If we write $D_{F\nu'\nu}'(q'^2)$ in the form

$$D_{F'\nu'\nu}'=D_1 g_{\nu'\nu}-D_2 q_{\nu'}'q_\nu',$$

where D_1 and D_2 are functions of q'^2 only, then Eq. (A17) implies

$$D_1 H_\nu - D_2(H\cdot q')q_\nu'=0. \tag{A18}$$

On multiplying Eq. (A17) by $(q')^2 q'^\nu$ and using Eq. (A14) we see that $H\cdot q'=0$, so that Eq. (A18) implies

$$H_\nu=0.$$

Thus, we have

$$q_\mu \Gamma^{\mu\nu}=0 \tag{A19a}$$

and by symmetry,

$$q_\nu' \Gamma^{\mu\nu}=0. \tag{A19b}$$

Equations (A16) and (A19) constitute the analogs for a neutral field of the generalized Ward identities for charged fields.

Invariance under particle-antiparticle conjugation is equivalent to the existence of a unitary operator U_c which leaves the vacuum invariant and is such that

$$U_c A_\mu(y) U_c^{-1}=-A_\mu(y),$$

and

$$U_c\phi(x)U_c^{-1}=\phi^\dagger(x).$$

It follows that

$$W_\mu(x',x;y)=-W_\mu(x,x';y),$$

but

$$W_{\mu\nu}(x',x;y,y')=W_{\mu\nu}(x,x';y,y').$$

We then have

$$\tilde{W}_\mu(p',p;q)=-\tilde{W}_\mu(-p,-p';q),$$

and

$$\tilde{W}_{\mu\nu}(p',p;q,q')=\tilde{W}_{\mu\nu}(-p,p';q,q'),$$

so that, using Eqs. (A15a) and (A15b), and recalling that $P=p+p'$,

$$\Gamma_\mu(q;P)=-\Gamma_\mu(q;-P), \tag{A20a}$$

$$\Gamma_{\mu\nu}(q,q';P)=\Gamma_{\mu\nu}(q,q';-P). \tag{A20b}$$

Finally, since $W_{\mu\nu}(x',x;y,y')=W_{\nu\mu}(x',x;y',y)$ so that $\tilde{W}_{\mu\nu}(p',p;q,q')=\tilde{W}_{\nu\mu}(p',p;q',q)$, we have also

$$\Gamma_{\mu\nu}(q,q';P)=\Gamma_{\nu\mu}(q',q;P). \tag{A20c}$$

Equations (A20a)–(A20c) are the symmetry properties of Γ_μ and $\Gamma_{\mu\nu}$ used in Sec. III of this paper.

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Lie Groups, Lie Algebras, and the Troubles of Relativistic $SU(6)$

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Some of the difficulties of relativistic $SU(6)$ are examined. Those arising from the use of continuous groups can be avoided by the use of algebras of finite sets of operators which are sufficient to give the desired properties of elementary particles. The nonconservation of probability associated with the relativistic separation of space and spin is pointed out. Quantum electrodynamics applied to atomic structure is shown to exhibit the type of peculiar symmetry which leaves the interaction invariant but is broken by free Dirac propagators. The implications of this analogy for $SU(6)$ are discussed. The mixing of physical and nonphysical states (positive- and negative-energy quark states) leads to noninvariance of the vacuum under the symmetry group, and to a degenerate vacuum in the exact symmetry limit. The existence of open inelastic channels for low-mass boson production is relevant to unitarity calculations and is implied in all energy regions where the symmetry is not badly broken.

INTRODUCTION

THE successes of the $SU(6)$-symmetry scheme for elementary particles[1] and its relativistic generalizations[2] have been accompanied by an assortment of difficulties in principle and also by some predictions in disagreement with experiment.[3] A better picture of the relation between the successes and the difficulties can be obtained by examining the general assumptions underlying the proposed theories to determine which are really necessary to obtain the desired results. Analysis of the sources of some of the troubles may help in finding ways to get around them.

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[1] F. Gürsey and L. A. Radicati, Phys. Rev. Letters **13**, 173 (1964); A. Pais, Phys. Rev. Letters **13**, 175 (1964); B. Sakita, Phys. Rev. **136**, B1756 (1964).

