Dynamics of a Broken SU_N Symmetry for the Oscillator

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All states of a one-dimensional harmonic oscillator are represented by a special unitary representation of the noncompact 2+1 Lorentz group. The direct product of N such representations leads to a degeneracy which is represented by the group SU_N, whereas all states of the N-dimensional oscillator are represented by the noncompact unitary group $NU_{N+1}{}^N$ whose Casimir operator determines the energy spectrum. The anharmonic oscillator is represented by a broken symmetry and mass-splitting formulas are obtained. It is shown how new quantum numbers arise from the direct products of basic dynamical groups, corresponding to composite structures.

I. INTRODUCTION

The occurrence of an SU_n symmetry is conveniently related to the existence of n basic objects (e.g., fields) and to invariance under the transformations between these objects.[1] This would be satisfactory if the symmetry were exact. In the case of broken symmetry it becomes necessary to consider the dynamics of the constituent basic objects. One is led to consider all energy or mass levels of the system, not only the levels of a given energy.

The example of the N-dimensional oscillator is admirably suited to study the dynamical origin of the broken symmetry. For this purpose we represent the one-dimensional oscillator by the simplest unitary representation of the 2+1 Lorentz group. This noncompact group gives *all* the energy levels of the oscillator and their spacing. We then take the direct products of N such oscillators (corresponding to forming a composite system) and show how the unitary symmetry occurs to describe the degeneracy of each energy level of the total system. *All* energy levels of the total system and their spacings are now described by the noncompact unitary group in $(N+1)$ dimensions which contain the group of degeneracy SU_N as a subgroup. The Casimir operator of this large group determines the energy spectrum. If the frequencies of the constituent oscillators are slightly different from each other, we obtain a broken symmetry. This clarifies the symmetry group of the anharmonic oscillator which is always a broken SU_N; in the special case when the ratio of frequencies has definite values the level splitting becomes so large that some levels coincide with the next energy level in which case the apparent symmetry is higher.

Finally, the special values of the Casimir operators that occur are discussed and the applications of the results to other systems are indicated.

[1] Permutation invariance of three basic fields and some additional conservation laws on the irreducible states of the permutation group plus some specific form of interactions also lead to SU_3: Y. Yamaguchi, Phys. Letters 9, 281 (1964); D. E. Neville (to be published) and J. Schechter, Y. Ueda, and S. Okubo (to be published).

II. DYNAMICAL GROUP OF THE OSCILLATOR

The *dynamical group* of a quantum-mechanical system has been defined to be that noncompact group which gives *all* states of the system, that is, energy (or mass) spectrum as well as the degeneracy of the levels.[2,3] It is a geometric representation of the Hamiltonian formalism. We shall discuss the dynamical group for the harmonic and anharmonic oscillator, show the relation of the group of degeneracy (symmetry group of the energy) to dynamics and obtain a mass splitting (or broken symmetry).

We represent the one-dimensional harmonic oscillator by the special unitary representations of the 2+1 Lorentz group as follows: The Lie algebra of the 2+1 Lorentz group[4] has three generators L_{12}, L_{13}, L_{23} with

$$[L_{\mu\nu},L_{\mu\lambda}]=-ig_{\mu\mu}L_{\nu\lambda};\quad g_{11}=g_{22}=-1,\, g_{33}=+1.$$

The group has unitary representations with a lower bound in the eigenvalue m of L_{12}, i.e., the discrete class $\mathfrak{D}^+(\phi)$, where ϕ is related to the eigenvalue of the Casimir operator Q: $Q=L_{12}(L_{12}+1)+2M^-M^+$, $M^\pm=(iL_{13}\pm L_{23})/\sqrt{2}$, the eigenvalue being $\phi(\phi+1)$. The lowest value of m in $\mathfrak{D}^+(\phi)$ is $\frac{1}{2}$ corresponding to $q=-\frac{1}{4}$ or $\phi=-\frac{1}{2}$. Similarly, in the representations $\mathfrak{D}^-(\phi)$, the highest possible eigenvalue of L_{12} is $m=-\frac{1}{2}$ with again $\phi=-\frac{1}{2}$. There is a special significance to these two representations: The states of the representation (Hilbert) space are obtained from the basic spinor representations ξ_1, ξ_2, by

