Schwinger equation immediately applies to Eq. (14.4). On the other hand, the proof of unitarity given in the Appendix for the Faddeev equations applies to Eq. (14.3), which is, in fact, extremely similar to the Faddeev equation. Equation (14.3) is, from a practical standpoint, better than Eqs. (11.12), since there are only three independent functions. On the other hand, their obvious defect is to be dependent upon the system of reference. However, the unitarity of t_1 is not affected if one replaces Eq. (14.4) by

$$t_1(\mathbf{p}_2,\mathbf{p}_3;\mathbf{p}_2',\mathbf{p}_3')=V(\mathbf{p}_2-\mathbf{p}_2')+\int V(\mathbf{p}_2-\mathbf{p}_2'')\frac{\pi^2}{\omega_1''\omega_2''}\frac{1}{(\omega_2''+\omega_3'')-(s-\omega_1)}$$
$$\times\delta(\mathbf{p}_2+\mathbf{p}_3-\mathbf{p}_2''-\mathbf{p}_3'')d\mathbf{p}_2''d\mathbf{p}_3''t_1(\mathbf{p}_2'',\mathbf{p}_3'';\mathbf{p}_2',\mathbf{p}_3'). \quad (14.6)$$

In that case $(s-\omega_1)$ appears as the energy of a two-particle bound state. The same approximation can also be made in the amplitudes of Sec. 6. However, this also spoils relativistic invariance.

15. CONCLUSIONS

We have presented in Sec. 12 a set of relativistic Faddeev equations which satisfy relativistic three-body "elastic" unitarity. The three-body scattering amplitude which in principle can be calculated with those equations represents the part of the Bethe-Salpeter amplitude for a set of ladder graphs which contains only the contribution of three-particle intermediate states. The final equations (11.11) are such that the usual mathematical techniques of potential scattering can be applied to them.

We must emphasize that we did not undertake this work guided by the belief that the Bethe-Salpeter equations represent something fundamental in high-energy physics. As a matter of fact, we used the ladder diagrams in order to obtain the final equations only; our main interest was to obtain practical relativistic equations to calculate the three-body amplitude once the solutions of the two-body problems were known. In the practical applications that we had in mind one does not start by solving the two-body problem; the experimental information is used and the two-body amplitudes are approximated by the direct channel bound states or resonance poles. That is the reason why this theory has to be regarded only as a semiphenomenological one.

Among the relativistic three-body systems, probably the most appealing is the system of three pseudoscalar mesons, because of the number of three-body resonances that are being found and because one can try to find the pseudoscalar mesons themselves as bound states. The computational procedure is in principle very simple. After separating angular momentum, parity, and isospin, one looks at the equation with the adequate J^p and I, and calculates the eigenvalues of the kernel. This gives enough information to determine the three-body bound states and resonances present in that channel. In particular, work is now in progress[16] in the 3π system in order to find the ω meson in the 1^- channel and the π meson in the 0^- channel.

However, there is something which is lost in the relativistic problem, and it is the simple relation that there exists in potential theory between the two-body t matrix with a two-body propagator $t_\alpha(z)$ and with a three-body propagator $t_\alpha(z)$,

$$\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|t_\alpha(z)|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle = \delta^{(3)}(\mathbf{p}_\alpha-\mathbf{p}_\alpha')\langle \mathbf{p}_\beta\mathbf{p}_\gamma|t_\alpha(z-\omega_\alpha)|\mathbf{p}_\beta'\mathbf{p}_\gamma'\rangle, \quad (15.1)$$
$$\omega_\alpha=\mathbf{p}_\alpha^2/2m_\alpha.$$

The dependence of our $t_\alpha(s)$ upon ω_α is not so simple; this point requires further investigation. This situation is reflected in the fact that when one makes the "pole" approximation for $t_\alpha(p_\beta p_\gamma, p_\beta' p_\gamma')$, the form factors will depend upon ω_α. However, they are still well-defined objects; from the practical point of view that is not an essential difficulty, because one always introduces phenomenological expressions for the bound-state and resonance form factors. Moreover, any form of the two-body amplitude which satisfies two-body unitarity will preserve three-body unitarity in the equations.