[2] A. Salam, R. Delbourgo, and J. Strathdee, Proc. Roy. Soc. (London) **284**, 146 (1965); M. A. B. Bég and A. Pais, Phys. Rev. **138**, B692 (1965); B. Sakita and K. C. Wali, *ibid.* **139**, B1355 (1965). A detailed list of references to earlier works on $SU(6)$ and its relativistic modifications is given by Sakita and Wali.

[3] S. Coleman, Phys. Rev. **138**, B1262 (1965); M. A. B. Bég and A. Pais, Phys. Rev. Letters **14**, 509 (1965); R. Blankenbecler, M. L. Goldberger, K. Johnson, and S. B. Treiman, *ibid.* **14**, 518 (1965).

THE POSSIBLE IRRELEVANCE OF THE CONTINUOUS GROUP

The continuous groups of transformations commonly used to introduce isospin, $SU(3)$, $SU(6)$, and $SU(12)$ symmetries are not really necessary to obtain these results and may be dispensed with if they give rise to difficulties in principle. All results usually attributed to these symmetries are obtainable from the following assumption: *There exists a finite set of operators which commute with the Lagrangian, Hamiltonian, or S matrix in some approximation, and which constitute the Lie algebra corresponding to the particular group.* The existence of the continuous group is not required. As an example, one can consider the case of isospin under the assumption that a superselection rule restricts the allowed states in Hilbert space to those which are eigenfunctions of the electric charge. The continuous isospin transformations are not defined in this case. However, the three isospin generators τ_+, τ_-, and τ_z are well-defined operators which take a state within the Hilbert space into another state which is also within the Hilbert space. The existence of these three operators and their commutation rules are sufficient to give all the results usually obtained from isospin; the continuous group is not required.[4]

The combination of Lorentz invariance and internal symmetries is quite straightforward if the latter are defined only by a finite set of operators, rather than by a continuous group. Lorentz covariance of the Lie algebra is sufficient. Note that the distinction between compact and noncompact internal symmetry groups becomes irrelevant if only commutation of the S matrix with the generators is required and not invariance under the continuous group. A Lie algebra can be transformed from that of a noncompact group to that of a compact group by multiplying generators by phase factors which are c numbers and do not affect the vanishing of the commutators with the S matrix. One can thus define multiplets of particle states from the Lie algebra without worrying about whether these are finite-dimensional nonunitary representations of a noncompact group or unitary representations of a compact group.[5] The problem of the existence of a larger continuous group containing both the Poincaré and internal-symmetry groups is avoided.[6] The Lie algebra of the larger group is defined, but only a subset of the continuous transformations generated by this algebra need have physical meaning, namely, those which produce continuous transformations in space-time and not in the space of the internal degrees of freedom.

SEPARATION OF SPACE AND SPIN

One set of troubles arising in relativistic $SU(6)$ theories stems from the separation of space-time from spin in order to define "spin independence." This requires the assumption of invariance under a group of transformations which is the direct product of two noncompact groups, one acting only on space-time and the other acting only on the internal degrees of freedom including the spinor indices.[6] Each of these groups must contain a group isomorphic to the homogeneous Lorentz group $\mathfrak{L}$ in any Lorentz-invariant theory. Thus any such theory is invariant under a subgroup which is the direct product $\mathfrak{L}\times\mathfrak{L}$ of two groups isomorphic to the Lorentz group, one acting on space-time and the other acting on the internal degrees of freedom. Inconsistencies are already inherent in this direct product. These can be seen from the well-known transformation properties of velocities and particle probability densities under ordinary Lorentz transformations. The density is not an invariant, but the relation between the transformations of densities and velocities is such that probability is conserved. The use of independent Lorentz transformations for space-time and spinor indices results in independent transformations of velocities and densities (a transformation in the space of spinor indices alone, for example, transforms the densities without changing the velocities at all). This can clearly lead to nonconservation of probability and difficulties with unitarity.

