compare this result with the results of the three-body K_{e3} decays,[24] which proceed through the vector interaction, because of the final-state interaction in K_{e4} decays. If we assume Eq. (1) to be dominated by the axial-vector current, as predicted by most theorists, and if there were no final-state interactions, then the violation parameter

$$X=[A(\Delta S/\Delta Q=-1)]/[A(\Delta S/\Delta Q=+1)]$$

corresponding to the parameter $a_0=0$ would be $X<0.25$. On the other hand, if there is no $T=2$ final-state interaction but the $T=0$ s-wave enhancement factor were as large as 4, then $X<0.5$.

[24] R. P. Ely, W. M. Powell, H. White, M. Baldo-Ceolin, E. Calimani, S. Ciampolillo, O. Fabbri, F. Farini, C. Filippi, H. Huzita, G. Miari, U. Camerini, W. F. Fry, and S. Natali, Phys. Rev. Letters **8**, 132 (1962); G. Alexander, S. P. Almeida, and F. S. Crawford, Jr., *ibid.* **9**, 69 (1962); B. Aubert, L. Behr, J. P. Lowys, P. Mittner, and C. Pascaud, Phys. Letters **10**, 215 (1964); M. Baldo-Ceolin, E. Calimani, S. Ciampolillo, C. Filippi, H. Huzita, F. Mattioli, and G. Miari, *Proceedings of the Sienna International Conference on Elementary Particles* (Societa Italionadi Fisica, Bologna, Italy, 1963); L. Kirsch, R. J. Plano, J. Steinberger, and P. Franzini, Phys. Rev. Letters **13**, 35 (1964).

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Spin Tests for Bosons

M. Ademollo, R. Gatto, and G. Preparata
Istituto di Fisica dell'Università, Firenze, Italy
and
Sezione di Firenze dell'I.N.F.N., Firenze, Italy
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Tests of spin for an unstable boson which decays into three spinless particles, or into a spin-1 and a spin-0 particle, or into two spin-$\frac{1}{2}$ particles, are presented. The proposed spin tests should be useful for the spin-parity determination of the new boson resonances. Spin tests linear in the experimental test functions are discussed in particular, in view of their general applicability independently of the production kinematics. Explicit expressions of the test functions are reported for the lower spin assignments.

I. INTRODUCTION

The recent discoveries of many resonant states which decay strongly or electromagnetically into baryons and bosons have stimulated the search for convenient spin-parity tests, which may allow for a determination of the spin and parity of the unstable particle, possibly avoiding dynamical hypotheses on the mechanisms of production and decay. Particularly useful have been the tests based on simultaneous analysis of angular and polarization distributions.[1]

In this paper we consider some possible methods for determining the spin of an unstable boson. We discuss its modes of decay, into three spinless particles, into a spin-1 and spin-0 particle, and into two spin-$\frac{1}{2}$ particles. In each case we look for relations among the coefficients of the final distributions which do not depend on the elements of the density matrix of the decaying boson. We obtain a general method for spin determination which appears to be more powerful than methods based on the reconstruction of the density matrix. The relations to be tested are in fact linear in the experimental averages and independent of the production process, making it possible to average on all the events, independent of the production kinematics. Such a possibility is especially useful when the number of events is relatively small.

In Sec. II we discuss the decay of a boson into three spinless particles. The spin tests we derive for this case could be of use for the spin-parity assignments to recently found three-body resonances such as 3π, $\eta\pi\pi$, $K\pi\pi$, etc. The final distributions are first expressed in terms of a suitable set of parameters which are subject to a number of constraints. Symmetry principles or possible identity between two of the final particles produce further relations. The different spin tests are discussed in Sec. 2.3 and are written down explicitly for spin one. Their explicit forms for spin two and three are reported in Appendix A. In Sec. III we discuss the mode of decay into a spin-1 and a spin-0 boson in view of applications to spin-parity tests of the $\pi\omega$ and $\pi\rho$

[1] R. Gatto and H. P. Stapp, Phys. Rev. **121**, 1553 (1961); N. Byers and S. Fenster, Phys. Rev. Letters **11**, 52 (1963); M. Ademollo and R. Gatto, Phys. Rev. **133**, B531 (1964); N. Byers and C. N. Yang, *ibid.* **135**, B796 (1964); S. M. Berman and R. J. Oakes, *ibid.* **135**, B1034 (1964).

resonances. The spin tests are elaborated in terms of directly measurable quantities which must be expressible in terms of geometrical parameters. In Sec. IV a very short account of possible tests applicable to decay into two fermions is given.

II. DECAY INTO THREE SPINLESS BOSONS

1. Angular Distributions

We consider here the decay of a boson B of spin j into three spinless bosons. Our results may be useful for the spin assignment to recently discovered boson resonances which decay into 3π, $\eta\pi\pi$, $K\pi\pi$, etc. We shall not discuss the properties of the Dalitz plots which have been widely discussed in the literature.[2] The spin tests that we shall present are based only on the angular-distribution analysis, and are, in a certain sense, complementary to the Dalitz-plot method.

We briefly describe the basic idea of the method. The final state depends in general on $2j+1$ dynamical coefficients which depend on the energies of the final bosons. When we integrate over the energies, all interference terms appear, giving a total of $(2j+1)^2$ parameters that we can arrange in a matrix of dimension $2j+1$. This matrix has the properties of a density matrix. The number of independent parameters is greatly reduced by the possible symmetries (such as parity conservation in the B decay, or symmetry under the exchange of two final bosons) and in many cases the parameters can be completely determined.

The final state of the three bosons in the center-of-mass system can be labeled by the following parameters: the direction of the normal $\mathbf{n}$ to the decay plane; the azimuth φ of one of the final momenta, say $\mathbf{p}_1$, around $\mathbf{n}$; and the energies ω_1 and ω_2 of two final bosons. Another equivalent set of parameters is: the unit vector $\mathbf{u}_1$ along $\mathbf{p}_1$, the azimuth ψ of $\mathbf{p}_2$ around $\mathbf{u}_1$, and the energies ω_1 and ω_2. According to the two choices we shall indicate the final-state vector by $|\mathbf{n}\varphi\omega_1\omega_2\rangle$ or by $|\mathbf{u}_1\psi\omega_1\omega_2\rangle$. We denote by P the complete density matrix of B and by M the transition matrix. The final distribution, for the first choice of parameters, is

$$I(\mathbf{n}\varphi\omega_1\omega_2)=\langle\mathbf{n}\varphi\omega_1\omega_2|MPM^\dagger|\mathbf{n}\varphi\omega_1\omega_2\rangle \qquad (2.1)$$

and is normalized such that

$$\int I(\mathbf{n}\varphi\omega_1\omega_2)d\varphi d\omega_1 d\omega_2 d\Omega_{\mathbf{n}}=1\,, \qquad (2.2)$$

the corresponding normalization factor being included in M. To evaluate the matrix element (2.1), it is convenient to make a rotation R of the frame of reference such that the normal $\mathbf{n}$ is directed along the z axis of the new frame. If $D(R)$ is the corresponding operator, we have

$$I(\mathbf{n}\varphi\omega_1\omega_2) =\langle\mathbf{n}=\mathbf{k}\;\varphi\omega_1\omega_2|D(R)MPM^\dagger D^{-1}(R)|\mathbf{n}=\mathbf{k}\;\varphi\omega_1\omega_2\rangle\,, \qquad (2.3)$$

where $\mathbf{k}$ is the unit vector along the polar axis, $D(R)=D(0,\xi,\eta)$ follows the convention of Edmonds,[3] and ξ, η are the polar coordinates of $\mathbf{n}$ in the original frame. By using the rotational invariance of M and inserting in (2.3) complete sets of angular momentum states of B, we obtain

$$\begin{aligned}I(\mathbf{n}\varphi\omega_1\omega_2)&=\sum_{\mu\mu'}\langle\mathbf{n}=\mathbf{k}\;\varphi\omega_1\omega_2|M|j\mu\rangle\langle j\mu|DPD^{-1}|j\mu'\rangle\\&\qquad\times\langle j\mu'|M^\dagger|\mathbf{n}=\mathbf{k}\;\varphi\omega_1\omega_2\rangle\\&=\sum_{\mu\mu'}f_\mu(\varphi\omega_1\omega_2)f_{\mu'}^*(\varphi\omega_1\omega_2)\\&\qquad\times\langle j\mu|DPD^{-1}|j\mu'\rangle,\end{aligned} \qquad (2.4)$$

where we have called

$$f_\mu(\varphi\omega_1\omega_2)=\langle\mathbf{n}=\mathbf{k}\;\varphi\omega_1\omega_2|M|j\mu\rangle. \qquad (2.5)$$

