Low-Energy Equivalent-Potential Approach for Strong Interactions. II

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In a previous paper a method was presented for constructing the long-range part of an effective potential at any given energy. In certain problems, this potential is singular at small distances, requiring a cutoff or some other parametrization. The parameters thereby introduced can be determined by requiring that the amplitude at threshold be the same as that given by the Singh-Udgaonkar sum rule.

1. INTRODUCTION

IN an earlier paper,[1] hereafter referred to as I, equations were given for constructing an effective potential at any given energy from the crossed-channel absorptive part. These equations are straightforward generalizations of the ones given by Charap and Fubini.[2] The absorptive part is calculated with the help of the strip approximation.[3] Arguments can be given for retaining only a few iterations in our equations.

In many cases, however, the potential is both singular and attractive at small distances. We therefore have to introduce a cutoff, or in some way parametrize the short-range part of the potential.

In order to calculate the parameters introduced in this fashion, we can use a sum rule introduced by Singh and Udgaonkar.[4] This is derived from the strip approximation[3] and resembles the Cini-Fubini approximation.[5] We can then demand that the amplitude at threshold as given by the effective potential be the same as that given by the sum rule. Although a cutoff is not actually needed in a lowest order bootstrap of the ρ in $\pi\pi$ scattering, a crude calculation is made in which this requirement is imposed. It is found that the results do not differ too much from one in which there is no cutoff.

2. THE EFFECTIVE POTENTIAL

For simplicity we shall consider the $\pi\pi$ problem, although our approach can be applied to other scattering problems. In I it was shown that the effective potential at any particular energy is given by

$$V^I(r,q^2)=-\pi^{-1}\int_{t_0}^{\infty}dt' v^I(t',q^2)r^{-1}e^{-r\sqrt{t'}}, \qquad (1)$$

where q = magnitude of the three-momentum at that energy, I = isotopic spin, r = radial distance, and t_0 = square of the lowest intermediate mass in the crossed channel (in this case, $t_0=4$). The function v^I is given by

$$v^I(t,q^2)=2s^{-1/2}\tilde{A}_t{}^I(t,s)-\frac{1}{\pi}\int_0^{\infty}dk'^2\frac{\alpha^I(k'^2,t)}{k'^2-q^2}, \qquad (2)$$

with $t=-2q^2(1-\cos\theta)$, θ = c.m. scattering angle, $s=4(q^2+1)$ = square of the total c.m. energy (we are taking pion mass = 1), and $\tilde{A}_t{}^I=\sum_{I'}\beta_{II'}A_t{}^{I'}$, where $\beta_{II'}$ is the isospin crossing matrix

$$\beta_{II'}=\begin{pmatrix}\frac{1}{3} & 1 & \frac{5}{3}\\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6}\\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6}\end{pmatrix}$$

and $A_t{}^I$ is the absorptive part in the t channel. To obtain α^I, we use unitarity, which gives

$$\alpha^I(k^2,t)=\frac{1}{\pi k}\int_{t_0}^{\infty}dt' \times\int_{t_0}^{\infty}dt''\frac{f_t{}^{I*}(t',k^2)f_t{}^I(t'',k^2)}{K^{1/2}(k^2;t,t',t'')}\theta(t-t_+), \qquad (3)$$

where

$$f_t{}^I(t,k^2)=2s^{-1/2}\tilde{A}_t{}^I(t,s)+\frac{k^2-q^2}{\pi}\int_0^{\infty}dk'^2\frac{\alpha^I(k'^2,t)}{(k'^2-q^2)(k'^2-k^2)}, \qquad (4)$$

$$K(k^2;t,t',t'')=t^2+t'^2+t''^2-2(tt'+tt''+t't'')-tt't''k^{-2}, \qquad (5)$$

and

$$t_+(k^2)=t'+t''+\frac{t't''}{k^2}+2\left[(t't'')\left(1+\frac{t'}{4k^2}\right)\left(1+\frac{t''}{4k^2}\right)\right]^{1/2}. \qquad (6)$$

The potential $V^I(r,q^2)$ then reproduces the absorptive part $\tilde{A}_t{}^I$ when the momentum $=q$.

