Three-Body Model for Stripping and the Validity of the Distorted-Wave Born Approximation

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A model is proposed, according to which a "deuteron," consisting of two identical particles, interacts with a "core nucleus," taken as an infinitely massive point particle which can form a bound state with either of the two particles. Assuming factorable interactions between the particles of all the three pairs, exact expressions for the scattering and stripping amplitudes are obtained. A comparison with the corresponding distorted-wave Born approximation (DWBA) amplitudes, which can also be obtained in the model, shows that DWBA is valid within the model. The model also predicts that the *partial-wave* scattering amplitudes a_l are small compared with the corresponding stripping amplitudes b_l. An (indirect) experimental test of this result is suggested on the basis of large angular momenta, which are needed to overcome the difficulties of finite nuclear size in actual situations.

1. INTRODUCTION

THE distorted-wave Born approximation (DWBA) for the treatment of direct nuclear reactions[1,2] and, in particular, (d,p) reactions[3] has been one of the most successful tools in nuclear theory. DWBA calculations which have been carried out by a large number of workers have found very impressive agreement with observations. The theory is continually being made more sophisticated in order to include more and more physical restrictions on the calculations, the latest being the inclusion of finite-range effects.[4] The DWBA approach, which has a strong intuitive appeal, was motivated mainly from considerations of practical interest, viz., to obtain improved fits to the data on nuclear reactions which the ordinary Born approximation could not provide. This might explain why a comparable degree of attention to the formal mathematical foundations of DWBA has not accompanied the development of the practical aspects of the theory during the last decade. In other words, the experimental success of the theory has generally tended to obscure the question as to precisely what effects are being ignored under this approximation. From a logical point of view, an estimation of the neglected effects could in principle lead to an understanding of DWBA. On the other hand, from a practical point of view, the mathematical formulation of a (d,p) reaction with a sizable nucleus, e.g., Zn^{60}, with all its size and structure effects, would be an almost impossible task. An "exact" formulation can at most be made for a highly idealized situation in which many effects must necessarily be neglected. The limitations imposed by such ideal conditions would of course tend to move the problem away from reality. Yet models have frequently provided very useful backgrounds for testing the validity of various approximations, the Lee model[5] being a good example. Such considerations have motivated us to consider a model stripping process which is essentially soluble, so that it may provide the necessary background for comparison with the results of a corresponding DWBA calculation within the same framework. In other words, the "experimental material" for such a situation is represented by the exact expressions for the amplitudes, against which the "theoretical" DWBA amplitudes can be tested. The model is of course not meant for application to actual stripping processes, though certain general features emerging from the model may be discussed in relation to experimental conditions.

Our model consists of an infinitely massive point nucleus A, with no internal structure, playing the role of the "core." A "deuteron" d, consisting of two identical spinless "nucleons" n_1 and n_2, interacting with A, can lead to any one of three possible processes of elastic scattering, stripping, or breakup reactions. For simplicity, it is further assumed that the core A can form only *one* bound state A' with either of n_1, or n_2, brought about by the potentials V_1 or V_2 acting between the pairs (An_1) or (An_2), respectively.[6] Similarly, a potential V_{12} acting between n_1 and n_2 leads to one bound state, viz., the deuteron (d). The problem is thus reduced to that of a three-body system under the influence of the three potentials V_1, V_2, and V_{12}. If these potentials are assumed to be factorable, our experience with the three-nucleon bound[7] and scattering problems[8] shows that an exact solution can be obtained in terms of certain single-parameter "spectator functions" satis-

[1] N. Austern, in *Fast Neutron Physics*, II, edited by J. B. Marion and J. L. Fowler (Interscience Publishers, Inc., New York, 1962).
[2] W. Tobocman, *Theory of Direct Nuclear Reactions* (Oxford University Press, London, 1961).
[3] S. T. Butler, *Nuclear Stripping Reactions* (John Wiley & Sons, Inc., New York, 1957).
[4] N. Austern, R. M. Drisko, E. C. Halbert, and G. R. Satchler, Phys. Rev. **133**, B3 (1964).
[5] T. D. Lee, Phys. Rev. **95**, 1329 (1954).
[6] This model has some points of similarity with a soluble one proposed by Amado [R. D. Amado, Phys. Rev. **132**, 485 (1963)], in connection with *n-d* scattering and stripping processes. However, Amado considers the force in only *one* of the pairs A_{n1}, A_{n2} (corresponding to the nucleon that is being captured), and ignores the force in the other pair. The present model takes account of potentials in *both* the pairs A_{n1} and A_{n2}, and is thus capable of generating many more connected graphs of higher order.
[7] A. N. Mitra, Nucl. Phys. **32**, 529 (1962); referred to as I.
[8] A. N. Mitra and V. S. Bhasin, Phys. Rev. **131**, 1265 (1963); referred to as II.

fying as many one-dimensional integral equations, and appropriate boundary conditions. The mathematical validity of such potentials, when a two-body system is dominated by bound states or resonances, has been established by the work of Lovelace,[9] on the basis of Faddeev's three-particle theory.[10] The numerical accuracy with which such potentials can represent the effect of static potentials has been found to be rather close by Sugar and Blankenbecler[11] on the basis of their theory of upper and lower bounds for scattering problems.