By performing a rotation of λ around z and using again the invariance of M, we easily obtain

$$f_\mu(\varphi+\lambda,\,\omega_1\omega_2)=e^{i\mu\lambda}f_\mu(\varphi,\omega_1\omega_2) \qquad (2.6)$$

and therefore

$$f_\mu(\varphi\omega_1\omega_2)=e^{i\mu\varphi}f_\mu(0\omega_1\omega_2)=e^{i\mu\varphi}f_\mu(\omega_1\omega_2). \qquad (2.6')$$

Using the properties of the rotation matrices, we have

$$\begin{aligned}\langle j\mu|DPD^{-1}|j\mu'\rangle&=\sum_{\nu\nu'}\mathfrak{D}_{\mu\nu}{}^{(j)}\rho_{\nu\nu'}\mathfrak{D}_{\nu'\mu'}{}^{(j)-1}\\&=\sum_{\nu\nu'}(-1)^{\mu'-\nu'}\rho_{\nu\nu'}\mathfrak{D}_{\mu\nu}{}^{(j)}\mathfrak{D}_{-\mu'-\nu'}{}^{(j)}\\&=\sum_{\nu\nu'}(-1)^{\mu'-\nu'}\rho_{\nu\nu'}\sum_{LMM'}(j\mu,j-\mu'|LM')\\&\qquad\times(j\nu,j-\nu'|LM)\mathfrak{D}_{M'M}{}^{(L)},\end{aligned} \qquad (2.7)$$

where we have denoted by ρ the spin-density matrix of B, $(j\mu,j-\mu'|LM')$ are the standard Clebsch-Gordan-Wigner coefficients and $\mathfrak{D}_{M'M}{}^{(L)}$ stands for $\mathfrak{D}_{M'M}{}^{(L)}(0,\xi,\eta)$. By virtue of (2.6′) and (2.7), the final distribution (2.4) can be written as

$$I(\mathbf{n}\varphi\omega_1\omega_2)=[(2j+1)/(8\pi^2)]\times\sum_{LMM'}\rho(L,M)F^*(L,M',\omega_1\omega_2)\mathfrak{D}_{M'M}{}^{(L)}(\varphi,\xi,\eta), \qquad (2.8)$$

where

$$\rho(L,M)=(\hat{L}/\hat{j})\sum_{\nu\nu'}\rho_{\nu\nu'}(j\nu',LM|j\nu)\,,\qquad[\hat{L}=(2L+1)^{1/2}] \qquad (2.9)$$

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[2] An extensive analysis of the 3π-decay mode can be found in the paper by C. Zemach, Phys. Rev. **133**, B1208 (1964). Many of the results of this paper can be extended to other three-body decays.

[3] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), Chap. 4.

$$F(L,M',\omega_1\omega_2)=(\hat{L}/\hat{j})\sum_{\mu\mu'}F_{\mu\mu'}(\omega_1\omega_2)\times(j\mu',LM'|j\mu)\,, \quad (2.10)$$

$$F^*(L,M',\omega_1\omega_2)=(-1)^{M'}F(L,-M',\omega_1\omega_2)\,, \quad (2.10')$$

$$F_{\mu\mu'}(\omega_1\omega_2)=f_\mu^*(\omega_1\omega_2)f_{\mu'}(\omega_1\omega_2)\,. \quad (2.11)$$

From the normalization condition (2.2), and since ρ is normalized to unit trace, we obtain

$$\sum_\mu\int F_{\mu\mu}(\omega_1\omega_2)d\omega_1d\omega_2=1\,. \quad (2.12)$$

From (2.8) we obtain in particular the distribution of the normal $\mathbf{n}$ by integrating over the other parameters:

$$I(\mathbf{n})=\sum_{LM}a(L,M)Y_L{}^M(\mathbf{n})\,, \quad (2.13)$$

$$a(L,M)=A(L)\rho(L,M)\,, \quad (2.14)$$

$$A(L)=[(2j+1)/4\pi]^{1/2}\sum_\mu F_{\mu\mu}(j\mu,L0|j\mu)\,, \quad (2.15)$$

$$F_{\mu\mu'}=\int F_{\mu\mu'}(\omega_1\omega_2)d\omega_1d\omega_2\,. \quad (2.16)$$

The energy distribution is

$$I(\omega_1\omega_2)=\sum_\mu F_{\mu\mu}(\omega_1\omega_2)\,. \quad (2.17)$$

The calculation of the final distribution $I(\mathbf{u}_1\psi\omega_1\omega_2)$ can be carried out exactly along the same lines. In the above equations we must only make the substitutions

$$\mathbf{n}(\xi,\eta)\to\mathbf{u}_1(\alpha,\beta)\,;\quad \varphi\to\psi\,;\quad D(0,\xi,\eta)\to D(0,\alpha,\beta)\,;$$
$$f_\mu(\omega_1\omega_2)\to g_\mu(\omega_1\omega_2)\,;\quad F_{\mu\nu}\to G_{\mu\nu}\,,$$

where

$$g_\mu(\omega_1\omega_2)=\langle\mathbf{u}_1=\mathbf{k},\psi=0,\omega_1\omega_2|M|j\mu\rangle\,. \quad (2.18)$$

For example the distribution of $\mathbf{u}_1$ will be

$$I(\mathbf{u}_1)=\sum_{LM}b(L,M)Y_L{}^M(\mathbf{u}_1)\,, \quad (2.19)$$

$$b(L,M)=B(L)\rho(L,M)\,, \quad (2.20)$$

$$B(L)=\left(\frac{2j+1}{4\pi}\right)^{1/2}\sum_\mu G_{\mu\mu}(j\mu,L0|j\mu)\,. \quad (2.21)$$

In the angular distributions (2.13) and (2.19), the coefficients $a(L,M)$ and $b(L,M)$ are products of a factor $\rho(L,M)$ depending only on the production process, and of a factor depending only on the decay. The matrix elements $F_{\mu\nu}$ or equivalently $G_{\mu\nu}$ play the role of decay parameters and are unknown, apart from relations among them, unless the decay process of B is specified.

The matrix F is a Hermitian non-negative definite matrix and its elements satisfy the conditions

$$0\le F_{\mu\mu}\le 1\,,\quad \mathrm{Tr}F=1\,, \quad (2.22)$$

and the Schwartz inequalities

$$|F_{\mu\nu}|^2\le F_{\mu\mu}F_{\nu\nu} \quad (2.22')$$

as it follows from (2.11) and (2.16). The same holds for G. Furthermore F and G are related by a similarity transformation. In fact by comparing (2.5) with $\varphi=0$ and (2.18), we see that the final-state vector of (2.18) is obtained from that of (2.5) by a rotation of the reference frame of $R\equiv(\pi,\pi/2,\pi/2)$. Therefore,

$$g_\mu(\omega_1\omega_2)=\sum_{\mu'}f_{\mu'}(\omega_1\omega_2)\mathfrak{D}_{\mu'\mu}{}^{(j)}(R)$$

or

$$G=D^{-1}FD\,, \quad (2.23)$$

where D stands for $\mathfrak{D}^{(j)}(\pi,\pi/2,\pi/2)$.

2. Symmetry Properties

The final angular distributions are thus dependent on the matrix elements of F. Taking into account the Hermiticity of F and the condition (2.22), the matrix elements depend in general on $4j(j+1)$ real independent parameters. However, this number is reduced in many cases because of symmetry properties.

Let us suppose, for instance, that parity is conserved in the decay of B. Then from (2.5), calling ϵ the product of the intrinsic parities of the initial and final particles, we have

$$f_\mu(\varphi,\omega_1\omega_2)=\epsilon f_\mu(\varphi+\pi,\omega_1\omega_2) \quad (2.24)$$

and, by virtue of Eq. (2.6), we have the selection rule

$$(-1)^\mu=\epsilon\,.$$

For the matrix elements $F_{\mu\nu}$ we thus have

$$F_{\mu\nu}=0\,,\quad \text{unless } (-1)^\mu=(-1)^\nu=\epsilon\,. \quad (2.25)$$

The corresponding property for G is

$$G_{\mu\nu}=\epsilon(-1)^{j-\mu}G_{-\mu\nu}=(-1)^{\mu+\nu}G_{-\mu-\nu}\,. \quad (2.26)$$

In this case we see that F depends on $j(j+2)$ real independent parameters for $\epsilon=(-1)^j$ and on j^2-1 parameters for $\epsilon=(-1)^{j+1}$.