$$|ab\rangle=N(ab)\xi_1{}^a\xi_2{}^b;\quad a=\phi+m,\, b=\phi-m. \qquad (2.1)$$

One can think of ξ_1 and ξ_2 as the basic constituents (or quarks) out of which all other states are obtained. Now for $\mathfrak{D}^-(-\frac{1}{2})$ or $\mathfrak{D}^+(-\frac{1}{2})$ we have one spinor component only because either a or b in Eq. (2.1) vanishes. In all other cases, we have some noninteger combination of both spinor components. In $\mathfrak{D}^+(-\frac{1}{2})$ the value of L_{12} ranges as

$$L_{12}=(n+\tfrac{1}{2}). \qquad (2.2)$$

[2] A. O. Barut, Phys. Rev. 135, B839 (1964).
[3] A. O. Barut and A. Böhm, Phys. Rev. (to be published).
[4] A. O. Barut and C. Fronsdal, Proc. Roy. Soc. (London) (to be published) and references therein.

We shall now interpret the energy of the oscillator as

$$E=\hbar\omega L_{12}. \tag{2.3}$$

The Hilbert space of the $\mathfrak{D}^+$ is spanned by the orthonormal set (2.1), or by the set

$$\langle n|n'\rangle=\delta_{nn'}. \tag{2.4}$$

The connection to ordinary quantum mechanics is obtained by introducing the new basis $|x\rangle$ such that

$$\langle x|n\rangle=N(n)e^{-x^2/2}H_n(x) \tag{2.5}$$

is the usual bound-state wave function. (See also Sec. III.)

The quantity $\frac{1}{2}$ in Eq. (2.2) is of course the zero-point energy and this factor cannot be anything else in $\mathfrak{D}^+(-\frac{1}{2})$. We remark that this factor has observable consequences.[5] In Hamiltonian dynamics this point is not unique: If one writes H as a Wick normal product

$$:H:=-(1/2m)(p+im\omega q)(p-im\omega q),$$

then the energy values are $\hbar\omega n$, $n=0, 1, 2, \cdots$, and the question arises as to which Hamiltonian is the correct one.[6]

The 2+1 Lorentz group is among the groups of motion of curved spaces (Riemann spaces). It is a two-dimensional space with constant curvature (embedded in a three-dimensional space). The generator L_{12} corresponds to rotations around the time axis. Thus, the simplest representation of the (2+1) Lorentz group is completely equivalent to the theory of one-dimensional harmonic oscillator.

The two other generators are related to L_{12} by the commutation relations

$$[L_{12},L_{13}]=iL_{23},\quad [L_{13},L_{23}]=-iL_{12},\quad [L_{23},L_{12}]=iL_{13}$$

and by the Casimir operator

$$Q=L_{12}^2-L_{13}^2-L_{23}^2.$$

In our case

$$Q=-\tfrac{1}{4}=L_{12}^2-L_{13}^2-L_{23}^2$$

or

$$E^2=(\hbar\omega L_{12})^2=-\tfrac{1}{4}\hbar^2\omega^2+\hbar^2\omega^2(L_{13}^2+L_{23}^2)$$
$$=-\tfrac{1}{4}\hbar^2\omega^2+(L'^2_{13}+L'^2_{23}).$$

If we now contract the group with respect to L_{12} by letting $L'_{13}=\epsilon L_{13}$, $L'_{23}=\epsilon L_{23}$, L_{12} unchanged and $\epsilon\to 0$ ($\hbar\omega\to 0$) we have

$$[L_{12},L'_{13}]=iL'_{23};\quad [L'_{23},L_{12}]=iL'_{13};\quad [L'_{13},L'_{23}]=0,$$

that is, the Euclidean group (i.e., flat space). Then we have separately

$$E=\text{invariant},\quad L'^2_{13}+L'^2_{23}=\text{invariant};$$

there is then no energy spectrum.

[5] For a review see K. Clusius, Die Chemie **56**, 241 (1943).
[6] C. P. Enz and A. Thellung, Helv. Phys. Acta **33**, 839 (1960).