Clues to the resolution of these difficulties may be found in an existing and well understood theory which has many of the same properties, namely, quantum electrodynamics and its application to the structure of *atoms*.[7] The interaction of a charged Dirac particle with the electromagnetic field $\bar{\psi}\gamma_\mu\psi A^\mu$ is invariant under the group $\mathfrak{L}\times\mathfrak{L}$ defined above. This is true for any Lorentz-

[4] H. J. Lipkin, *Lie Groups for Pedestrians* (North-Holland Publishing Company, Amsterdam, 1965), shows in detail how all these results are obtainable from the Lie algebra without the use of the continuous group. Note also that the use of isospin and Wigner supermultiplets in nuclear physics does not require the existence of the continuous transformations or give them any physical meaning. Rather the relation between representations of continuous unitary groups and finite permutation groups makes the Lie algebra convenient for describing systems of nucleons with charge- (and spin)-independent interactions under the additional requirement that neutrons and protons must separately satisfy the Pauli principle.

[5] Whether finite- or infinite-dimensional representations arise in a particular case depends upon the physical interpretation of the operators which constitute the Lie algebra. Hermitian operators have real eigenvalues and their squares are positive definite. Thus if an invariant of the Lie algebra, such as the quadratic Casimir operator, is expressed as a function of Hermitian operators, one can examine the restrictions which Hermiticity and positive definiteness impose on the allowable eigenvalues of operators of the Lie algebra. Examples where such considerations distinguish between finite- and infinite-dimensional multiplets are given in Ref. 4. This question is discussed in more detail in the Appendix using a specific example.

[6] L. Michel and B. Sakita, Ann. Inst. Henri Poincaré (to be published).

[7] H. J. Lipkin, International Center for Theoretical Physics, Trieste, Report No. ICTP 65/52 (unpublished). Note that the spectrum of the hydrogen atom for example can be obtained by treating two Dirac particles coupled to the electromagnetic field with an interaction invariant under $\mathfrak{L}\times\mathfrak{L}$. (This neglects the anomalous magnetic moment of the proton on the hyperfine splitting, but these effects are irrelevant to the present argument.) The "spin-independent" symmetry is indeed present to a very good approximation.

invariant interaction which has no explicit spatial dependence (i.e., no explicit derivative couplings). Many of the low-lying bound states of this theory exhibit this symmetry to a very good approximation. It appears as a decoupling of the spin and orbital angular momenta commonly known as Russell-Saunders or *LS* coupling. The splitting within the multiplets is small and is known as fine and hyperfine structure. The formulas for these splittings are the ancestors of the Gell-Mann–Okubo mass formula.[8] The symmetry is also exhibited in scattering processes treated in the Born approximation. A particularly relevant example is the absence of polarization in the scattering of relativistic electrons in a Born approximation.

Although many difficulties have worried the investigators of quantum electrodynamics, the difficulties associated with $\mathfrak{L}\times\mathfrak{L}$ were never noticed and cause no trouble. This is one of the advantages of having a complete theory from which explicit dynamical calculations can be made, rather than trying to draw general conclusions from approximate symmetry properties. However, the $\mathfrak{L}\times\mathfrak{L}$ invariance appears in a very similar way in quantum electrodynamics and elementary particles and the comparison is instructive. Examination of the former case from the point of view of this invariance reveals the following properties:

1. $\mathfrak{L}\times\mathfrak{L}$ is only an approximate symmetry. It is broken by the spin-orbit coupling in the propagators of the Dirac particles. Thus agreement with predictions from the symmetry is found in certain areas where the approximation is good, and not in others where the approximation is bad.

2. The symmetry holds very well in two areas: (1) properties of bound states in which the motion of the constituent particles is nonrelativistic to a good approximation; (2) scattering processes treated in Born approximation.

The assumption that the relativistic $SU(6)$ theory of elementary particles also has the above properties avoids the difficulties of the noncompact groups, and also the disagreements with experiment in predicting no polarization for certain scattering processes. One can question, however, whether this assumption does not also throw out the cases where agreement is found. A crucial point is the nonrelativistic nature of the bound states. Since quark masses should be high to explain the failure to observe them, the binding energy of mesons and baryons in a quark model must be of the same order as the masses of the constituent quarks; this implies that the bound state is highly relativistic. However, there are two independent parameters which characterize the degree to which a particular bound state must be considered as relativistic. One is the ratio of the binding energy to the total rest mass of the constituents; the other is the velocity of the particles. It is the velocity which is relevant to the neglect of the "symmetry-breaking" effects of Dirac propagators. This velocity can be small in a quark model for mesons and baryons, provided that the size of the bound state (i.e., the range of the forces) is large in comparison with the Compton wavelength of the quark.[9] For this case, the spin-orbit coupling can be considered to be small.