Furthermore, in many cases, at least two of the final particles are identical or are simply related by charge conjugation or isospin rotations. We can then exchange the two particles in the amplitudes (2.5). We can see this in two examples. The resonance $K\bar{K}\pi$ of 1410 MeV[4] with $Q=S=0$ is supposed to decay strongly satisfying charge independence. In each of the two observed decay modes, $K^0K^-\pi^+$ and $K^+\bar{K}^0\pi^-$, the $K\bar{K}$ system is in a pure triplet state of isotopic spin and therefore the decay amplitude is even under the exchange of K and $\bar{K}$. We also remark that since the resonance is neutral, the final states of the two decay modes are related by charge conjugation in a definite way and therefore, the

[4] R. Armenteros, *et al.*, *Proceedings of the Sienna International Conference on Elementary Particles* (Società Italiana di Fisica, Bologna, Italy, 1963), Vol. 1, p. 287.

amplitudes are equal, apart from the sign. Next, let us consider the $\pi^+\pi^-\eta$ resonance of 960 MeV[5] called X^0. Charge conjugation exchanges the two pions and the decay amplitude takes a factor of ± 1 under this exchange depending on the X^0 charge-conjugation number.

Now, if the final distribution is symmetrical under the exchange of two final particles, say 1 and 2, we must have

$$I(\mathbf{n}\varphi\omega_1\omega_2)=I(-\mathbf{n},-\varphi-\theta,\omega_2\omega_1), \quad (2.27)$$

where $\theta=\theta(\omega_1\omega_2)$ is the angle between the two particles and ω_1,ω_2 are their energies. From (2.8), (2.10) we then obtain the conditions

$$F_{\mu\mu'}(\omega_1\omega_2)=e^{-i(\mu-\mu')\theta}F_{-\mu-\mu'}(\omega_2,\omega_1). \quad (2.28)$$

The same conditions could also have been obtained directly from (2.5) and (2.11). However, when Eq. (2.28) is integrated over the energies any symmetry property of the nondiagonal matrix elements of F disappears, and we get only

$$F_{\mu\mu}=F_{-\mu-\mu}, \quad (2.29)$$

which in turn implies $A(L)=0$ for odd L in Eq. (2.15), as necessary for $I(\mathbf{n})=I(-\mathbf{n})$.

To exhibit simpler symmetries it is convenient to consider the distribution of the bisector of the momenta of the particles 1 and 2, instead of the two momenta separately. Referring to Eq. (2.8) we label the final state by the azimuth of the bisector, $\chi=\varphi+\frac{1}{2}\theta$, and we have

$$I(\mathbf{n}\chi\omega_1\omega_2)=[(2j+1)/8\pi^2] \times \sum_{LMM'} \rho(L,M)\Phi^*(L,M',\omega_1\omega_2)\mathfrak{D}_{M'M}{}^{(L)}(\chi,\xi,\eta), \quad (2.30)$$

where

$$\Phi(L,M',\omega_1\omega_2)=(\hat{L}/\hat{j})\sum_{\mu\mu'}\Phi_{\mu\mu'}(\omega_1\omega_2)(j\mu',LM'|j\mu), \quad (2.31)$$

$$\Phi_{\mu\mu'}(\omega_1\omega_2)=e^{i(\mu-\mu')\theta/2}F_{\mu\mu'}(\omega_1\omega_2). \quad (2.32)$$

From (2.28) we have for the new matrix Φ:

$$\Phi_{\mu\mu'}=\Phi_{-\mu-\mu'} \quad (2.33)$$

to which we must add the analog of (2.25) when parity is conserved. These relations greatly reduce the number of decay parameters. The number of real independent parameters is now $\frac{1}{4}[2(j+1)^2+(-1)^j-3]$ for $\epsilon=(-1)^j$ and $\frac{1}{4}[2j^2-(-1)^j-3]$ for $\epsilon=(-1)^{j+1}$.

The angular distribution of the bisector is given by a formula analogous to (2.19) which is obtained by replacing G in Eq. (2.21) with a matrix Γ given by

$$\Gamma=D^{-1}\Phi D, \quad (2.34)$$

where D is the same as in Eq. (2.23).

[5] G. R. Kalbfleisch *et al.*, Phys. Rev. Letters **12**, 527 (1964); **13**, 349 (1964); M. Goldberg *et al.*, *ibid.* **12**, 546 (1964); **13**, 249 (1964); P. M. Dauber *et al.*, *ibid.* **13**, 449 (1964).

2.3. Spin Tests

There are different ways of using the results of the preceding subsections to derive tests for the spin of B. Let us consider for example the angular distribution of the normal $\mathbf{n}$ given by Eq. (2.13). The conditions that ρ and F be non-negative and of unit trace, give some limitations on the angular distribution coefficients with $M=0$. From (2.14), (2.15), and (2.9) we have explicitly

$$a(L,M)=[(2L+1)/4\pi]^{1/2}\sum_{\mu}F_{\mu\mu}(j\mu,L0|j\mu) \times \sum_{\nu\nu'}\rho_{\nu\nu'}(j\nu',LM|j\nu), \quad (2.35)$$

from which we easily obtain

$$|a(L,0)|\leq[(2L+1)/4\pi]^{1/2}\max_\mu|(j\mu,L0|j\mu)| \times\max_\nu|(j\nu,L0|j\nu)| \quad (2.35')$$

and the stronger limitation

$$\sum_M|a(L,M)|^2\leq[(2L+1)/4\pi]\max_\mu|(j\mu,L0|j\mu)|^2 \times\max_\nu|(j\nu,L0|j\nu)|^2, \quad (2.35'')$$

where $\max_{\mu,\nu}$ means the maximum value of the argument with respect to the allowed values of μ and ν in (2.35). For example, if parity is conserved in the B decay, μ must satisfy $(-1)^\mu=\epsilon$, as we know from (2.25). Analogous limitations hold for the coefficients $b(L,0)$ of Eq. (2.19).

More precise tests can be obtained by determining the parameters $F_{\mu\nu}$ as we shall see in a moment. We define the test functions

$$T(L,M,M')=\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\xi d\xi\int_0^{2\pi}d\eta \times I(\xi,\eta,\varphi)\mathfrak{D}_{M'M}{}^{(L)*}(\varphi,\xi,\eta), \quad (2.36)$$

where $I(\xi,\eta,\varphi)$ is the distribution (2.8) integrated over the energies ω_1,ω_2. As is clear from the derivation of Eq. (2.8), the angles ξ, η, φ have to be measured as follows: ξ, η are the polar coordinates of the normal to the decay plane,

$$\mathbf{n}=(\mathbf{p}_1\times\mathbf{p}_2)/|\mathbf{p}_1\times\mathbf{p}_2|,$$

in a B rest frame defined from the production reaction[6]; φ is the azimuth of $\mathbf{p}_1$ measured in the frame obtained from the B rest frame by a successive rotation of η around the z axis and of ξ around the new y axis, in such a way as to carry the z axis of the new frame in the direction of $\mathbf{n}$.

[6] For example, if B is produced in a two-body reaction such as $\pi+N\rightarrow N+B$, one first takes the reaction center-of-mass system with the x axis along the incident pion and the z axis normal to the production plane; then one goes to the B rest frame by a pure time-like Lorentz transformation.

From Eq. (2.8) we obtain

$$T(L,M,M')=[(2j+1)/(2L+1)]\rho(L,M)F^*(L,M'). \quad (2.37)$$

The coefficients $T(L,M,M')$ determine in particular the angular distributions of $\mathbf{n}$ and of $\mathbf{u}_1$ as given by Eqs. (2.13) and (2.19), respectively. In fact, by comparing (2.14), (2.15), and (2.37) we have simply

$$a(L,M)=[(2L+1)/4\pi]^{1/2}T(L,M,0). \quad (2.38)$$

Also using (2.20), (2.21), and (2.23) we find after some manipulations

$$b(L,M)=\sum_{M'} T(L,M,M')Y_L^{M'}(\pi/2,\pi), \quad (2.39)$$

where

$$Y_l^m(\pi/2,\pi)=\cos[(l+m)\pi/2] \times\frac{1}{2^l}\left(\frac{2l+1}{4\pi}\right)^{1/2}\frac{[(l+m)!(l-m)!]^{1/2}}{[\frac{1}{2}(l+m)]![\frac{1}{2}(l-m)]!}. \quad (2.40)$$

To determine the matrix elements $F_{\mu\nu}$ one can proceed as follows. One first determines the experimental values of the ratios

$$R(L,M',M'')=T(L,M,M')/T(L,M,M'') \quad (2.41)$$

for all the independent values of L, M', and M''. These ratios are independent of the B density matrix and are given, from (2.37), by

$$R(L,M',M'')=F^*(L,M')/F^*(L,M''). \quad (2.42)$$

By virtue of Eq. (2.10) this is equivalent to a system of linear equations in the $F_{\mu\nu}$ of the kind[7]

$$\sum_{\mu\nu} F_{\mu\nu}(j\mu,LM'|j\nu)=R(L,M',M'') \times\sum_{\mu'\nu'} F_{\mu'\nu'}(j\mu',LM''|j\nu'). \quad (2.43)$$