Our three-body model thus automatically takes into account certain features of the stripping process, viz., the influence of the target on the deuteron internal wave function, the deuteron polarizability or breakup, and the residual proton-target interaction. The most serious drawback of the model in simulating the actual physical situation perhaps lies in the assumption of a point nucleus and (factorable) s-wave interactions between pairs. In an actual stripping calculation, on the other hand, the optical potentials that are used for the calculation of the (Ad) and (Ap) wave functions are not only dependent on the finite size of the nucleus, but strongly affect many partial waves in any pair of particles.

In Sec. 2 we spell out the model in some detail and obtain the exact amplitudes for stripping as well as elastic d-A scattering. A partial-wave analysis of these amplitudes is also carried out. In Sec. 3, the DWBA is first defined within the model and then used to calculate the corresponding scattering and stripping amplitudes. It is shown that it is possible to obtain the DWBA amplitudes without explicit reference to any optical potential, unlike the customary procedures.[1,2] In Sec. 4, a comparison of the two results leads to a simple condition on the validity of DWBA within the model. A possible way of testing this condition on actual physical systems is suggested.

2. EXACT AMPLITUDES ON THE MODEL

Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be the momenta of the two nucleons n_1 and n_2 (distinguished only by the magnitudes of these momenta, but otherwise identical) in the laboratory frame in which the infinitely heavy nucleus A is at rest. The separable interactions V_1, V_2, and V_{12} in this 3-particle space, have the following structures

$$\langle \mathbf{P}_1\mathbf{P}_2|V_1|P_1'P_2'\rangle = -\tfrac{1}{2}(\lambda_1/M)g(P_1)g(P_1')\delta(\mathbf{P}_2-\mathbf{P}_2'), \quad (2.1)$$

with a similar expression for V_2, and

$$\langle P_1P_2|V_{12}|P_1'P_2'\rangle = -(\lambda/M)f(p)f(p')\delta(\mathbf{P}-\mathbf{P}'), \quad (2.2)$$

where

$$2\mathbf{p}=\mathbf{P}_1-\mathbf{P}_2, \quad \mathbf{P}=\mathbf{P}_1+\mathbf{P}_2, \quad (2.3)$$

with corresponding definitions for $\mathbf{p}'$ and $\mathbf{P}'$. The binding energy α^2/M of the deuteron is given in terms of (2.2), in the usual way[12] by

$$\lambda^{-1}=4\pi\int_0^\infty q^2dq\, f^2(q)(q^2+\alpha^2)^{-1}. \quad (2.4)$$

Similarly, the binding energy $\alpha_1^2/2M$ of either nucleon in the (single) bound state A' of A and n is given by the solution of the two-body equation

$$(P_1^2+\alpha_1^2)\phi(\mathbf{P}_1)=\lambda_1 g(P_1)\int d\mathbf{q}g(q)\phi(\mathbf{q}), \quad (2.5)$$

whence

$$\lambda_1^{-1}=4\pi\int_0^\infty q^2dq\, g^2(q)(q^2+\alpha_1^2)^{-1}. \quad (2.6)$$

In terms of these binding energies, the total energy E of the full three-body system is given by

$$2ME=\tfrac{1}{2}k^2-2\alpha^2=k_1^2-\alpha_1^2, \quad (2.7)$$

where $\mathbf{k}$ is the separation momentum between A and d, and $\mathbf{k}_1$ is the corresponding quantity between the stripped nucleon and A'. The three-body wave function $\Psi(\mathbf{P}_1,\mathbf{P}_2)$ can now be determined from the Schrödinger equation

$$(\tfrac{1}{2}M^{-1}P_1^2+\tfrac{1}{2}M^{-1}P_2^2+V_1+V_2+V_{12}-E)\Psi=0, \quad (2.8)$$

where, for any operator K,

$$K\Psi(\mathbf{P}_1,\mathbf{P}_2)=\int d\mathbf{Q}_1d\mathbf{Q}_2\langle\mathbf{P}_1\mathbf{P}_2|K|\mathbf{Q}_1\mathbf{Q}_2\rangle\Psi(\mathbf{Q}_1,\mathbf{Q}_2). \quad (2.9)$$

Following the techniques of I, the solution of (1.8) which is symmetrical in the momenta $\mathbf{P}_1$ and $\mathbf{P}_2$ may be read off as

$$D(E)\Psi=g(P_1)G(\mathbf{P}_2)+g(P_2)G(\mathbf{P}_1)+2f(p)F(\mathbf{P}), \quad (2.10)$$

where

$$D(E)=P_1^2+P_2^2-2EM=\tfrac{1}{2}P^2+2p^2-2EM. \quad (2.11)$$

The single-parameter functions G and F satisfy the equations

$$[\lambda_1^{-1}-h_1(P_2)]G(\mathbf{P}_2)=2\int d\mathbf{q}K(\mathbf{P}_2,\mathbf{q})F(\mathbf{q})$$
$$+\int d\mathbf{q}g(P_2)g(q)G(\mathbf{q})(q^2+P_2^2-2ME-i\epsilon)^{-1}, \quad (2.12)$$

$$[\lambda^{-1}-h(P)]F(\mathbf{P})=2\int d\mathbf{q}K(\mathbf{q},\mathbf{P})G(\mathbf{q}), \quad (2.13)$$

[9] C. Lovelace, Phys. Rev. **135**, B1225 (1964). This paper also contains a very complete list of references to calculations with separable potentials.