If a "low-velocity–tight-binding" quark model is assumed to underlie the relativistic $SU(6)$ theories, an important difference should be noted between such a model and the "low-velocity–weak-binding" which characterizes atoms and quantum electrodynamics. Although the tight binding does not affect the use of the symmetry in treating the low-velocity bound state, it is important in dynamical calculations of all processes in which virtual free quark-antiquark pairs can appear in intermediate states. The high quark mass drastically reduces the contribution of such diagrams.

Consider, for example, Compton scattering in quantum electrodynamics and its counterpart meson-baryon scattering for elementary particles. In Compton scattering calculated by second-order perturbation theory, summation over *all* electron intermediate states *includes those of negative energy.* These represent the contribution of virtual pair production. In meson-baryon scattering calculated to second order with a Yukawa-type three-point vertex, summation over *all* three-quark intermediate states includes, in addition to the bound baryon states, also states in which one or more of the quarks is in a state of negative energy. This can be interpreted as including contributions from virtual quark-antiquark pairs. Such contributions are automatically included in some calculations which do not form the sum explicitly but obtain the result by using closure relations or symmetry properties. These calculations can be expected to give erroneous results if the formalism used is one of "low-velocity–weak-binding" and includes exaggerated contributions from the quark-antiquark pair states. Examples of such calculations are meson-baryon scattering in the $SU(12)$ theory, which use only symmetry properties and no dynamical model.[10] If the dominant contribution to these processes involves an intermediate baryon state, the assumption of complete $SU(12)$ symmetry includes contributions from states which do not satisfy the Bargmann-Wigner equations. These can be interpreted as three-quark states in which one or more quarks have negative energy. If these contributions are to be considered as those involving quark-antiquark pair production, by analogy with Compton scattering, then their magnitudes are highly erroneous, since the quark mass does not appear explicitly in the theory and is effectively assumed to be of the same order as the baryon mass in the propagators for these states. One should not be surprised if predictions from such calculations are not in agreement with

[8] M. Gell-Mann and Y. Ne'eman, *The Eightfold Way* (W. A. Benjamin and Company, Inc., New York, 1964).

[9] The author is indebted to Y. Nambu for first pointing out the possibility of a tightly bound system which is nonrelativistic.

[10] J. M. Cornwall, P. G. O. Freund, and K. T. Mahanthappa, Phys. Rev. Letters, **14**, 515 (1965).

experiment.[10] A similar argument holds if the contribution is due to one-meson exchange.

MIXING OF PHYSICAL AND NONPHYSICAL STATES

Another problem which arises in the $U(12)$ descriptions arises from the redundant description of states and the use of transformations which mix physical and nonphysical spinors. A 12-component spinor describes a quark which has six possible states, a 143-component object describes 36 meson states, a 364-component object describes 56 baryon states. For each value of the momentum, there are 6 quark spinors, 36 meson spinors, and 56 baryon spinors which describe physical particle states; while there are 6 quark spinors, 107 meson spinors, and 308 baryon spinors which are orthogonal to the physical spinors and which do not describe physical particle states. In a quark model the latter are states containing negative-energy quarks. These nonphysical states are removed by some prescription, such as the requirement that physical states satisfy the Bargmann-Wigner equation. However, the $U(12)$-symmetry operations transform spinors representing physical states into those representing nonphysical states.