Let us briefly discuss the utilization of this method in the different situations. In the general case of parity nonconservation, and provided that $\rho(L,M)\neq 0$ for all the values of L, $1\leq L\leq 2j$, the independent equations are at most $2j(2j+1)$ (taking into account also the complex conjugate of each equation) and they are not enough to determine the $4j(j+1)$ parameters. A total of $2j$ real parameters will remain undetermined and we may choose them to be the diagonal elements $F_{\mu\mu}$. The test will then consist in the verification that a solution exists such that the conditions (2.22) and (2.22′) are satisfied and the whole matrix is non-negative definite. The situation is quite different for a parity-conserving decay. In this case, because of (2.25), we have

$$T(L,M,M')=0, \quad \text{for odd } M' \quad (2.44)$$

and the independent complex Eqs. (2.43) are j^2 or, equivalently, $2j^2$ real equations, which will in general be sufficient to determine all the parameters. In some cases the number of equations is larger than the number of parameters and we have a number of consistency relations to be verified. Finally, we consider the case of parity-conserving decay with an additional symmetry under exchange of two final particles, as discussed in subsection 2.2. Here we may refer to the matrix Φ instead of F, replacing φ in (2.36) by the azimuth of the bisector χ. From (2.31) and (2.33) we have now, in addition to (2.44):

$$\Phi(L,-M)=(-1)^L\Phi(L,M). \quad (2.45)$$

The Eqs. (2.43) are therefore real and their number is $j(j-1)+1$, $(j\geq 1)$. The parameters $\Phi_{\mu\nu}$ can in general be determined, except for special cases. The matrices F and Φ constructed by the above method, must be non-negative and have to satisfy the limitations (2.22) and (2.22′).

We illustrate the method for the 3-body decay of a $j=1$ boson. For positive parity $\epsilon=+1$ (e.g., $\omega\to 3\pi$), by virtue of (2.25), the only nonzero matrix element is $F_{00}=1$. The angular distributions are completely determined from the density matrix and contain only the even-L terms. From (2.14) and (2.15) we get

$$a(2,M)=-(3/10\pi)^{1/2}\rho(2,M). \quad (2.46)$$

From (2.26) the diagonal matrix elements of G are $G_{11}=G_{-1-1}=\frac{1}{2}$ and we obtain, by use of (2.20) and (2.21)

$$b(2,M)=\tfrac{1}{2}(3/10\pi)^{1/2}\rho(2,M). \quad (2.47)$$

These results essentially coincide with those found by several authors,[8] using different techniques, for the decay $\omega\to 3\pi$. From (2.46) and (2.47) we also obtain

$$a(2,M)/b(2,M)=-2 \quad (2.48)$$

independent of the production process.

Let us consider the case $\epsilon=-1$. $F_{\mu\nu}$ has now four nonzero matrix elements, those with $\mu,\nu\neq 0$. We have only one ratio of the kind (2.41), namely, $R(2,2,0)=\alpha$ and the corresponding Eq. (2.43) gives

$$F_{1-1}=\alpha/\sqrt{6}. \quad (2.49)$$

Since (2.22) and (2.22′) imply

$$|F_{\mu\nu}|\leq\tfrac{1}{2}, \quad \mu\neq\nu \quad (2.50)$$

we have the condition

$$|\alpha|\leq\sqrt{\tfrac{3}{2}}. \quad (2.51)$$

The coefficients of the angular distribution of the normal are given by

$$a(1,M)=(3/8\pi)^{1/2}(F_{11}-F_{-1-1})\rho(1,M), \quad (2.52)$$

$$a(2,M)=\tfrac{1}{2}(3/10\pi)^{1/2}\rho(2,M). \quad (2.53)$$

[7] The system of Eqs. (2.43) is actually nonhomogeneous because of the condition $\sum_\mu F_{\mu\mu}=1$.

[8] G. Feldman, T. Fulton, and K. C. Wali, Nuovo Cimento 24, 278 (1962); M. Jacob and A. Morel, Phys. Letters 7, 350 (1963); see also Ref. 9, Eq. (3) and footnote 13.

We can find the limitation:

$$\sum_M |a(1,M)|^2 \leq (1/16\pi)(3-2|\alpha|^2). \quad (2.53')$$

We notice that, if there is exchange symmetry, we have, by use of (2.29), $F_{11}=F_{-1-1}=\frac{1}{2}$ and $a(1,M)$ vanishes. The angular distribution of the momentum $\mathbf{u}_1$ can be expressed in terms of the ratio α. From (2.39), (2.38), and (2.40) we obtain

$$b(2,M)=\tfrac{1}{2}a(2,M)[(\sqrt{6})\,\mathrm{Re}\alpha-1]. \quad (2.54)$$

The limitation (2.51) gives

$$-2\leq b(2,M)/a(2,M)\leq 1 \quad (2.55)$$

and the value of the ratio b/a is the same as for $\epsilon=+1$ when $\mathrm{Re}\alpha=0$.

In Appendix A the discussion is extended to $j=2$ and $j=3$. We conclude this Section by the following remarks:

(1) We have always disregarded the energy dependence in the final distribution. However, the same analysis can be carried out for each value of the energies ω_1,ω_2. Of course Eqs. (2.22) and (2.22′) will then be replaced by (2.17) and (2.11), respectively. In this case all the parameters $F_{\mu\nu}(\omega_1,\omega_2)$ are expressible by the $f_\mu(\omega_1\omega_2)$.

(2) Spin tests additional to those considered above can be carried out from analysis of the density matrix. Once the parameters $F_{\mu\nu}$ have been determined, the density matrix itself can be obtained, by use of Eq. (2.37), for events having a given production kinematics.

III. DECAY INTO A SPIN-1 AND A SPIN-0 BOSON

We consider the decay process

$$B\rightarrow b_0+b_1, \quad (3.1)$$

where B is boson of spin j, b_0 is a spin-0 boson, and b_1 is a spin-1 boson. We also assume that parity is conserved in the decay process (3.1). Our considerations apply in particular to the $\pi\omega$ resonance and to the $\pi\rho$ resonances. We have already discussed the decay of the B meson in a recent letter[9] and we shall mainly refer to the results contained there.

We call ρ and σ the density matrices of B and b_1 in their respective rest systems and we denote by M the transition matrix in spin space. We have

$$I(\mathbf{v})\sigma=M\rho M^\dagger, \quad (3.2)$$

where $\mathbf{v}$ is the unit vector along the b_1 momentum in the B rest system and

$$I(\mathbf{v})=\mathrm{Tr}(MM^\dagger) \quad (3.3)$$

is the b_1 angular distribution. Since M is invariant under spatial rotations, its matrix elements have the general expression

$$\langle\mathbf{v},\nu|M|j\mu\rangle=\sum_{lm} T_l(lm,1\nu|j\mu)Y_l^m(\mathbf{v}), \quad (3.4)$$

where $|\nu\rangle$, $(-1\leq\nu\leq1)$, and $|j\mu\rangle$ denote the spin state of b_1 and of B, respectively, and T_l are the reduced matrix elements for B decay. They satisfy

$$\sum_l |T_l|^2=1. \quad (3.4')$$

From angular-momentum and parity conservation we have

$$l=j \quad \text{for} \quad \epsilon=(-1)^j,$$
$$l=j\pm1 \quad \text{for} \quad \epsilon=(-1)^{j+1}, \quad (3.5)$$

having denoted by ϵ the product of the intrinsic parities of B, b_0, and b_1. From (3.2) we have, by use of (3.4)

$$I(\mathbf{v})\sigma_{\nu\nu'}=\sum_{ll'} T_lT_{l'}^*\sum_{\mu\mu'}\rho_{\mu\mu'}\sum_{mm'}(lm,1\nu|j\mu)\times(l'm',1\nu'|j\mu')Y_l^m(\mathbf{v})Y_{l'}^{m'*}(\mathbf{v}). \quad (3.6)$$

It is convenient to introduce the expansion of the density matrices ρ and σ in terms of irreducible tensor operators, in analogy with (2.9):

$$\rho(f,\varphi)=(\hat{f}/\hat{j})\sum_{\mu\mu'}\rho_{\mu\mu'}(j\mu',f\varphi|j\mu), \quad (3.7)$$

$$\sigma(k,\kappa)=(\hat{k}/\sqrt{3})\sum_{\nu\nu'}\sigma_{\nu\nu'}(1\nu',k\kappa|1\nu). \quad (3.8)$$