[10] L. D. Faddeev, Zh. Eksperim i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].

[11] R. Sugar and R. Blankenbecler, Phys. Rev. **135**, B472 (1964).

[12] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954).

where

$$h_1(P_2)=\int d\mathbf{q}g^2(q)(P_2^2+q^2-2ME-i\epsilon)^{-1}, \quad (2.14)$$

$$h(P)=\int d\mathbf{q}f^2(q)(q^2+\tfrac{1}{4}P^2-ME-i\epsilon)^{-1} \quad (2.15)$$

and

$$K(\mathbf{x},\mathbf{y})=g(\mathbf{x}-\mathbf{y})f(\mathbf{x}-\tfrac{1}{2}\mathbf{y}) \times(y^2-2\mathbf{x}\cdot\mathbf{y}+2x^2-2ME-i\epsilon)^{-1}. \quad (2.16)$$

Using (2.4), (2.7), and (2.15), it is easily seen that the factor $\lambda^{-1}-h(P)$ multiplying $F(\mathbf{P})$ in (2.13) vanishes when $P^2=k^2$ so that one may write

$$\lambda^{-1}-h(P)\equiv(P^2-k^2)l(P). \quad (2.17)$$

Similarly, the definition

$$\lambda_1^{-1}-h_1(P_2)\equiv(P_2^2-k_1^2)l_1(P_2) \quad (2.18)$$

extracts the zero of the corresponding factor on the left-hand side of (2.12).

The meaning of the poles in the functions $F(p)$ and $G(p)$ becomes clear in terms of their respective interpretations on the lines of I. Thus $F(\mathbf{P})$ is the wave function of d with respect to A, and $G(\mathbf{P}_2)$ that of n_2 with the composite A' of A and n_1. The internal wave functions of the composites d and A' are of course described by the factors $2f(p)/D(E)$ and $g(P_1)/D(E)$, respectively. The pole of $F(\mathbf{P})$ at $P^2=k^2$ therefore incorporates the physical condition that at infinite separation, the momentum of d with respect to A is $\mathbf{k}$. Similarly, the pole of $G(\mathbf{P}_2)$ at $P_2^2=k_1^2$ is associated with a momentum $\mathbf{k}_1$ at infinite separation of n_2 from A'. For an incident deuteron of momentum $\mathbf{k}$ we have thus the boundary condition

$$F(\mathbf{P})=(2\pi)^3\delta(\mathbf{P}-\mathbf{k})+4\pi a(\mathbf{P})(P^2-k^2-i\epsilon)^{-1}, \quad (2.19)$$

where $a(\mathbf{P})$ with $P^2=k^2$ is the elastic scattering amplitude of d by A. The deuteron stripping amplitude $b(\mathbf{k}_1)$ is similarly defined by the boundary condition

$$G(\mathbf{P}_2)=4\pi b(\mathbf{P}_2)(P_2^2-k_1^2-i\epsilon)^{-1}, \quad (2.20)$$

in conjunction with (2.19). Similarly, the elastic scattering amplitude $b(\mathbf{P}_2)$ with $P_2^2=k_1^2$ of a nucleon by A' is incorporated in the boundary condition

$$G(\mathbf{P}_2)=(2\pi)^3\delta(\mathbf{P}_2-\mathbf{k}_1)+4\pi b(\mathbf{P}_2)(P_2^2-k_1^2-i\epsilon)^{-1}, \quad (2.21)$$

where the plane wave term of (2.19) must now be absent from $F(\mathbf{P})$.

These results, being all exact within the model, enable the elastic scattering [$a(\mathbf{P})$] and stripping [$b(\mathbf{k}_1)$] amplitudes of d with respect to A to be calculated from the following integral equations obtained from the substitution of (2.19) and (2.20) in (2.12) and (2.13)

$$l_1(P_2)b(\mathbf{P}_2)=4\pi^2K(\mathbf{P}_2,\mathbf{k}) +\int d\mathbf{q}K_0'(\mathbf{P}_2,\mathbf{q})b(\mathbf{q}) +2\int d\mathbf{q}K(\mathbf{P}_2,\mathbf{q})a(\mathbf{q})(q^2-K^2-i\epsilon)^{-1}, \quad (2.22)$$

$$l(P)a(\mathbf{P})=2\int d\mathbf{Q}K(\mathbf{Q},\mathbf{P})b(\mathbf{Q})(Q^2-k_1^2-i\epsilon)^{-1}, \quad (2.23)$$

where $K(\mathbf{x},\mathbf{y})$ is already defined as in (2.16), and $K_0'(\mathbf{P}_2,\mathbf{q})$ is a pure s-wave kernel given by

$$K_0'(\mathbf{P}_2,\mathbf{q})=g(P_2)g(q)(q^2-k_1^2-i\epsilon)^{-1} \times(q^2+P_2^2+\alpha_1^2-k_1^2-i\epsilon)^{-1}. \quad (2.24)$$