III. COMPOSITE SYSTEMS—DIRECT PRODUCTS

We consider in this Section the direct product

$$\mathfrak{D}^{(+)}(-\tfrac{1}{2})\otimes\cdots\otimes\mathfrak{D}^{(+)}(-\tfrac{1}{2}). \tag{3.1}$$

The states of the direct product are clearly degenerate in L_{12}.

$$L_{12}=\sum_{i=1}^{N} L_{12}^{(i)}. \tag{3.2}$$

The degeneracy of the level can be obtained by simply counting the ways in which a value of L_{12} results from those of $L_{12}^{(i)}$. For $N=3$, the degeneracy of the nth level is

$$d=\tfrac{1}{2}(n+1)(n+2); \tag{3.3}$$

in general it is

$$d=(N+n-1)!/[n!(N-1)!]. \tag{3.4}$$

Equation (3.3) is the dimension of SU_3 representations for which one Casimir operator is zero: The dimension of the SU_3 representations is

$$d=\tfrac{1}{2}(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2) \tag{3.5}$$

and we obtain in (3.3) the representations $(\lambda_1,0)$ or $(0,\lambda_2)$. For each n there is a definite representation of SU_3. For $N=2$ we get all the representations of SU_2: $d=2j+1$; $j=0, \frac{1}{2}, 1, \cdots$.

We can also show explicitly how SU_N appears and identify the generators. From

$$[M^+,M^-]=L_{12},\quad [L_{12},M^\pm]=\pm M^\pm \tag{3.6}$$

and (3.2) we have [Note: $(M^+)^\dagger=-M^-$ for unitary representations.]

$$L_{12}=\textstyle\sum_i [M_i^+,M_i^-]=\sum_i (M_i^{+\dagger}M_i^+-M_i^+M_i^{+\dagger}), \tag{3.7}$$

which is essentially equivalent to the usual representation of the oscillator Hamiltonian in terms of the creation and annihilation operators. The usual creation and annihilation operators a and a^* are related to the generators M^+, M^- of the 2+1 Lorentz group by the equations

$$M^+=-(a^+/\sqrt{2})(L_{12}+\tfrac{1}{2})^{1/2},$$
$$M^-=(a/\sqrt{2})(L_{12}-\tfrac{1}{2})^{1/2}. \tag{3.8}$$

Hence

$$[L_{12},a]=a,\quad [L_{12},a^*]=-a^*,\quad [a_i,a_j^*]=\delta_{ij}; \tag{3.6'}$$

a and a^* in turn are essentially given by $(\partial/\partial x)+x$ and $(\partial/\partial x)-x$. Equation (3.7) is, as is well known, invariant under the transformation

$$M_i^{+\prime}=U_{ji}M_j^+, \tag{3.9}$$

where U_{ij} is an $N\times N$ complex unitary matrix, and

$$(M_i'^+)^\dagger=U_{ji}^*M_j^{+\dagger}, \tag{3.9'}$$

because

$$\textstyle\sum_i M_i'^+M_i'^{+\dagger}=\sum_{ijk} U_{ji}U_{ki}^*M_j^+M_k^{+\dagger}=\sum_j M_j^+M_k^{+\dagger}$$

and similarly for the second term. The commutation relations (3.6′) (but not 3.6) are clearly invariant under the unitary transformations.

Let us label the states in the product by $|n,\alpha\rangle$ where n is the eigenvalue of L_{12}. Then U acts on the degeneracy index α. Hence, instead of $L_{12}^{(i)}$, $L_{13}^{(i)}$, $L_{23}^{(i)}$ [$3N$ operators with N fixed Casimir operators $Q^{(i)}$] we can introduce the total operators L_{12}, L_{13}, L_{23} and the generators of U_N, or for infinitesimal transformations the generators of SU_N. The fundamental representation of SU_N is N dimensional, with the generators represented, say, by λ_j. We combine these with the N-dimensional (nonunitary) representation of the 2+1 group: μ_k. The commutators can be evaluated in this representation and the group generated by $\mu_k\lambda_j$ (with only one of the μ_k commuting with all the λ_j, namely L_{12}) is the noncompact unitary group of rank N

$$NU_{N+1}{}^{N}\,, \tag{3.10}$$

which appears then as the dynamical group of the N-dimensional oscillator whose special representations give *all* the levels with degeneracy as well as their spacing.

Example

Two-Dimensional Oscillator

We can express H in terms of the SU_2 generators[7]

$$L_{12}=2(\mathbf{J}^2+\tfrac{1}{4})^{1/2}=[(2J)^2+1]^{1/2}, \tag{3.11}$$

where $\mathbf{J}$ are the generators of $SU_2\sim\bar{S}\bar{O}_3$ (covering group).