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APPENDIX. PROOF OF UNITARITY FOR THE FADDEEV EQUATIONS

1. Nonrelativistic Equations

The Faddeev equations for the three-body scattering operator $T(z)$ are[1]

$$T_\alpha(z)=t_\alpha(z)+\sum_\gamma t_\alpha(z)\frac{1}{H_0-z}(1-\delta_{\alpha\gamma})T_\gamma(z), \quad (A.1)$$

where H_0 is the Hamiltonian of the three-particle system and $t_\alpha(z)$ is the two-body t matrix in a three-body Hilbert system. The meaning of the subindices is similar to that explained in Sec. 11 for the relativistic case. Here we assume that there are no two-body bound states, so the only amplitude is the $3\to 3$ scattering amplitude. We also have

$$T_{\alpha'}(z)=t_{\alpha'}(z)+\sum_{\gamma'} T_{\gamma'}(z)(1-\delta_{\alpha'\gamma'})\frac{1}{H_0-z}t_{\alpha'}(z), \quad (A.2)$$

$$T_{\alpha\alpha'}(z)=\delta_{\alpha\alpha'}t_\alpha(z)+\sum_\gamma t_\alpha(z)\frac{1}{H_0-z}(1-\delta_{\alpha\gamma})T_{\gamma\alpha'}(z), \quad (A.3)$$

$$T_{\alpha\alpha'}(z)=\delta_{\alpha\alpha'}t_{\alpha'}(z)+\sum_{\gamma'} T_{\alpha\gamma'}(z)(1-\delta_{\gamma'\alpha'})\frac{1}{H_0-z}t_{\alpha'}(z), \quad (A.4)$$

and

$$T(z)=\sum_\alpha T_\alpha(z)=\sum_{\alpha'} T_{\alpha'}(z)=\sum_{\alpha,\alpha'} T_{\alpha\alpha'}(z).$$

The proof now follows the same steps explained in Sec. 4.

(a) Taking the adjoint of (A.4), we have

$$T_{\alpha\alpha'}^\dagger(z)=\delta_{\alpha\alpha'}t_{\alpha'}^\dagger(z)+\sum_{\gamma'} t_{\alpha'}^\dagger(z)\frac{1}{H_0-z}(1-\delta_{\alpha'\gamma'})T_{\alpha\gamma'}^\dagger(z). \quad (A.5)$$

(b) Relabeling the indices $\alpha\leftrightarrow\alpha'$, and calling the summation variable γ in place of γ', which gives

$$T_{\alpha'\alpha}^\dagger(z)=\delta_{\alpha'\alpha}t_\alpha^\dagger(z)+\sum_\gamma t_\alpha^\dagger(z)\frac{1}{H_0-\bar{z}}(1-\delta_{\alpha\gamma})T_{\alpha'\gamma}^\dagger(z). \quad (A.6)$$

(c) Introducing $\Delta T_{\alpha\alpha'}(z)=T_{\alpha\alpha'}(z)-T_{\alpha'\alpha}^\dagger(z)$ and using (A.3), (A.6) and the unitarity condition for $t_\alpha(z)$ gives

$$t_\alpha(z)-t_\alpha^\dagger(z)=+2\pi i t_\alpha(z)\delta(H_0-z)t_\alpha^\dagger(z);$$

we get

$$\Delta T_{\alpha\alpha'}(z)=A_{\alpha\alpha'}(z)+\sum_\gamma t_\alpha(z)\frac{1}{H_0-z}(1-\delta_{\alpha\gamma})T_{\gamma\alpha'}(z), \quad (A.7)$$

where

$$A_{\alpha\alpha'}(z)=2\pi i t_\alpha(z)\delta(H_0-z)\times[T_{\alpha'\alpha}^\dagger(z)+\sum_\gamma(1-\delta_{\alpha\gamma})T_{\alpha'\gamma}^\dagger(z)],$$

$$A_{\alpha\alpha'}(z)=2\pi i t_\alpha(z)\delta(H_0-z)(\sum_\gamma T_{\alpha'\gamma}^\dagger(z))=2\pi i t_\alpha(z)\delta(H_0-z)T_{\alpha'}^\dagger(z). \quad (A.8)$$

[16] J. L. Basdevant and R. Kreps (private communication).