A partial-wave decomposition of Eqs. (2.22) and (2.23) now goes through in the usual way according to

$$K(\mathbf{x},\mathbf{y})=\sum_0^\infty K_l(x,y)(2l+1)P_l(\hat{x}\cdot\hat{y}), \quad (2.25)$$

$$b(\mathbf{P}_2)=\sum_0^\infty b_l(P_2)(2l+1)P_l(\hat{P}_2\cdot\hat{k}), \quad (2.26)$$

$$a(\mathbf{P})=\sum_0^\infty a_l(P)(2l+1)P_l(\hat{P}\cdot\hat{k}), \quad (2.27)$$

where $\hat{k}$, the direction of the incident deuteron, is taken as the polar axis. The corresponding equations for the lth partial-wave amplitudes $b_l(P_2)$ and $a_l(P)$ are

$$l_1(P_2)b_l(P_2) =4\pi\delta_{l0}\int_0^\infty q^2dq\,K_0'(P_2,q)b_l(q) +4\pi^2K_l(P_2,k)+8\pi\int_0^\infty q^2dqK_l(P_2,q) \times a_l(q)(q^2-K^2-i\epsilon)^{-1}, \quad (2.28)$$

$$l(P)a_l(P) =8\pi\int_0^\infty Q^2dQ\,K_l(Q,P)b_l(Q)(Q^2-k_1^2-i\epsilon)^{-1}. \quad (2.29)$$

The physical partial-wave amplitudes for d-A scattering and stripping are now, respectively, $a_l(k)$ and $b_l(k_1)$. The functions $K_l(x,y)$ can be explicitly calculated for simple shapes of $f(p)$ and $g(p)$. For example, with

$$f(p)=(\beta^2+p^2)^{-1}, \quad g(p)=(\beta_1^2+p^2)^{-1}, \quad (2.30)$$

the formulas for K_l, $l(P)$, and $l_1(P_2)$ are listed in the Appendix.[13]

[13] Similar formulas have been given by R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. **136**, B650 (1964).

3. THE DWBA STRIPPING AMPLITUDE

To simulate the distorted-wave Born approximation within this model, we may again appeal to the interpretation of the various terms of Ψ, as given by (2.10). Thus, the term

$$\psi^{(1)}=2f(p)D^{-1}(E)F(\mathbf{P}) \tag{3.1}$$

represents the part of the wave function which takes full account of the distortion of the deuteron wave function in the field of the nucleus. However, (3.1) still contains some effect of the *coupling* of the internal motion of d with its bodily motion, through the P dependence of the factor

$$\tfrac{1}{2}D(E)=\tfrac{1}{4}(P^2-k^2)+(p^2+\alpha^2), \tag{3.2}$$

which goes against the spirit of DWBA. This P dependence may be dropped through the approximation of putting P^2 on the "energy-shell," viz., $P^2=k^2$ in (3.2), so that the (unnormalized) DWBA wave function of the deuteron is finally given by

$$\psi^{\mathrm{DW}}(d)=\phi_d(p)[(2\pi)^3\delta(\mathbf{P}-\mathbf{k})+4\pi a(\mathbf{P})(P^2-k^2-i\epsilon)^{-1}], \tag{3.3}$$

where

$$\phi_d(p)=f(p)(p^2+\alpha^2)^{-1} \tag{3.4}$$

is the deuteron internal wave function and the boundary condition (2.19) has been incorporated in (3.3). In a similar way, the distorted wave function of the nucleon n_2 with respect to the bound state of A and n_1 is obtainable from the term

$$\psi^{(2)}=g(P_1)D^{-1}(E)G(\mathbf{P}_2). \tag{3.5}$$

The decoupling of the internal motion of A' is again effected through the replacement

$$D(E)=\alpha_1^2+P_1^2+(P_2^2-k_1^2)\approx\alpha_1^2+P_1^2, \tag{3.6}$$

so that using the boundary condition (2.21), we have

$$\psi^{\mathrm{DW}}(n_2)=\phi_{A'}(P_1)[(2\pi)^3\delta(\mathbf{P}_2-\mathbf{k}_1)+4\pi b(\mathbf{P}_2)(P_2^2-k_1^2-i\epsilon)^{-1}], \tag{3.7}$$

where

$$\phi_{A'}(P_1)=g(P_1)(P_1^2+\alpha_1^2)^{-1} \tag{3.8}$$

is the internal wave function of A'.

The foregoing considerations show how the present model can accommodate DWBA without explicitly invoking an "optical model" for each of the deuteron and nucleon motions, in the sense that it is traditionally used for practical stripping calculations. The "optical potential" in this approach is of course present, but in a highly implicit form. Thus the inelastic effects on the distorted deuteron wave function $F(\mathbf{P})$ are taken into account to the extent of inclusion of the stripping and breakup reaction channels, in addition to the elastic channel. Similarly, the "optical potential" for the distorted function $G(\mathbf{P}_2)$ of n_2 with respect to A' includes the (inelastic) effects of the pickup and breakup reaction channels. On the other hand, in a practical stripping calculation with a medium-sized nucleus, most of the contribution to the optical potential arises from the finite size and internal structure of the nucleus, and the latter has no analog in this model.