Equations (3) and (4) constitute a nonlinear integral equation for $f_t{}^I$ if we are given $\tilde{A}_t{}^I$. This can be solved by iteration, where, in lowest approximation, we would drop the integral term in Eq. (4).[6] As was shown in I, the nth iteration would give $f_t{}^I$ exactly in the region

[1] L. A. P. Balázs, Phys. Rev. **137**, B1510 (1965).
[2] J. M. Charap and S. P. Fubini, Nuovo Cimento **14**, 540 (1959); **15**, 73 (1960).
[3] G. F. Chew and S. C. Frautschi, Phys. Rev. **123**, 1478 (1961).
[4] V. Singh and B. M. Udgaonkar, Phys. Rev. **130**, 1177 (1963). For a more detailed derivation of this sum rule, see L. A. P. Balázs, Phys. Rev. **134**, B1315 (1964).
[5] M. Cini and S. P. Fubini, Ann. Phys. (N. Y.) **10**, 352 (1960).
[6] Actually this iteration procedure breaks down near $s=0$. Since, however, there are no zero-mass particles in strong-interaction physics, this does not give rise to any difficulties.

$t_0<t<(n+1)^2t_0$. Once we know $f_t{}^I$ and α^I, we can find v^I through Eq. (2). We can then solve

$$\nabla^2\psi+[q^2-V^I(r,q^2)(1+(-1)^IP_X)]\psi=0 \qquad (7)$$

to obtain a unitarized amplitude at the energy corresponding to q^2. Here P_X is the space-exchange operator.

The above method obviously is useful only if a few iterations are needed, i.e., if only the low-t part of $v^I(t,q^2)$ is important. This nearest-singularity approximation is of course the usual sort of assumption one makes in strong-interaction calculations. In the present method such an assumption is not unreasonable if we assume that both $\tilde{A}_t{}^I$ and α^I are dominated at large t by a few Regge poles,[7] and if the residue and position of these poles are assumed to have only right-hand cuts in the k^2 plane.[8] This is because we can then write in this region

$$f_t{}^I(t,k^2)=\frac{1}{\pi}\int_0^\infty dk'^2\frac{\alpha^I(k'^2,t)}{k'^2-k^2} \qquad (8)$$

which, together with Eq. (2) leads to $v^I(t,q^2)=0$ for large t.[9]

To implement the above procedure we must have a way of calculating the absorptive part $\tilde{A}_t{}^I$. In I, this was done by using a simplified version of the strip approximation.[3] We first approximate $A_t{}^I$ by keeping only a few partial waves in the t channel.

$$A_t{}^I\simeq B_t{}^I(t,s)=\sum_{l=0}^{L}(2l+1)\times\frac{1+(-1)^{I+l}}{2}\,\mathrm{Im}A_t{}^I\left(\frac{t}{4}-1\right)P_l\left(1+\frac{2s}{t-4}\right). \qquad (9)$$

Since the amplitude has been assumed to satisfy a Mandelstam representation,

$$\tilde{A}_t{}^I(t,s)=\tilde{B}_t{}^I(t,s)+\frac{1}{\pi}\int_4^\infty ds'\rho^I(s',t)\times\left[\frac{1}{s'-s}-\frac{2}{t-4}\sum_{l=0}^{L}(2l+1)Q_l\left(1+\frac{2s'}{t-4}\right)P_l\left(1+\frac{2s}{t-4}\right)\right], \qquad (10)$$

where we have subtracted out the lowest order term $\tilde{B}_t{}^I=\sum_{I'}\beta_{II'}B_t{}^{I'}$. This has the effect of suppressing the contribution of the third double spectral function which would otherwise have to be included in Eq. (10). It should also suppress the high-s' part of the integral in Eq. (10), and so we need consider only elastic s-channel unitarity. This gives

$$\rho^I(s,t)=\frac{2}{\pi q\sqrt{s}}\int_{t_0}^\infty dt'\int_{t_0}^\infty dt''\times\frac{\tilde{A}_t{}^{I*}(t',s)\tilde{A}_t{}^I(t'',s)}{K^{1/2}(q^2;t,t',t'')}\theta[t-t_+(q^2)] \qquad (11)$$

for the double spectral function.

The Eqs. (10) and (11) closely resemble Eqs. (3) and (4) and can be solved in exactly the same way. In lowest order we again drop the integral term in Eq. (10). The nth iteration then gives $\tilde{A}_t{}^I$ in the region $t_0<t<(n+1)^2t_0$. It was shown in I that if we go to the same order in the iterations of both Eqs. (3) and (4), and Eqs. (10) and (11), we obtain real v^I.

In the above scheme, Eq. (10) will in general diverge so we must either put in a cutoff or subtract out additional partial waves at each iteration. Mandelstam[10] has demonstrated that a cutoff version of the strip-approximation equations leads to Regge behavior at large t. This, as we saw above, can be used to argue that $v^I(t,q^2)$ should vanish for large t.