In (3.6) we also use the formula

$$Y_l^m(\mathbf{v})Y_{l'}^{m'*}(\mathbf{v})=(-1)^{m'}[\hat{l}\hat{l}'/(4\pi)^{1/2}]\times\sum_{LM}\frac{1}{\hat{L}}(l0,l'0|L0)(lm,l'-m'|LM)Y_L^M(\mathbf{v}) \quad (3.9)$$

and we obtain, after summation on the magnetic quantum numbers in the standard way, the following expression for the polarization distribution of b_1:

$$I(\mathbf{v})\sigma(k,\kappa)=\sum_{LM}a(k\kappa,LM)Y_L^M(\mathbf{v}). \quad (3.10)$$

$$a(k\kappa,LM)=(4\pi)^{-1/2}\hat{k}(2j+1)\times\sum_{f\varphi}\rho(f,\varphi)(k\kappa,LM|f\varphi)\times\sum_{ll'}(-1)^lT_lT_{l'}^*\hat{l}\hat{l}'(l0,l'0|L0)X\begin{Bmatrix}l & j & 1\\ l' & j & 1\\ L & f & k\end{Bmatrix}, \quad (3.11)$$

where X is the Wigner $9-j$ coefficient. We notice that the angular distribution (3.10) contains only even-L terms, due to the presence of the Clebsch-Gordan coefficient $(l0,l'0|L0)$ in (3.11), as it must for a parity-

[9] M. Ademollo, R. Gatto, and G. Preparata, Phys. Rev. Letters **12**, 462 (1964). See also S. U. Chung, Phys. Rev. **138**, B1541 (1965). The coefficients 0.3586 in the expression for $A(42,2\varphi)$ in Table II of the first paper should be replaced by 0.4811. We are grateful to Dr. Chung for pointing out this error to us.

conserving two-body decay. For $k=0$, $\sigma(0,0)=1/\sqrt{3}$, Eqs. (3.10), (3.11) give the b_1 angular distribution.

The coefficients $a(k\kappa,LM)$ satisfy the relation

$$a^*(k\kappa,LM)=(-1)^{\kappa+M}a(k-\kappa,L-M). \quad (3.12)$$

They can be obtained directly from experiment by the following procedure. To each event one associates a B rest frame and a b_1 rest frame. These rest frames have to be obtained from the center-of-mass system for the reaction in which B is produced, by two successive pure time-like Lorentz transformations,[6] in order that our noncovariant formalism be relativistically correct. In the b_1 rest frame one measures the polarization coefficients $\sigma(k,\kappa)$ from the b_1 decay. In the B rest frame the coefficients $a(k\kappa,LM)$ are obtained as the averages

$$a(k\kappa,LM)=\langle\sigma(k,\kappa)Y_L{}^{M*}(\mathbf{v})\rangle_{\mathbf{v}} \quad (3.13)$$

over the b_1 angular distribution. The coefficients $\sigma(k,\kappa)$ are also obtained as averages over the angular distribution of the b_1 decay products. We consider the examples of ρ decay and ω decay. For $\rho\to 2\pi$ we have[10]

$$\sigma(2,\kappa)=-(10\pi/3)^{1/2}\langle Y_2{}^{\kappa*}(\mathbf{u})\rangle_{\mathbf{u}}, \quad (3.14)$$

where $\mathbf{u}$ is the direction of any one of the final pions. For the decay $\omega\to 3\pi$ we can apply the equations (2.46) and (2.47), respectively, for the distribution of the normal and of the momentum. In both cases the coefficients vanish for $k=1$. With the aim of obtaining spin tests independent of the density matrix, it is convenient to introduce the test functions[11]

$$T(Lk,f\varphi)=(12\pi)^{1/2}\sum_{M\kappa}a(k\kappa,LM)(k\kappa,LM|f\varphi), \quad (3.15)$$

the normalization being such that $T(00,00)=1$. From (3.11) we have

$$T(Lk,f\varphi)=\hat{\jmath}B(Lkf)\rho(f,\varphi), \quad (3.16)$$

$$B(Lkf)=\sqrt{3}\hat{k}\hat{\jmath}\times\sum_{ll'}(-1)^lT_lT_{l'}{}^*\hat{l}\hat{l}'(l0,l'0|L0)X\begin{Bmatrix} l & j & 1\\ l' & j & 1\\ L & f & k\end{Bmatrix}, \quad (3.17)$$

[the normalization is $B(000)=1$]. The spin and parity assignment can be made by verifying the following conditions: If $\epsilon=(-1)^j$ one has $l=l'=j$ and $|T_j|^2=1$; the $B(Lkf)$ are known numbers depending on the assumed spin j. The following conditions must be satisfied by the experimental test functions $T(Lk,f\varphi)$: (i) $T(Lk,f\varphi)=0$ for $f+k=$odd, due to the vanishing of the X coefficients: (ii) $T(22,00)=1/\sqrt{2}$, for any value of j; (iii) the ratios $T(Lk,f\varphi)/T(L'k',f\varphi)$ are known numbers depending on j. If $\epsilon=(-1)^{j+1}$ it is convenient to introduce the decay parameters

$$\begin{aligned}\alpha&=2\,\mathrm{Re}(T_{j-1}T_{j+1}{}^*),\\ \beta&=2\,\mathrm{Im}(T_{j-1}T_{j+1}{}^*),\\ \gamma&=|T_{j-1}|^2-|T_{j+1}|^2,\end{aligned} \quad (3.18)$$

satisfying the condition

$$\alpha^2+\beta^2+\gamma^2=1. \quad (3.19)$$

Equation (3.17) takes then the form

$$B(Lkf)=a\alpha+ib\beta+c\gamma+d \quad (3.20)$$

where a, b, c, and d are real geometrical coefficients depending on L, k, f, and j. The symmetry properties of the X coefficients require that $b=0$ for $f+k=$even and $a=c=d=0$ for $f+k=$odd. The ratios $T(Lk,f\varphi)/T(L'k',f\varphi)$ are independent of the production process and give a set of linear equations in the decay parameters, in a way similar to that discussed in the previous section. The consistency of the solution, taking also into account the condition (3.19), indicates the correct value of j, and the decay parameters themselves can then be determined. The coefficients $B(Lkf)$ for $j=0,1,2$ are tabulated in Table I for $\epsilon=(-1)^j$ and in Table II for $\epsilon=(-1)^{j+1}$.

IV. DECAY INTO TWO FERMIONS

In this Section we include for completeness a brief discussion of the decay of a spin-j boson into two spin-$\frac{1}{2}$ fermions. We write the density matrix of the final fermions, in the rest frame of the boson B, in the form

$$\rho^{(f)}=\sum\rho_{\nu\nu'}{}^{(B)}T_{ls}T_{l's'}{}^*(s\mu|\tfrac{1}{2}\nu_1,\tfrac{1}{2}\nu_2)\times(s\mu'|\tfrac{1}{2}\nu_1',\tfrac{1}{2}\nu_2')(j\nu|lm,s\mu)(j\nu'|l'm',s\mu')\times Y_l{}^m(\mathbf{v})Y_{l'}{}^{m'*}(\mathbf{v})|\nu_1\nu_2\rangle\langle\nu_1'\nu_2'|, \quad (4.1)$$

where $\rho^{(B)}$ is the boson-density matrix, $\mathbf{v}$ is the decay direction, and T_{ls} are the reduced decay-matrix elements.

TABLE I. The coefficients $B(Lkf)$ for $\epsilon=(-1)^j$.

j^ϵ	
0^+	$B(0\,0\,0)=\ \ 1$
1^-	$B(0\,0\,0)=\ \ 1$
	$B(2\,2\,0)=\ \ 0.7071$
	$B(2\,0\,2)=\ \ 0.3162$
	$B(0\,2\,2)=-0.5000$
	$B(2\,2\,2)=-2.2136$
2^+	$B(0\,0\,0)=\ \ 1$
	$B(2\,2\,0)=\ \ 0.7071$
	$B(2\,0\,2)=-0.2673$
	$B(0\,2\,2)=-0.5916$
	$B(2\,2\,2)=-0.5051$
	$B(4\,2\,2)=-0.2711$
	$B(4\,0\,4)=-0.3564$
	$B(2\,2\,4)=\ \ 0.2020$
	$B(4\,2\,4)=\ \ 3.3128$

[10] M. Peshkin, Phys. Rev. **123**, 637 (1961).