To proceed further, the DWBA stripping amplitude which is defined by

$$b^D(\mathbf{k}_1)=C\langle\psi^{\mathrm{DW}}(n_2)|V_{12}|\psi^{\mathrm{DW}}(d)\rangle, \tag{3.9}$$

is after some trivial integrations reducible to the form

$$b^D(\mathbf{k}_1)=C\int\int d\mathbf{P}d\mathbf{P}_2 g(P_1)f(p)(P_1^2+\alpha_1^2)^{-1}\times[(2\pi)^3\delta(\mathbf{P}_2-\mathbf{k}_1)+4\pi b^*(\mathbf{P}_2)(P_2^2-k_1^2+i\epsilon)^{-1}]\times[(2\pi)^3\delta(\mathbf{P}-\mathbf{k})+4\pi a(\mathbf{P})(P^2-k^2-i\epsilon)^{-1}], \tag{3.10}$$

where

$$\mathbf{p}=\tfrac{1}{2}\mathbf{P}-\mathbf{P}_2,\quad \mathbf{P}_1=\mathbf{P}-\mathbf{P}_2.$$

The constant C must be adjusted so that the lowest Born approximation to (3.10) normalizes to[14]

$$b^B(\mathbf{k}_1)=4\pi^2 l_1^{-1}(k_1)K(\mathbf{k}_1,\mathbf{k}), \tag{3.11}$$

in agreement with the inhomogeneous term of (2.22). This gives, according to Eqs. (2.4) and (2.16),

$$C=(2\pi)^{-4}l_1^{-1}(k_1). \tag{3.12}$$

A straightforward integration of (3.10) yields

$$l_1(k_1)b^D(\mathbf{k}_1)=l_1(k_1)b^B(\mathbf{k}_1)+2\int d\mathbf{P}\,K(\mathbf{k}_1,\mathbf{P})a(\mathbf{P})(P^2-k^2-i\epsilon)^{-1}+2\int d\mathbf{P}_2K''(\mathbf{P}_2,\mathbf{k})b^*(\mathbf{P}_2)(P_2^2-k_1^2+i\epsilon)^{-1}+\pi^{-2}\int\int d\mathbf{P}d\mathbf{P}_2K''(\mathbf{P}_2,\mathbf{P})b^*(\mathbf{P}_2)a(\mathbf{P})\times(P_2^2-k_1^2+i\epsilon)^{-1}(P^2-k^2-i\epsilon)^{-1}, \tag{3.13}$$

where

$$K''(\mathbf{x},\mathbf{y})=f(\mathbf{x}-\tfrac{1}{2}\mathbf{y})g(\mathbf{x}-\mathbf{y})[(\mathbf{x}-\mathbf{y})^2\pm\alpha_1^2]^{-1}. \tag{3.14}$$

Using Eq. (2.22), the first two terms on the right of (3.13) are re-expressible as

$$l_1(k_1)b(\mathbf{k}_1)-\int d\mathbf{q}K_0'(k_1,q)b(\mathbf{q}). \tag{3.15}$$

As for the last two terms of (3.13), their meaning be-

[14] The reason why a normalization is required in Eq. (3.9) is that the functions $\psi^{\mathrm{DW}}(d)$ and $\psi^{\mathrm{DW}}(n)$ defined by (3.3) and (3.7), respectively, are not normalized as they stand. On the other hand, the exact stripping amplitude $b(\mathbf{P}_2)$ defined by (2.20) is a fully normalized quantity, whose Born term, (3.11) can be read off from Eq. (2.22). The constant C in Eq. (3.9) which expresses the normalization of the DWBA amplitude is most easily determined from the requirement that the Born approximations to the exact and DWBA amplitudes agree.

comes slightly more transparent if an approximation is made in the K'' functions. A comparison of (2.16) and (3.14) shows that the "propagator factor" in $K''(\mathbf{P}_2,\mathbf{k})$ differs from the corresponding factor in $K(\mathbf{P}_2,\mathbf{k})$ to the extent of an additive term $(P_2^2-k_1^2)$ which is present in the latter. The same additive term accounts for the difference between $K''(\mathbf{P}_2,\mathbf{P})$ and $K(\mathbf{P}_2,\mathbf{P})$. Replacement of K'' by K in the last two terms of (3.13), therefore, amounts to a modification of the propagators of K'' to the extent indicated, thus making them different only off the energy shell for $\mathbf{P}_2$. This may not be a serious error since (1) $\mathbf{P}_2$ is an integration variable in these terms and (2) the poles of the modified propagators of K'' that would now be present in the region of integration are far removed from the poles of the other factors $(P_2^2-k_1^2+i\epsilon)^{-1}$ or $(P^2-k^2-i\epsilon)^{-1}$ which produce the dominant effects. Making these replacements, the resultant expressions can be simplified with the use of Eq. (2.23), so that (3.13) reduces to the more transparent form

$$
\begin{aligned}
l_1(k_1)b^D(\mathbf{k}_1)
&=l_1(k_1)b(\mathbf{k}_1)+l(k)a^*(\mathbf{P})\,|_{(P^2=k^2)} \\
&\quad-\int d\mathbf{q}\,K_0'(k_1,q)b(\mathbf{q}) \\
&\quad+\tfrac{1}{2}\pi^{-2}\int d\mathbf{Q}l^*(Q)\,|a(\mathbf{Q})|^2(Q^2-k^2-i\epsilon)^{-1}.
\end{aligned}
\tag{3.16}
$$