3. DETERMINATION OF PARAMETERS DESCRIBING SHORT-RANGE EFFECTS

In a lowest order ρ bootstrap, we would keep only the ρ contribution to Eqs. (9) and (10). This gives

$$\tilde{A}_t{}^1(t,s)=3\beta_{11}\,\mathrm{Im}A_1{}^1\left(\frac{t}{4}-1\right)P_1\left(1+\frac{2s}{t-4}\right) \qquad (12)$$

in the s-channel $I=1$ state. As we saw in I, a delta-function approximation for the resonance

$$\mathrm{Im}A_1{}^1\left(\frac{t}{4}-1\right)=4\pi\Gamma_1{}^1\nu_R\delta(t-m^2) \qquad (13)$$

gives a potential

$$V^1(r,q^2)=-12\beta_{11}\Gamma_1{}^1s^{-1/2}(s+2\nu_R)r^{-1}e^{-mr}, \qquad (14)$$

where m=mass of the ρ, $\nu_R=q_R{}^2=\frac{1}{4}m^2-1$, and $(2q_R{}^2\Gamma_1{}^1/m)$ is the half-width in the q^2 variable. We see that Eq. (14) does not lead to any difficulties at $r=0$ when we solve the Schrödinger equation (7).

Suppose we now turn to πN scattering. If we only consider nucleon exchange we are led to the effective potential

$$V(r,q^2)=\binom{-1}{2}\frac{1}{2W}\left(\frac{g^2}{4\pi}\right)\frac{e^{-mr}}{r}\left\{(E+M)(W-M)+\frac{(E-M)(W+M)}{q^2}\left[\frac{m^2}{2}+q^2+\frac{1+mr}{r^2}\boldsymbol{\sigma}\cdot\mathbf{L}\right]\right\}P_X \qquad (15)$$

[7] T. Regge, Nuovo Cimento **14**, 951 (1959).
[8] V. Singh, Phys. Rev. **127**, 632 (1962).
[9] Because of oscillations, only the low-k'^2 part of the integral is likely to be important in Eq. (8).
[10] S. Mandelstam, Ann. Phys. (N. Y.) **21**, 302 (1963).

for $I=\frac{1}{2}$ and $\frac{3}{2}$, respectively (see the Appendix). Here M=mass of nucleon, $g=\pi N$ coupling constant ($g^2/4\pi=14.4$), W=total c.m. energy, E=nucleon energy, $\mathbf{L}$=orbital-angular-momentum operator, and $m^2=M^2-(2E-W)^2$. In the $I=\frac{3}{2}$, $J=\frac{3}{2}$ state, where nucleon exchange is expected to provide the dominant force,[11] Eq. (15) becomes

$$V(r,q^2)=-\frac{1}{W}\left(\frac{g^2}{4\pi}\right)\frac{e^{-mr}}{r}\left\{(E+M)(W-M)+\frac{(E-M)(W+M)}{q^2}\left[\frac{m^2}{2}+q^2+\frac{1+mr}{r^2}\right]\right\}. \quad (16)$$

This is both attractive and singular at $r=0$. It therefore requires some kind of parametrization at small r, where higher order effects are obviously important. This can be done, for instance, by introducing some kind of cutoff or hard core. The actual form is not likely to be important.[12]

In order to calculate the parameters representing short-range effects we shall use the Singh-Udgaonkar sum rule.[4] This is similar to the Cini-Fubini approximation[5] but differs from it in several details. In the $\pi\pi$ case, it gives for the total amplitude

$$A^I(s,t)=\frac{1}{\pi}\int_L ds'\frac{A_s{}^I(s',t)}{s'-s}+\frac{1}{\pi}\int_L dt'\tilde{A}_t{}^I(t',s)\left[\frac{1}{t'-t}+\frac{(-1)^I}{t-4+s+t}\right], \quad (17)$$

where the L indicates that the integrals run over the low-energy resonance regions. If we make a partial-wave expansion of the s-channel absorptive part $A_s{}^I$ and then project out the lth partial wave from Eq. (17), we obtain

$$A_l{}^I(\nu)=\frac{\nu^l}{\pi}\int_L d\nu'\frac{\mathrm{Im}A_l{}^I(\nu')}{\nu'^l(\nu'-\nu)}+(\text{Contribution of waves}>l)+\frac{1}{\pi\nu}\int_L dt'\times\tilde{A}_t{}^I(t;4\nu+4)Q_l\left(1+\frac{t'}{2\nu}\right), \quad (18)$$

where $\nu=q^2$. The contribution of higher nonresonant waves in Eq. (18) as well as some further corrections to Eq. (17) can be determined with the help of the strip approximation.[13] These effects are expected to be small, however.