The nonphysical transformations cannot be avoided here as was done for isospin by replacing the continuous group by a judiciously chosen finite set of operators. For particles at rest the nonrelativistic $SU(6)$ generators can be defined to have nonvanishing matrix elements only between physical states. For particles moving in a single direction, the W spin and $SU(6)_W$ generators can be defined.[11] However, the generators of the full $SU(12)$ group include operators having nonvanishing matrix elements between physical and nonphysical spinors. An S matrix which is invariant under $SU(12)$ can have nonvanishing matrix elements connecting physical and nonphysical spinors. The restriction to physical states of all intermediate states arising in unitarity relations causes difficultities.[3]

One very simple and peculiar manifestation of these difficulties is the behavior of the vacuum under $SU(12)$. *The vacuum is not invariant under* $SU(12)$. The vacuum state satisfies the relations

$$b_{k-}^{*}|0\rangle=0\,, \qquad (1)$$

where b_{k-}^{*} is a creation operator for a quark in a negative-energy state. These relations (1) are not invariant under those $SU(12)$ transformations which mix positive and negative energy states. Thus if G is an operator which generates such a transformation, then $G|0\rangle$ does not vanish and does not satisfy the relations (1). This gives rise to the following consequences:

1. If a limit of exact $SU(12)$ symmetry exists (i.e., the symmetry-breaking can be neglected in a consistent way), the vacuum is infinitely degenerate in this limit. Successive operation on the vacuum with operators which are generators of $SU(12)$ give other states different from the vacuum. Thus in this limit the theory probably contains zero-mass bosons.[12] Inelastic boson production processes are therefore always present, and *must be included* in any calculations of unitarity. Note that in the real world, bosons of essentially zero mass are indeed present, namely pions. The pion mass is small compared to energies characteristic of symmetry breaking (i.e., the φ-π mass difference). Thus, scattering amplitudes can be expected to satisfy relations obtained from the symmetry to a good approximation only at energies where inelastic pion production channels are open. Thus inelastic channels must always be considered in any discussions of unitarity.

2. The physical particle states are *not members of simple $SU(12)$ multiplets even in the limit of exact $SU(12)$ symmetry*. The field operators which create the particles have simple $SU(12)$ transformation properties, but neither the vacuum nor the particle states are simple. Thus the standard type of Clebsch-Gordan analysis must be used with care to verify that the results only depend upon the assumption of simple transformation properties for the particle-creation operators and not for the states themselves. The situation is very similar to that arising in the many-body problem, where simple elementary excitations are treated although neither the ground state nor the excited states is simple.[13]

CONCLUSIONS

The present indications are that $SU(6)$ predictions agree with experiment for static properties and for "one-dimensional" relativistic properties which are natural generalizations of nonrelativistic $SU(6)$ by the use of W spin. The latter include two-body decays and forward scattering processes.[14] In other processes, disagreements are found and should be expected. Detailed studies and comparisons with experimental data should reveal where symmetry-breaking is important and perhaps indicate how it should be taken into account.

The introduction of symmetry-breaking into calculations of processes should be different from the case of $SU(3)$. There is no evidence for a particular symmetry-breaking interaction having definite $SU(6)$ or $SU(12)$ properties which should be added as a spurion in first order, by analogy with the octet symmetry breaking of $SU(3)$. The mass splittings in the $SU(6)$ multiplets are not in agreement with the assumption of a first-order symmetry-breaking interaction transforming like a member of a single representation[15] of

[11] H. J. Lipkin and S. Meshkov, Phys. Rev. Letters **14**, 670 (1965).

[12] J. Goldstone, Nuovo Cimento **19**, 154 (1961).

[13] D. Pines, *Elementary Excitations in Solids* (W. A. Benjamin and Company, Inc., New York, 1963).

[14] K. Johnson and S. B. Treiman, Phys. Rev. Letters **14**, 189 (1965).

[15] M. A. B. Bég and V. Singh, Phys. Rev. Letters **13**, 418 (1964); H. Harari and H. J. Lipkin, *ibid*. **14**, 570 (1965).

$SU(6)$. The kinetic energy spurion which is a member of a 35 in $SU(6)$ and a 143 in $SU(12)$ does not explain *in any order* important symmetry-breaking properties like the ρ-π mass difference. A better approach to symmetry breaking might be to assume a specific dynamical model and to use the correct propagators for the intermediate states, while retaining the symmetry in the vertex functions. An important and easily calculated effect would be the removal of contributions from nonphysical intermediate states which do not satisfy the Bargmann-Wigner equations. Such a prescription is not equivalent to the introduction of a spurion with specific transformation properties.