A partial-wave analysis of this equation as in (2.25)–(2.29), yields the DWBA partial amplitudes $b_l^D(k_1)$ in the form

$$
\begin{aligned}
l_1(k_1)b_l^D(k_1)
&=l_1(k_1)b_l(k_1)+l(k)a_l^*(k) \\
&\quad-4\pi\delta_{l0}\int_0^\infty q^2dq\,K_0'(k_1,q)b_l(q) \\
&\quad+2\pi^{-1}\delta_{l0}\int_0^\infty Q^2dQ\,l^*(Q)\sigma_{\rm el}(Q)(Q^2-k^2-i\epsilon)^{-1},
\end{aligned}
\tag{3.17}
$$

where

$$\sigma_{\rm el}(Q)=\sum_0^\infty(2l+1)\,|a_l(Q)|^2 \tag{3.18}$$

may be interpreted as the total elastic scattering cross-section *off the energy shell.* This is obvious since according to Eq. (2.29) the quantities $a_l(Q)$ of Eq. (3.18) are defined off the energy shell except when $Q^2=k^2$. The analogy of the last term of (3.17) with a dispersion formula is thus only a formal one.

4. VALIDITY OF DWBA FOR THE MODEL

The relations (3.16) and (3.17) express the deviations of the DWBA amplitudes from their exact counterparts. The "correction terms" in these formulas are therefore convenient for a discussion of the validity of DWBA within the model. The first correction term of either equation is the one that is proportional to the complete elastic scattering amplitude $a(\mathbf{k})$ in (3.16) and the one with the corresponding partial amplitude in (3.17). The last two corrections are pure isotropic effects ($l=0$), as is clear from (3.17). Of these the last one involves an integration over the total elastic cross section on and off the energy shell. An interesting feature of this term is that it does not involve the potentials at all [except through the normalization factor $l(Q)$], so that this part of the "correction" may well be independent of the particular model considered. On the other hand, the third term on the right of Eq. (3.16) or (3.17), is a model-dependent "correction." Indeed, the latter may be interpreted as arising from an off-shell *exchange* scattering of an s-wave nucleon (n_2) by the bound state of A with the other nucleon (n_1). While the appearance of the s wave in this term is traceable to the assumption of factorable potentials between pairs, the isotropy of the last term in (3.16) or (3.17) is a more fundamental effect, involving, as it does, the *total* cross section.

As for the magnitudes of these corrections, it is instructive and indeed possible to draw certain qualitative conclusions based on the mere assumption of usual short-range potentials. For this purpose, it is useful to recall the numerical results obtained recently by Bhasin, Schrenk, and Mitra[15] for low-energy n-d scattering. The n-d system, which represents a true three-body problem, can be counted upon to provide important information as to the nature of the results expected from "three-body approximations" to more complicated systems. The present model has some obvious points of similarity to the n-d system except for the neglect of recoil effects and nonidentity of the nucleus A with either nucleon. For an n-d system of course, stripping and elastic scattering are formally identical processes, and cannot be distinguished experimentally. On the other hand, there is a profound difference in their physical mechanisms, a correct interpretation of which should be of great value in the present context. Indeed, it was shown in III that n-d scattering proceeds via either of two mechanisms, (1) exchange of a nucleon line (Fig. 1 of III), termed as "exchange scattering," and (2) a "triangle diagram" (Fig. 2 of III) in which two nucleon lines are simultaneously exchanged between the two particles, termed "potential scattering." The matrix elements of potential scattering are characterized by the appearance of integrals with a greater number of "shape factors" [like $g(p)$ or $f(p)$] than those of exchange scattering. Because of the short range of the forces concerned, the former are expected to be much smaller than the latter. This was indeed found to be the case in III, where the

[15] V. S. Bhasin, G. L. Schrenk, and A. N. Mitra, Phys. Rev. **137**, B398 (1965); referred to as III.