[11] The test functions defined in (3.15) are slightly different from the analogous test functions $A(Lk,f\varphi)$ defined in Ref. 9. The relation among them is

$$T(Lk,f\varphi)=(-1)^{f+k}A^*(Lk,f\varphi)=(-1)^\varphi A(Lk,f-\varphi).$$

TABLE II. The coefficients $B(Lkf)$ for $\epsilon=(-1)^{j+1}$.

j^ϵ	
0^-	$B(0\,0\,0)=\ \ 1$
	$B(2\,2\,0)=-1.4142$
1^+	$B(0\,0\,0)=\ \ 1$
	$B(2\,2\,0)=\ \ \alpha+0.3536\gamma-0.3536$
	$B(2\,2\,1)=-0.8660i\beta$
	$B(2\,0\,2)=\ \ 0.4472\alpha+0.1581\gamma-0.1581$
	$B(0\,2\,2)=\ \ 0.4500\gamma+0.5500$
	$B(2\,2\,2)=\ \ 0.5916\alpha-0.0598\gamma+0.0598$
	$B(4\,2\,2)=-0.4811\gamma+0.4811$
2^-	$B(0\,0\,0)=\ \ 1$
	$B(2\,2\,0)=\ \ 1.0392\alpha+0.2121\gamma-0.3536$
	$B(2\,2\,1)=\ \ 0.8744i\beta$
	$B(2\,0\,2)=\ \ 0.1309\alpha-0.4009\gamma+0.0267$
	$B(0\,2\,2)=\ \ 0.2133\gamma+0.3803$
	$B(2\,2\,2)=\ \ 0.5677\alpha+0.1611\gamma+0.0196$
	$B(4\,2\,2)=-0.2179\alpha-0.2711\gamma+0.2711$
	$B(2\,2\,3)=\ \ 0.2498i\beta$
	$B(4\,2\,3)=-0.3938i\beta$
	$B(4\,0\,4)=-0.4364\alpha+0.0891\gamma-0.0891$
	$B(2\,2\,4)=\ \ 0.0495\alpha-0.4377\gamma-0.4108$
	$B(4\,2\,4)=-0.3245\alpha-0.3333\gamma+0.3333$
	$B(6\,2\,4)=\ \ 0.2205\gamma-0.2205$

The polarization of one of the two fermions, say f_1, is described by the density matrix $\rho^{(1)}$, obtained from $\rho^{(f)}$ by taking the trace with respect to the variables of f_2. The angular distribution and polarization of f_1 are given by

$$I(\mathbf{v})=\mathrm{Tr}[\rho^{(1)}],$$
$$I(\mathbf{v})\mathbf{P}=\mathrm{Tr}[\rho^{(1)}\boldsymbol{\sigma}^{(1)}]. \tag{4.2}$$

One finds

$$I(\mathbf{v})=\sum_{LM}a(L,M)Y_L{}^M(\mathbf{v}), \tag{4.3}$$

where

$$a(L,M)=(1/(2j+1))\times[(2L+1)/(4\pi)]^{1/2}A(L)S(L,M) \tag{4.4}$$

and

$$S(L,M)=\sum_{\nu\nu'}\rho_{\nu\nu'}{}^{(B)}(j\nu',LM|j\nu) \tag{4.5}$$

are the coefficients of the multipole expansion of the density matrix of B; furthermore, we have put

$$A(L)=\sum_{s\mu}(l0,s\mu|j\mu)A_s{}^\mu A_s{}^{\mu*}$$

with

$$A_s{}^\mu=\sum_l \hat{l}T_{ls}(l0,s\mu|j\mu).$$

An explicit expression of these quantities in terms of the matrix elements T_{ls} is

$$A_0{}^0=T_{j0}(2j+1)^{1/2};$$
$$A_1{}^0=T_{j-1,1}j^{1/2}-T_{j+1,1}(j+1)^{1/2},$$
$$A_1{}^1=(1/\sqrt{2})[T_{j+1,1}\sqrt{j}-T_{j,1}(2j+1)^{1/2}+T_{j-1,1}(j+1)^{1/2}],$$
$$A_1{}^{-1}=(1/\sqrt{2})[T_{j+1,1}\sqrt{j}+T_{j,1}(2j+1)^{1/2}+T_{j-1,1}(j+1)^{1/2}]. \tag{4.6}$$

Normalization in the solid angle is expressed by

$$\sum_{ls}|T_{ls}|^2=1. \tag{4.7}$$

An expression for $A(L)$, quite convenient for spin tests, is

$$A(L)=(L0,j0|j0)[|A_0{}^0|^2+|A_1{}^0|^2]+(L0,j1|j1)[|A_1{}^1|^2+(-1)^L|A_1{}^{-1}|^2]. \tag{4.8}$$

In terms of the three orthogonal unit vectors

$$\mathbf{v}_1=\mathbf{v},\quad \mathbf{v}_2=\mathbf{n}\times\mathbf{v}/|\mathbf{n}\times\mathbf{v}|,\quad \mathbf{v}_3=\mathbf{v}_1\times\mathbf{v}_2,$$

(where $\mathbf{n}$ is the normal of the production plane), we obtain

$$I(\mathbf{v})\mathbf{P}\cdot\mathbf{v}_1=\sum_{LM}b(L,M)Y_L{}^M(\mathbf{v}), \tag{4.9}$$

$$b(L,M)=(-1)^j[2\pi(2j+1)]^{-1/2}D_0(L)S(L,M),$$
$$I(\mathbf{v})\mathbf{P}\cdot\mathbf{v}_2=-(i/\sqrt{2})[\sum_{LM}c_1(L,M)\mathfrak{D}_{1M}{}^{(L)}(\omega)+c_{-1}(L,M)\mathfrak{D}_{-1M}{}^{(L)}(\omega)],$$
$$I(\mathbf{v})\mathbf{P}\cdot\mathbf{v}_3=(1/\sqrt{2})[\sum_{LM}c_1(L,M)\mathfrak{D}_{1M}{}^{(L)}(\omega)-c_{-1}(L,M)\mathfrak{D}_{-1M}{}^{(L)}(\omega)], \tag{4.10}$$

where

$$c_{\pm1}(L,M)=(-1)^j(\sqrt{2}/4\pi)\times[(2L+1)/(2j+1)]^{1/2}D_{\pm1}(L)S(L,M)$$

and $\mathfrak{D}_{\pm1M}{}^{(L)}(\omega)$ are the well-known elements of the rotation matrices, depending on the Euler angles $\omega\equiv(\varphi,\theta,0)$, $[\mathbf{v}\equiv(\theta,\varphi)]$. The quantities D_i appearing in Eqs. (4.9) and (4.10) are defined as

$$D_0(L)=(1/\sqrt{2})[(j0,j0|L0)2\,\mathrm{Re}(A_0{}^0A_1{}^{0*})-(j1,j-1|L0)(|A_1{}^1|^2-|A_1{}^{-1}|^2)],$$
$$D_1(L)=D_{-1}(L)^*=-(1/\sqrt{2})(j0,j1|L1)\times[(A_0{}^0+A_1{}^0)A_1{}^{-1*}+(-1)^L(A_0{}^0-A_1{}^0)^*A_1{}^1]. \tag{4.11}$$

When f_2 is a left-handed neutrino, one has the condition

$$\sum_s A_s{}^\mu(s\mu|\tfrac{1}{2}\nu_1,\tfrac{1}{2}-\tfrac{1}{2})=0, \tag{4.12}$$

and the preceding equations become in this case

$$a(L,M)=2[(2L+1)/(4\pi)]^{1/2}S(L,M)\times[(L0,j0|j0)|T_{j0}|^2+(L0,j1|j1)|T_{j,1}|^2], \tag{4.13}$$

$$b(L,M)=-2[(2L+1)/(4\pi)]^{1/2}S(L,M)\times[(L0,j0|j0)|T_{j0}|^2-(L0,j1|j1)|T_{j,1}|^2], \tag{4.14}$$

$$c_1(L,M)=[(-1)^{L+j}/2\pi][2(2L+1)(2j+1)]^{1/2}\times(j0,j1|L1)T_{j1}T_{j0}{}^*S(L,M), \tag{4.15}$$

$$c_{-1}(L,M)=[(-1)^{L+j}/2\pi][2(2L+1)(2j+1)]^{1/2}\times(j0,j1|L1)T_{j1}{}^*T_{j0}S(L,M). \tag{4.16}$$

The results agree with those of Durand, Landovitz and Leitner.[12] When parity is conserved in the decay, we distinguish, according to the two values of $P_B P_{f_1 f_2}$, between two cases:

(a) $P_B P_{f_1 f_2} = (-1)^j$

The only nonvanishing amplitudes are T_{j0} and T_{j1}, so that we have

$$\begin{aligned} a(L,M) &= [(2L+1)/4\pi]^{1/2}[(L0,j0|j0)|T_{j0}|^2 \\ &\quad + (L0,j1|j1)|T_{j1}|^2]S(L,M) \qquad \text{for even } L, \\ &= 0 \qquad \text{for odd } L; \end{aligned} \tag{4.17}$$

$$\begin{aligned} b(L,M) &= [(2L+1)/4\pi]^{1/2}(L0,j1|j1)|T_{j1}|^2 S(L,M) \qquad \text{for odd } L, \\ &= 0 \qquad \text{for even } L; \end{aligned} \tag{4.18}$$

$$\begin{aligned} c_1(L,M) &= [(-1)^j/4\pi] \\ &\quad \times[\tfrac{1}{2}(2L+1)(2j+1)]^{1/2}(j0,j1|L1) \\ &\quad \times[T_{j0}T_{j1}^* - (-1)^L T_{j0}^* T_{j1}]S(L,M), \end{aligned} \tag{4.19}$$

$$\begin{aligned} c_{-1}(L,M) &= [(-1)^j/4\pi] \\ &\quad \times[\tfrac{1}{2}(2L+1)(2j+1)]^{1/2}(j0,j1|L1) \\ &\quad \times[T_{j0}^* T_{j1} - (-1)^L T_{j0}T_{j1}^*]S(L,M). \end{aligned} \tag{4.20}$$

(b) $P_B P_{f_1 f_2} = (-1)^{j+1}$

In this case we have

$$\begin{aligned} a(L,M) &= [(2L+1)/4\pi]^{1/2}[S(L,M)/(2j+1)] \\ &\quad \times[(L0,j0|j0)|A_1^0|^2 + 2(L0,j1|j1)|A_1^1|^2] \\ &\qquad \text{for even } L, \\ &= 0 \qquad \text{for odd } L; \end{aligned} \tag{4.21}$$

$$\begin{aligned} b(L,M) &= [(2L+1)/4\pi]^{1/2}[S(L,M)/(2j+1)] \\ &\quad \times 2(L0,j1|j1) \qquad \text{for odd } L, \\ &= 0 \qquad \text{for even } L. \end{aligned} \tag{4.22}$$

$$\begin{aligned} c_1(L,M) &= \frac{(-1)^{j+1}}{4\pi}\left(\frac{2L+1}{2j+1}\right)^{1/2} S(L,M) \\ &\quad \times (j0,j1|L1)\begin{cases} 2i\,\mathrm{Im}(A_1^0 A_1^{1*}) & \text{even } L \\ 2\,\mathrm{Re}(A_1^0 A_1^{1*}) & \text{odd } L, \end{cases} \\ c_{-1}(L,M) &= (-1)^{L+1} c_1(L,M). \end{aligned} \tag{4.23}$$

From proper averages on the angular and polarization distributions we can derive the various coefficients $a(L,M)$, $b(L,M)$, and $c_{\pm 1}(L,M)$. Putting

$$\begin{aligned} \xi^2 &= |A_0^0|^2 + |A_1^0|^2, \\ \eta^2 &= |A_1^1|^2, \\ \zeta^2 &= |A_1^{-1}|^2, \\ \alpha &= (A_0^0 + A_1^0)A_1^{-1*}, \\ \beta &= (A_0^0 - A_1^0)^* A_1^1, \\ \theta &= 2\,\mathrm{Re}(A_0^0 A_1^{0*}), \end{aligned}$$

it is easy to show that in the general case

$$\frac{a(L,M)}{b(L,M)} = \frac{(\eta^2-\zeta^2)}{(\eta^2+\zeta^2)} \qquad \text{for odd } L, \tag{4.24}$$

$$\frac{a(L,M)}{b(L,M)} = \frac{((L0,j0|j0)\xi^2 + (L0,j1|j1)(\eta^2+\zeta^2))}{((L0,j0|j0)\theta + (L0,j1|j1)(\eta^2-\zeta^2))} \qquad \text{for even } L. \tag{4.25}$$

From the solution of these equations one can obtain the parameters ξ^2, η^2, ζ^2 and θ. Furthermore we have

$$c_1(L,M)/a(L,M) = (\text{known number}) \times (\alpha+\beta) \qquad \text{for even } L, \tag{4.26}$$

$$c_1(L,M)/a(L,M) = (\text{known number}) \times (\alpha-\beta) \qquad \text{for odd } L. \tag{4.27}$$

From (4.26) and (4.27) we can get α and β and test the consistency of the spin assignment by the relations

$$|\alpha|^2 = (\xi^2+\theta)\zeta^2, \tag{4.28}$$

$$|\beta|^2 = (\xi^2-\theta)\eta^2. \tag{4.29}$$

These equations provide us with a general test for the spin of B.

Note added in proof. We have received an unpublished report by C. Zemach in which a similar analysis is developed. There is a strong correspondence in basic formalism between the two approaches. In particular, Eq. (2.28) of Zemach's report is essentially the same as our Eq. (2.8), with φ_{kM} corresponding to our $F^*(LM,\omega_1\omega_2)$, with Zemach's Eq. (2.25) corresponding to our Eq. (2.10), and so on. Also $d\sigma_{kM'M}$ in Eq. (2.29) of Zemach's paper is equivalent to our $T(L,M,M')$ in our Eq. (2.36). We thank Charles Zemach for sending us this information before its publication.

APPENDIX A

Three-Body Decay of a Higher-Spin Boson

In this Appendix the method is applied to the three-body decay of a boson of spin j and parity ϵ (ϵ denotes the product of initial and final intrinsic parities) and the relevant spin tests are derived. The case $j=1$, for both cases of parity, is discussed in the text (Sec. 2.3). The discussion is here extended to $j=2$ and $j=3$.

[12] L. Durand, L. F. Landovitz and J. Leitner, Phys. Rev. **112**, 273 (1958).

$j^{\epsilon}=2^{-}$

The nonzero matrix elements $F_{\mu\nu}$ are those with $\mu,\nu=\pm1$. There are four independent ratios (2.41), and the Eqs. (2.43) give

$$R(2,2,0)=-2.450F_{-11}, \quad (A1)$$

$$R(4,2,0)=1.581F_{-11}, \quad (A2)$$

$$R(3,2,0)=R(4,4,0)=0. \quad (A3)$$

From (A1), (A2), and (2.50) we have

$$R(2,2,0)=-1.55R(4,2,0), \quad (A4)$$

$$|R(2,2,0)|\leq1.22. \quad (A5)$$

The coefficients of the distribution of the normal [Eq. (2.13)] are

$$a(1,M)=[0.915/(4\pi)^{1/2}](F_{11}-F_{-1-1})\rho(1,M), \quad (A6)$$

$$a(2,M)=-[0.600/(4\pi)^{1/2}]\rho(2,M), \quad (A7)$$

$$a(3,M)=-[1.197/(4\pi)^{1/2}](F_{11}-F_{-1-1})\rho(3,M), \quad (A8)$$

$$a(4,M)=-[0.800/(4\pi)^{1/2}]\rho(4,M), \quad (A9)$$

and those of the momentum distribution [Eq. (2.19)] are

$$b(2,M)=-\tfrac{1}{2}a(2,M)(6\,\mathrm{Re}F_{-11}+1), \quad (A10)$$

$$b(4,M)=a(4,M)(0.375-1.250\,\mathrm{Re}F_{-11}). \quad (A11)$$

For the $\sum_M |a(L,M)|^2$ we can give limitations analogous to (2.53′) by using

$$\sum_M |\rho(L,M)|^2\leq\frac{(2L+1)}{(2j+1)}\max_\mu |(j\mu,L0|j\mu)|^2 \quad (A12)$$

$$(F_{11}-F_{-1-1})^2\leq1-4|F_{1-1}|^2. \quad (A13)$$

We find

$$(4\pi)\sum_M |a(1,M)|^2\leq0.335(1-4|F_{1-1}|^2), \quad (A14)$$

$$(4\pi)\sum_M |a(2,M)|^2\leq0.103, \quad (A15)$$

$$(4\pi)\sum_M |a(3,M)|^2\leq0.573(1-4|F_{1-1}|^2), \quad (A16)$$

$$(4\pi)\sum_M |a(4,M)|^2\leq0.329. \quad (A17)$$

For the ratios $b(L,M)/a(L,M)$ we find the following limitations

$$-2\leq b(2,M)/a(2,M)\leq1, \quad (A18)$$

$$-0.250\leq b(4,M)/a(4,M)\leq1. \quad (A19)$$

$j^{\epsilon}=2^{+}$

The nonzero matrix elements $F_{\mu\nu}$ are those with $\mu,\nu=0,\pm2$. The Eqs. (2.43) give in this case:

$$(F_{02}+F_{-20})=R(2,2,0)(1-2F_{00}), \quad (A20)$$

$$F_{02}-F_{-20}=0.447R(3,2,0)(F_{22}-F_{-2-2}), \quad (A21)$$

$$(F_{02}+F_{-20})=R(4,2,0)(0.259+1.289F_{00}), \quad (A22)$$

$$F_{-2,2}=R(4,4,0)(0.120+0.597F_{00}). \quad (A23)$$

These equations together with the relation $F_{00}+F_{22}+F_{-2-2}=1$, are sufficient to determine the matrix $F_{\mu\nu}$ completely. The ratios R have to satisfy the limitations

$$|R(4,2,0)|\leq3.87, \quad (A24)$$

$$|R(4,4,0)|\leq4.20, \quad (A25)$$

$$|R(4,2,0)/R(2,2,0)|\leq3.87. \quad (A26)$$

The coefficients $a(L,M)$ are

$$a(1,M)=[1.823/(4\pi)^{1/2}](F_{22}-F_{-2-2})\rho(1,M), \quad (A27)$$

$$a(2,M)=[1.195/(4\pi)^{1/2}](1-2F_{00})\rho(2,M), \quad (A28)$$

$$a(3,M)=[0.600/(4\pi)^{1/2}](F_{22}-F_{-2-2})\rho(3,M), \quad (A29)$$

$$a(4,M)=[1/(4\pi)^{1/2}](0.200+F_{00})\rho(4,M). \quad (A30)$$

The coefficients $b(L,M)$ are given by

$$b(2,M)=a(2,M)[-0.500+1.225\,\mathrm{Re}\{R(2,2,0)\}], \quad (A31)$$

$$b(4,M)=a(4,M)[0.375-0.791\,\mathrm{Re}\{R(4,2,0)\}+0.956\,\mathrm{Re}\{R(4,4,0)\}]. \quad (A32)$$

The $a(L,M)$ are subjected to the limitations

$$(4\pi)\sum_M |a(1,M)|^2\leq1.329(1-4|F_{2-2}|^2), \quad (A33)$$

$$(4\pi)\sum_M |a(2,M)|^2\leq0.408, \quad (A34)$$

$$(4\pi)\sum_M |a(3,M)|^2\leq0.144(1-4|F_{2-2}|^2), \quad (A35)$$

$$(4\pi)\sum_M |a(4,M)|^2\leq0.741. \quad (A36)$$

We have also the limitation

$$-6.6\leq b(4,M)/a(4,M)\leq7.4. \quad (A37)$$

$j^{\epsilon}=3^{+}$

There are nine (eight independent) matrix elements $F_{\mu\nu}$ for $\mu,\nu=0,\pm2$. The Eqs. (2.43) give explicitly:

$$F_{02}+F_{-20}=0.894R(2,2,0)(F_{22}+F_{-2-2}-1), \quad (A38)$$

$$F_{02}-F_{-20}=-R(3,2,0)(F_{22}-F_{-2-2}), \quad (A39)$$

$$F_{02}+F_{-20}=R(4,2,0)\times[7.486(F_{22}+F_{-2-2})-3.457], \quad (A40)$$

$$F_{-22}=-R(4,4,0)\times[1.555(F_{22}+F_{-2-2})-0.718], \quad (A41)$$

$$F_{02}-F_{-20}=0.755R(5,2,0)(F_{22}-F_{-2-2})\,, \tag{A42}$$

$$F_{02}+F_{-20}=-R(6,2,0)\times[0.936(F_{22}+F_{-2-2})-1.341]\,, \tag{A43}$$

$$F_{-22}=-R(6,4,0)\times[0.556(F_{22}+F_{-2-2})-0.796]\,, \tag{A44}$$

and also

$$R(5,4,0)=R(6,6,0)=0\,. \tag{A45}$$

Equations (A38)–(A44) are, in general, sufficient to determine the matrix elements $F_{\mu\nu}$. Furthermore the following consistency relations have to be satisfied among the experimental ratios:

$$R(5,2,0)/R(3,2,0)=-1.324\,, \tag{A46}$$

$$R(4,4,0)/R(4,2,0)=-2.9R(6,4,0)/R(6,2,0)\,, \tag{A47}$$

$$\frac{0.894R(2,2,0)-3.457R(4,2,0)}{0.894R(2,2,0)-7.486R(4,2,0)}=\frac{1.341R(6,2,0)+3.457R(4,2,0)}{0.936R(6,2,0)+7.486R(4,2,0)}\,. \tag{A48}$$

In addition, we have the following limitations:

$$|R(6,4,0)|\leq 2.12\,, \tag{A49}$$

$$|R(6,2,0)/R(2,2,0)|\leq 0.63\,, \tag{A50}$$

$$|R(6,2,0)/R(4,2,0)|\leq 10\,. \tag{A51}$$

$j^\epsilon=3^-$

We have in this case 15 independent matrix elements $F_{\mu\nu}$ with μ, ν odd. The Eqs. (2.43) are

$$F_{-11}+0.644(F_{-3-1}+F_{13})=R(2,2,0)[1.632(F_{33}+F_{-3-3})-0.611]\,,$$

$$F_{-3-1}-F_{13}=R(3,2,0)[0.707(F_{33}-F_{-3-3})+(F_{-11}-F_{-11})]\,,$$

$$-F_{-11}+1.161(F_{-3-1}+F_{13})=R(4,2,0)[0.316(F_{33}+F_{-3-3})+0.159]\,,$$

$$F_{-31}+F_{-13}=R(4,4,0)[0.308(F_{33}+F_{-3-3})+0.155]\,,$$

$$F_{-3-1}-F_{13}=R(5,2,0)[0.267(F_{33}-F_{-3-3})+1.129(F_{11}-F_{-1-1})]\,,$$

$$F_{-31}-F_{13}=R(5,4,0)[0.307(F_{33}-F_{-3-3})+1.300(F_{11}-F_{-1-1})]\,,$$

$$F_{-11}+0.259(F_{-3-1}+F_{13})=R(6,2,0)[0.731-0.683(F_{33}+F_{-3-3})]\,,$$

$$F_{-31}+F_{-13}=R(6,4,0)[1.034-0.966(F_{33}+F_{-3-3})]\,,$$

$$F_{-33}=R(6,6,0)[0.5-0.460(F_{33}+F_{-33})]\,.$$

The matrix F can be determined, apart from one relation among the diagonal matrix elements. The following consistency relations have to be satisfied among the coefficients R:

$$\frac{0.043R(4,2,0)+0.611R(2,2,0)+0.929R(6,2,0)}{1.632R(2,2,0)-0.066R(4,2,0)+0.868R(6,2,0)}=\frac{1.034R(6,4,0)-0.155R(4,4,0)}{0.966R(6,4,0)+0.308R(4,4,0)}\,,$$

$$\frac{0.611R(2,2,0)+0.586R(4,2,0)+3.427R(6,2,0)}{1.632R(2,2,0)-3.378R(4,2,0)-1.936R(6,2,0)}=\frac{1.034R(6,4,0)-0.155R(4,4,0)}{0.966R(6,4,0)+0.308R(4,4,0)}\,.$$

We add a few remarks for the case of exchange symmetry, considered in Sec. 2.2. It is convenient to consider the matrix Φ instead of F. The matrix elements $\Phi_{\mu\nu}$ satisfy the same conditions that $F_{\mu\nu}$ and, in addition, the symmetry condition of Eq. (2.33). The test functions $T(L,M,M')$ will satisfy Eq. (2.44) and, in addition,

$$T(L,M,0)=0\quad\text{for odd }L\,, \tag{A52}$$

deriving from (2.37) and (2.45). The ratios $R(L,M,M')$ must be real. For the specific cases of $j\leq 3$, the equations for the $\Phi_{\mu\nu}$ can be obtained as particular cases of the corresponding equations for the $F_{\mu\nu}$, by taking into account the preceding considerations. Some of the limitations on the ratios R are actually stronger. Specifically, we have

for $j^\epsilon=2^+$, instead of (A24), we get

$$|R(4,2,0)|\leq 1.12\,; \tag{A53}$$

for $j^\epsilon=3^+$, in addition to (A49)–(A51), we have

$$|R(6,2,0)|\leq 0.96\,; \tag{A54}$$

for $j^\epsilon=3^-$, the limitations are the following—

$$\begin{gathered}|R(4,4,0)|\leq 1.84\,,\\ |R(4,2,0)|\leq 6.3\,,\\ |R(6,2,0)|\leq 1.43\,,\\ |R(6,4,0)|\leq 1.86\,,\\ |R(6,6,0)|\leq 15\,,\\ 0.15\leq|R(6,4,0)/R(4,4,0)|\leq 6.5\,.\end{gathered} \tag{A55}$$