Now both the equivalent-potential approach and Eq. (18) are likely to be reasonable at low energies, although they fail at higher energies. In particular, they should both give a fair approximation to the amplitude at the physical threshold. We can therefore demand that they give the same amplitude (and perhaps one or more derivatives) at that point. This requirement should enable us to determine any parameters in our calculation.

4. A CRUDE CALCULATION OF THE ρ MESON

We have seen that we do not run into any difficulties in a lowest order ρ bootstrap in $\pi\pi$ scattering. To illustrate the procedure of the preceding section, however, we shall nevertheless parametrize short-range effects in a crude way and impose the Singh-Udgaonkar normalization at threshold. This normalization is not automatically satisfied. In a simple ρ bootstrap,[14] for instance, one gets $b_1{}^1=0.130$ from Eq. (7) and $b_1{}^1=0.204$ from Eq. (18), where $b_l{}^I=[A_l{}^I(\nu)/\nu^l]_{\nu=0}$. The situation is improved a little if one does a simultaneous ρ-f^0 bootstrap. In the calculation of Ref. 14, one has $b_1{}^1=0.077$, $b_2{}^0=0.0044$ from Eq. (7) and $b_1{}^1=0.107$, $b_2{}^0=0.0045$ from Eq. (18). Nevertheless, one still has a discrepancy in the $I=1$, $l=1$ state, where the centrifugal term is not sufficiently strong to completely shield off short-range effects.

In I a calculation was made of the ρ meson using a variational formula. An effective-range approximation using the potential (14) gave

$$\mathrm{Re}[\nu/A_1{}^1(\nu)]=a_e-r_e\nu, \quad (19)$$

where

$$a_e=m^3\left[\frac{128m}{81\Gamma_1{}^1(m^2+4)}-\frac{1}{2}\right] \quad (20)$$

and

$$r_e=\frac{m}{32}(37-8m^2)-\frac{128m^2}{81\Gamma_1{}^1(m^2+4)}\left[1-\frac{8m^2}{m^2+4}\right]. \quad (21)$$

Now, in general, short-range effects are likely to be more important at higher energies. We shall therefore roughly take them into account by using

$$\Phi(\nu)=\mathrm{Re}[\nu/A_1{}^1(\nu)]=a_e-r_e\nu-c\nu^2. \quad (22)$$

Since a_e and r_e are more important at low energies, we assume that they depend primarily on long-range forces. We shall therefore use Eqs. (20) and (21) to determine them. On the other hand, c becomes felt only at higher energies and so we assume that it absorbs all the short-range effects in our simple model. We shall determine it by requiring that

$$b_1{}^1=1/a_e, \quad (23)$$

where $b_1{}^1$ will be given by Eq. (18), which, in the approximation of dropping everything except the ρ meson and using Eqs. (12) and (13), gives

$$b_1{}^1=\frac{\Gamma_1{}^1}{\nu_R}+\frac{\Gamma_1{}^1}{2}\frac{\nu_R+2}{(\nu_R+1)^2}. \quad (24)$$

[11] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

[12] L. A. P. Balázs, Phys. Rev. **124**, 602 (1961). As is well known, the one-pion exchange potential in nucleon-nucleon scattering is also attractive and singular at $r=0$.

[13] L. A. P. Balázs, Phys. Rev. **137**, B168 (1965).

[14] L. A. P. Balázs and S. M. Vaidya (unpublished).

The position of the resonance $\nu=\nu_R$ is given by the condition

$$\Phi(\nu_R)=0, \tag{25}$$

while its reduced width is

$$\Gamma_1{}^1=-1/\Phi'(\nu_R). \tag{26}$$

For a given $\nu_R=\frac{1}{4}m^2-1$, Eqs. (20), (23), and (24) can be trivially solved to find $\Gamma_1{}^1$. Now Eqs. (25) and (26), when combined with (22), can be used to eliminate c to give a single relation between a_e, r_e, ν_R, and $\Gamma_1{}^1$; using Eqs. (20) and (21) we can therefore try various values of m and check whether this relation is satisfied. Approximate consistency was obtained with $\Gamma_1{}^1=0.57$ and $m=4.0$.[15] This result does not differ too much from the ones obtained in I or Ref. 14 without the imposition of the Singh-Udgaonkar formula. This seems to suggest that short-range effects do not play a very important role in the ρ bootstrap.