contribution of potential scattering to the quartet scattering length turned out to be almost entirely negligible ($\ll 1\%$) compared with that of exchange scattering which already gives beautiful agreement with experiment.[15,16] This result has a direct bearing on the present model when it is recognized that our stripping amplitude $b(\mathbf{k}_1)$ corresponds precisely to the n-d exchange scattering amplitude, and the elastic A-d scattering is the exact analog of "potential scattering" in the n-d problem. Now, since the Yamaguchi-type potential shapes[12] listed in the appendix are essentially the same as those used in III, except for recoil effects, it is clear that for such shapes at least, the relative numerical magnitudes of the various terms in the present problem should have a close parallel to the corresponding estimates in the n-d case. Thus it should be possible to estimate the order of magnitude of each term in (3.16) or (3.17) simply by examining the structure of the integral in each (in terms of shape factors). The stripping amplitude $b(\mathbf{k}_1)$ involving the smallest number of shape factors must be the dominant term. The elastic scattering amplitude $a(\mathbf{k})$ which, according to (2.23), is given by an integral involving $b(\mathbf{Q})$ as well as the shape factors, must, on the three-body model, be an order of magnitude smaller than the stripping amplitude. The same remarks apply to the third and fourth terms on the right of (3.16), vis-à-vis the stripping amplitude. Thus we conclude that *within the framework of the three-body model, DWBA is a valid approximation.* An identical result holds for (3.17) expressing the corrections to the DWBA partial amplitudes, including the (algebraically worst) case of $l=0$.

The next question is whether this model has any bearing on the validity of DWBA stripping calculations in *actual* nuclei. The principal difficulty of confronting this model with experiment lies in the absence of size or structure in our model nucleus. A direct manifestation of nuclear size in the context of experiment is the appearance of elastic cross sections several orders of magnitude larger than the stripping cross sections.[17] Even for a d-d reaction, the elastic cross section bears a ratio of 10–12 to the stripping cross section.[18] To understand this fact it must be remembered that most of the elastic cross section is a result of diffraction scattering from the rim of the nucleus, a process to which many l values contribute. This feature is absent in our model, not only because of the assumption of a point nucleus but also because of the assumption of s-wave interactions in pairs. As a result of these assumptions, the conclusion in the previous paragraph, concerning the relative magnitudes of the elastic and stripping cross sections, is the exact opposite of what holds in practice. This is a formidable handicap which must be overcome before any conclusion bearing on the actual situation may be attempted on the basis of the model.

A possible solution may lie in the consideration of the partial wave amplitudes rather than of the complete amplitudes. For this purpose we must fall back on Eq. (3.17) which, for $l\neq 0$, takes a particularly simple form, viz.,

$$b_l^D(k_1)=b_l(k_1)+N_1^2N^{-2}a_l(k)\,, \tag{4.1}$$

where N^2 and N_1^2 are just the normalization constants for the internal wave functions of the d and A' states according to

$$N^{-2}=l(k)=\int d\mathbf{q}\, f^2(q)(q^2+\alpha^2)^{-2}\,, \tag{4.2}$$

$$N_1^{-2}=l_1(k_1)=\int d\mathbf{Q}\, g^2(Q)(Q^2+\alpha_1^2)^{-2}\,. \tag{4.3}$$

From (4.1), the criterion for the validity of DWBA is deduced as

$$N^{-2}|a_l(k)|\ll N_1^{-2}|b_l(k_1)|\,, \tag{4.4}$$

i.e., the lth partial amplitude for elastic scattering should be small compared with the corresponding stripping amplitude. Of course, within our three-body model, the inequality (4.4) is valid. However, we now want to explore the possibility of using (4.4) as a probe in actual physical situations. A particular advantage of this form lies in the appearance of physically measurable quantities only (possible only for $l\neq 0$). Moreover, unlike the conditions on the *complete* amplitudes (which, as we have just seen, can never be satisfied for physical situations), condition (4.4) involves amplitudes for only a particular l state, and it should be useful to see if this condition can be reconciled with the difficulties of finite-size effects. Now a basic difference between the stripping and elastic-scattering mechanisms is that the former depends on the shape of the wave functions in the nuclear interior while the latter depends only on the logarithmic derivatives of the radial wave functions at the nuclear surface. One may therefore expect that the l dependence of a partial scattering amplitude is much stronger than that of a partial stripping amplitude. Thus it is likely that elastic-scattering amplitudes die off faster with l than do the stripping amplitudes. In this way one may try to overcome the size effects by choosing a sufficiently large l in (4.4) before putting it to "experimental test." The precise magnitude of the critical l needed for a particular case must, of course, depend on the nuclear size, the larger the nucleus the larger the l value required. Of course, it might then be argued that since, for large l, the partial-wave scattering amplitude falls off rapidly anyway, the condition (4.4), which would be automatically satisfied at some stage, would hardly serve the purpose intended for it. To answer this objection it should be remembered that the effects due to the size and structure of the deuteron,

[16] A. G. Sitenko and V. F. Kharchenko, Nucl. Phys. **49**, 15 (1963); see also, R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. Letters **13**, 574 (1964).

[17] See, for a recent reference, J. Testoni *et al.*, Nucl. Phys. **50**, 479 (1964).