(d) Next, we define $\Delta T_\alpha(z)=\sum_{\alpha'}\Delta T_{\alpha\alpha'}(z)$. As α' is a dummy index in Eq. (A.7), we can sum over α' and get

$$\Delta T_\alpha=2\pi i t_\alpha(z)\delta(H_0-z)T^\dagger(z)+t_\alpha(z)\frac{1}{H_0-z}(1-\delta_{\alpha\gamma})\Delta T_\gamma(z). \quad \text{(A.9)}$$

Comparing this equation with Eq. (A.1), we find that the kernel is the same, and its inhomogeneous term is gotten by applying on the left of the inhomogeneous term of (A.1) the operator $2\pi i\delta(H_0-z)T^\dagger(z)$. Therefore, as the Faddeev equations are Fredholm equations, we obtain

$$\Delta T_\alpha(z)=2\pi i T_\alpha(z)\delta(H_0-z)T^\dagger(z). \quad \text{(A.10)}$$

(e) Finally, $\Delta T(z)=T(z)-T^\dagger(z)=\sum_\alpha \Delta T_\alpha(z)$;

$$\Delta T(z)=\sum_\alpha \Delta T_\alpha(z)=2\pi i T(z)\delta(H_0-z)T^\dagger(z), \quad \text{(A.11)}$$

which is the off-the-energy-shell unitarity condition. If there are two-body bound states, there are also additional cuts in the z plane which represent the right-hand cuts for bound-state scattering. In that case it is necessary to introduce more scattering operators because one faces a multichannel problem. However, the right-hand side of (A.11) is still the discontinuity of $T(s)$ across the three-particle cut in the z plane.

2. Relativistic Equations

Here, in place of Eqs. (A.1) through (A.4) one has to start from Eqs. (11.1) through (11.4). The only difference now is that the Green function $1/(H_0-z)$ is replaced by $E^\alpha(s)$. We remember that the function $E^\alpha(s)$ is just the function $E_3(s)$ calculated according to a certain prescription. That distinction is very important in the manipulation of the equations, but for the proof of unitarity it is irrelevant because $E^{(\alpha)}(s)$, $\alpha=1, 2, 3$ has the same discontinuity as $E_3(s)$, given by

$$\text{Disc}E^{(\alpha)}(s)=2\pi i\Delta_3(s); \quad \text{(A.12)}$$

here $\Delta_3(s)$ can be thought of as an operator diagonal in four-momentum space, with matrix element given by $\delta^+(q_1^2-m^2)\delta^+(q_2^2-m^2)\delta^+(q_3^2-m^2)$; $s=(q_1+q+q_3)^2$. By repeating the same steps as before, and using the unitarity condition for the relativistic $t_\alpha(s)$ discussed in Sec. 7,

$$\Delta t_\alpha(s)=2\pi i t_\alpha(s)\Delta_3(s)t_\alpha{}^\dagger(s), \quad \text{(A.13)}$$

one can get

$$T(s)-T^\dagger(s)=2\pi i T(s)\Delta_3(s)T^\dagger(s). \quad \text{(A.14)}$$

This is an off-shell unitarity relation; when the integration over the intermediate variables $q_1q_2q_3$ in the right-hand side is carried out, the presence of $\Delta_3(s)$ yields the usual relativistic three-body phase space. When the external variables are put on the mass shell, Eq. (A.14) reduces to the usual mass-shell unitarity condition.