General arguments regarding the unitarity of the S-matrix must take into account the possibility of infrared divergences in inelastic channels in the limit of exact symmetry. Such effects of zero-mass bosons are probably irrelevant to the practical use of the symmetry. The presence of obvious symmetry-breaking effects such as those discussed above must also be interpreted before drawing conclusions about apparent violations of unitarity.

APPENDIX: AN EXAMPLE OF AN $\mathcal{L}\times\mathcal{L}$ THEORY

Consider a quantum-mechanical system having a Hamiltonian H and states ψ satisfying the time-dependent Schrödinger equation,

$$(H-i\partial/\partial t)\psi=0. \tag{A1}$$

Let G be any generator of the homogeneous proper Lorentz group. Then if the dynamics of the system are Lorentz invariant, a state obtained from ψ by a Lorentz transformation is also a solution of the equation (A1).

$$(H-i\partial/\partial t)e^{i\alpha G}\psi=0, \tag{A2}$$

where α is any real c number.

Let us now write

$$G=G_x+G_s, \tag{A3}$$

where G_x acts only upon the space-time variables and G_s acts only upon spinor indices.

The assumption of Lorentz invariance [Eqs. (A1) and (A2)] is equivalent to the assumption of the vanishing of the commutator

$$[(H-i\partial/\partial t),G]=[(H-i\partial/\partial t),(G_x+G_s)]=0. \tag{A4}$$

We now make the further assumption that the commutators in Eq. (A4) vanish individually for the space-time and spinor parts,

$$[(H-i\partial/\partial t),G_x]=[(H-i\partial/\partial t),G_s]=0. \tag{A5}$$

We shall see that the assumption (A5) is sufficient to give the desired results of $\mathcal{L}\times\mathcal{L}$ invariance, but does not lead to the infinite multiplets commonly associated with the direct product of two noncompact groups.

From the relation (A5) we can construct multiplets of states which are degenerate eigenfunctions of the Hamiltonian. Since G_s does not act on space-time, it commutes with the time derivative. Thus

$$[H,G_s]=0. \tag{A6}$$

Let ψ be a stationary state of the Hamiltonian, with the eigenvalue E. Then, it follows from the vanishing of the commutator (A6) that successive operation on ψ with the various generators G_s generates a set of degenerate eigenfunctions of H. Let us now investigate the structure of these multiplets generated by operating with the operators G_s which constitute a Lie algebra isomorphic to that of the noncompact homogeneous proper Lorentz group.

We shall see that the multiplets generated by the operators G_s are *finite*, and that they are *not* the infinite-dimensional unitary representations of the Lorentz group. Their character is determined by the *positive definiteness of the quadratic Casimir operator of the Lie algebra*, as defined by the action of these operators within the Hilbert space. The Casimir operator is a quadratic form in the six generators of the Lorentz group. If we consider the six generators G_s acting only on the spinor indices, we see that the three which generate spatial rotations are *Hermitian*, while the three which generate Lorentz transformations are *anti-Hermitian*. Since the square of a Hermitian operator is positive definite, while that of an anti-Hermitian operator is negative definite, the Casimir operator turns out to be positive definite. Thus, for any set of states of the physical system corresponding to a given irreducible multiplet of this Lie algebra, the square of the eigenvalue of any generator must be smaller in absolute magnitude than the eigenvalue of the quadratic Casimir operator. The eigenvalues of the generators are therefore bounded within a given representation and only the finite representations occur.

The algebra of the operators G_s is better described by a redefinition of the operators of the Lie algebra, choosing the phases *to make all operators Hermitian*. One then obtains the Lie algebra of the real Euclidean four-dimensional rotation group and everything follows simply.

We now see that the two vanishing commutators of Eq. (A5) and the designation $\mathcal{L}\times\mathcal{L}$ indicate a formal similarity which is misleading. The multiplet structures obtained from these two apparently isomorphic algebras are *different*. One is compact, the other is noncompact. The physical reasons determining the compactness are completely different in the two cases. The operators G_x generate transformations in space-time. The phases of the generators are determined by the requirement that they transform real space-time into real space-time; i.e., x and t are real variables in a Minkowski space. The operators G_s generate transformations in a Hilbert space. The only guide to the choice of phases for the definition of the algebra and the correct multiplet structure is the Hermiticity of the operators.