[18] e.g., L. Lyons *et al.*, Phys. Letters **3**, 359 (1963).

which have been fully taken into account in our model, have also played a role in the derivation of (4.1)–(4.4). Indeed, in a sense our model is largely complementary to the assumptions made in normal DWBA calculations, where the greatest emphasis has generally been on the proton or deuteron optical potential with respect to the (finite-sized) nucleus, with successive refinements like spin-orbit, tensor, finite-range, nonlocal, collective, etc., effects. On the other hand, conventional DWBA calculations have hardly taken account of the deuteron internal structure and polarization effects on the stripping and scattering amplitudes. Our purpose in choosing a large l is just to overcome the nuclear size effects but not the effects of deuteron structure which in fact we want to test. We claim that this is possible through the artifice of partial-wave amplitudes if we consider a fairly *small* and *tight* nucleus. A good example might be provided by the α particle, for which scattering and stripping data are available,[19,20] though much more data exist for medium-heavy nuclei.

Having decided on some reasonably large l values, in relation to a particular nucleus, an indirect experimental test of (4.4) may be suggested on the following lines. Suppose that a "good" optical potential has been fitted to each of the data for elastic deuteron and proton scattering by a given nucleus. Suppose further that a successful DWBA fit, using the same potentials, has been found for the (d,p) reaction data on the same nucleus. The partial wave DWBA amplitudes (with the above restrictions on l), which can now be *calculated* for both scattering and stripping, may be taken to represent the "experimental data" for testing (4.4). While such a program is not in the conventional DWBA spirit (DWBA calculations have been generally concerned with differential cross sections rather than partial waves), it can certainly be regarded as well within its scope. A possible snag could arise out of the ambiguities in the optical potential parameters all of which fit elastic-scattering data equally well. The sensitivity of the stripping results to such ambiguities was studied for medium and heavy nuclei by Smith[21] who found that these could cause deviations in the so-called "spectroscopic factors" to the extent of as much as 200%. A second source of ambiguity is that in many cases the available data are incomplete, leading, e.g., to uncertainties in the normalizations chosen for the scattering data.[21] A point in favor of a condition like (4.4), however, is that it is a highly qualitative statement, not likely to be strongly affected by such variations. As such it may yet be worthwhile to confront it with experiment in the sense described above.

To summarize, we have shown that within our three-body model of stripping, DWBA is a valid approximation. The model also predicts a condition (4.4), in terms of partial-wave amplitudes for scattering and stripping, which is amenable to an indirect experimental test for actual situations. A possible way of overcoming the nuclear size effect in actual cases is suggested through the use of large l values in the above condition, which is most likely to remain unaffected by ambiguities in the conventional DWBA parameters.

[19] See, e.g., H. J. Erramuspe and R. J. Slobodrian, Nucl. Phys. **49**, 65 (1963), which gives references to the earlier experimental papers.

[20] For a three-body optical potential approach to scattering, see J. L. Gammel, B. J. Hill, and R. M. Thaler, Phys. Rev. **119**, 267 (1960).

[21] W. R. Smith, Phys. Rev. **137**, B913 (1965).

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APPENDIX

The results of this appendix refer to the form factors defined by Eq. (2.30) of the text.

With the notation

$$A_1(x,y)=x^2+y^2+\beta_1^2\,,$$
$$A_2(x,y)=x^2+\tfrac{1}{4}y^2+\beta^2\,, \qquad (A1)$$
$$A_3(x,y)=2x^2+y^2-2ME-i\epsilon\,,$$

and

$$a_i=\tfrac{1}{2}A_i/xy\,, \qquad (A2)$$

Eqs. (2.16) and (2.30) of the text yield

$$4x^3y^3K(\mathbf{x},\mathbf{y})=\sum_{ijk}(a_i-a_j)^{-1}(a_i-a_k)^{-1}(a_i-\mu)^{-1}$$
$$(i,\,j,\,k=1,\,2,\,3)\,, \qquad (A3)$$

where

$$\mu=\hat{x}\cdot\hat{y}\,.$$

The expansion

$$(a-\mu)^{-1}=\sum_l(2l+1)Q_l(a)P_l(\mu) \qquad (A4)$$

finally leads to the expression

$$K_l(x,y)=x^{-1}y^{-1}\sum_{ijk}(A_i-A_j)^{-1}(A_i-A_k)^{-1}Q_l(a_i)\,. \qquad (A5)$$

Here only $Q_l(a_3)$ has a branch cut on which it must be evaluated in the sense (A1). The remaining functions are evaluated as

$$l(P)=\tfrac{1}{4}\pi^2\beta^{-1}(\alpha+\gamma+2\beta)(\alpha+\beta)^{-2}$$
$$\times(\beta+\gamma)^{-2}(\alpha+\gamma)^{-1}\,, \qquad (A6)$$

$$l_1(P_2)=\pi^2\beta_1^{-1}(\alpha_1+\gamma_1+2\beta_1)(\alpha_1+\beta_1)^{-2}$$
$$\times(\beta_1+\gamma_1)^{-2}(\alpha_1+\gamma_1)^{-1}\,, \qquad (A7)$$

where

$$\gamma=\gamma(k^2+i\epsilon)=[\alpha^2+\tfrac{1}{4}(P^2-k^2-i\epsilon)]^{1/2}\,, \qquad (A8)$$

$$\gamma_1=\gamma_1(k_1^2+i\epsilon)=[\alpha_1^2+P_2^2-k_1^2-i\epsilon]^{1/2}\,, \qquad (A9)$$

thus specifying how these functions are defined on their respective branch cuts.