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APPENDIX: NUCLEON-EXCHANGE POTENTIAL IN πN SCATTERING

The iteration procedure of I described in Sec. 2 for constructing an effective potential from $\tilde{A}_t{}^I$ is completely equivalent to requiring that the Born expansion of the potential reproduce the correct amplitude to any given order. Now in πN scattering, the physical amplitude f can be written[16]

$$f=f_1-q^{-2}\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{q}'f_2, \tag{A1}$$

where $\mathbf{q}$ and $\mathbf{q}'$ are the initial and final three-momenta in the c.m. system and

$$f_1=\frac{E+M}{2W}[A+(W-M)B], \tag{A2}$$

$$f_2=\frac{E-M}{2W}[-A+(W+M)B]. \tag{A3}$$

The functions A and B satisfy the Mandelstam representation and can therefore be written in the form

$$A=\frac{1}{\pi}\int_4^\infty dt'\frac{A_t(t',s)}{t'-t}+\frac{1}{\pi}\int_{(M+1)^2}^\infty du'\frac{A_u(u',s)}{u'-u}, \tag{A4}$$

$$B=\begin{pmatrix}1\\-2\end{pmatrix}\frac{(g^2/4\pi)}{M^2-u}+\frac{1}{\pi}\int_4^\infty dt'\frac{B_t(t',s)}{t'-t}+\frac{1}{\pi}\int_{(M+1)^2}^\infty du'\frac{B_u(u',s)}{u'-u}, \tag{A5}$$

for $I=\frac{1}{2}$ and $\frac{3}{2}$, where

$$u=(2E-W)^2+w;\ w=-(\mathbf{q}+\mathbf{q}')^2.$$

Consider now a potential

$$V=(V_0{}^D+V_1{}^D\boldsymbol{\sigma}\cdot\mathbf{L})+(V_0{}^E+V_1{}^E\boldsymbol{\sigma}\cdot\mathbf{L})P_X, \tag{A6}$$

where

$$V_0{}^i=-\frac{1}{\pi}\int_{t_i}^\infty dt'v_0{}^i(t',q^2)\frac{e^{-r\sqrt{t'}}}{r} \tag{A7}$$

and

$$V_1{}^i=-\frac{1}{\pi}\int_{t_i}^\infty dt'v_1{}^i(t',q^2)\frac{e^{-r\sqrt{t'}}}{r^3}(1+r\sqrt{t'}) \tag{A8}$$

for $i=D$, E. If we take the Born approximation of the potential (A6), we therefore have a contribution to the physical amplitude

$$f^B=g_1+\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{q}'g_2, \tag{A9}$$

where

$$g_1=\frac{1}{\pi}\int_{t_D}^\infty\frac{dt'}{t'-t}[v_0{}^D(t',q^2)-(q^2+\tfrac{1}{2}t')v_1{}^D(t',q^2)]+\frac{1}{\pi}\int_{t_E}^\infty\frac{du'}{u'-w}\times[v_0{}^E(u',q^2)-(q^2+\tfrac{1}{2}u')v_1{}^E(u',q^2)] \tag{A10}$$

and

$$g_2=\frac{1}{\pi}\int_{t_D}^\infty\frac{dt'}{t'-t}v_1{}^D(t',q^2)-\frac{1}{\pi}\int_{t_E}^\infty\frac{du'}{u'-w}v_1{}^E(u',q^2). \tag{A11}$$

We see that Eqs. (A1) and (A9) have the same structure. By comparing these expressions we can therefore determine the lowest order $v_0{}^i$ and $v_1{}^i$.

If we keep only nucleon exchange, i.e., drop all the integral terms in Eqs. (A4) and (A5), we get

$$v_0{}^D=v_1{}^D=0 \tag{A12}$$

and

$$v_1{}^E(u,q^2)=\begin{pmatrix}1\\-2\end{pmatrix}\frac{(E-M)(W+M)}{2Wq^2}\times\left(\frac{q^2}{4\pi}\right)\delta(u-m^2), \tag{A13}$$

$$v_0{}^E(u,q^2)=\left(\frac{m^2}{2}+q^2\right)v_1{}^E(u,q^2)+\begin{pmatrix}1\\-2\end{pmatrix}\times\frac{(E+M)(W-M)}{2W}\left(\frac{g^2}{4\pi}\right)\delta(u-m^2). \tag{A14}$$

If we combine these results with Eqs. (A7) and (A8), we obtain Eq. (15).

[15] The corresponding experimental values are $\Gamma_1{}^1=0.18$ and $m=5.5$ if we take width $=110$ MeV and mass $=765$ MeV. See A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **36**, 977 (1964).

[16] S. W. MacDowell, Phys. Rev. **116**, 774 (1960); W. Frazer and J. Fulco, *ibid.* **119**, 1420 (1960); S. Frautschi and D. Walecka, *ibid.* **120**, 1486 (1961).