The infinitely many states of the two-dimensional oscillator are represented by a special representation of $SU_3{}^2$, in which each and every representation of the SU_2 subgroup occurs only once.[8]

It is important that we get not only all the states but all the energy values of the spectrum as well, because L_{12} is in the algebra and one can change the energy by one of the group operations. The Casimir operator Q of $SU_3{}^2$ is of the form

$$L_{12}{}^2-(2\mathbf{J})^2+A^2=Q\,, \tag{3.12}$$

where A^2 is the square of the remaining generators of $SU_3{}^2$ and commutes with $L_{12}{}^2$ and J^2. In the special representation Eq. (3.12) is equivalent to (3.11). This is an explicit example of the method proposed earlier[2,9] that the energy or mass spectrum can be determined by the Casimir operator of a larger dynamical group.

IV. MASS SPLITTING. ANHARMONIC OSCILLATOR

Let us now assume that the frequencies ω in (2.3) of the constituents are different. Then, if we add the energies

$$E=\textstyle\sum_i E_i=\hbar\sum_i\omega_i L_{12}{}^{(i)}=\hbar\sum_i\omega_i[M_i{}^+,M_i{}^-]\,, \tag{3.13}$$

we do not have the strict unitary invariance (3.9) except for very special ratios of ω_i (see below). On the other hand, if we operate with $L_{12}{}^{(i)}$, rather than $E^{(i)}$, we can still talk of an exact SU_n symmetry, which is however broken if expressed in terms of $E^{(i)}$. For example, in the case $N=2$, we obtain the mass-splitting formula

$$\begin{aligned}E&=\omega_1[(2J)^2+1]^{1/2}+\lambda E^{(2)}\,,\quad \lambda=(\omega_2-\omega_1)/\omega_1\\&=E_0+\lambda' J_3\,,\end{aligned} \tag{3.14}$$

where E_0 is the energy of the state with $J_3=0$. Thus the energy splitting is proportional to J_3, the quantum number of the only subgroup of SU_2. In the special case, $\omega_2=2\omega_1$, for example, we have $\lambda=1$; the mass splitting becomes equal to $E^{(2)}$, which means that the level $E^{(2)}=1$, for example, coincides with one of the levels of the next value of the total energy. This explains why the degeneracy of the levels is now as 1, 1, 2, 2, 3, 3, $\cdots$, i.e., two series of representations of SU_2, one for n even, the other for n odd.[10]

A convenient way to express the broken symmetry is to write

$$(E_1/\omega_1)+(E_2/\omega_2)+\cdots+(E_N/\omega_N)=\text{invariant of } SU_N$$

with the subsidiary condition

$$E=E_1+E_2+\cdots+E_N\,. \tag{3.15}$$

Geometrically, if all ω_i are equal, all energy levels are on one plane of constant energy $E=\sum_i E_i$, otherwise the plane $\sum_i (E_i/\omega_i)$ is tilted with respect to the planes of constant energy, giving energy splitting.

V. BASIC SPINORS AND ANTISPINORS (QUARKS AND ANTIQUARKS)

Finally we have to ask why only special representaltions of the degeneracy group SU_N, or of the dynamica-group $NU_{N+1}{}^N$ occur and how these special representations are characterized. This is a rather general feature of the symmetry problem. Also for a symmetric rotator or the nonrelativistic H atom only special representations of O_4 symmetry, namely, those with one Casimir operator zero, occur.

The direct product of the representations of the 2+1 group is obtained from the multispinors.[4]

$$\xi_1{}^a\xi_2{}^b\eta_1{}^c\eta_2{}^d\cdots.$$

As we have noted, in the case of the representations $\mathfrak{D}^+(-\tfrac{1}{4})$, however, only one of the components enter: $\xi_1{}^a\eta_1{}^c$, $\cdots$. Thus, we do not generate all the representa-

[7] See also S. P. Alliluev, Zh. Eksperim. i Teor. Fiz. **33**, 200 (1957) [English transl.: Soviet Phys.—JETP **6**, 156 (1958)].

[8] See, for example, C. Fronsdal, Proc. Roy. Soc. (London) (to be published).

[9] A. O. Barut, in *Conference on Symmetry Principles at High Energy* (W. H. Freeman and Company, San Francisco, 1964); and Ref. 2.

[10] Ya. N. Demkov, Zh. Eksperim. i Teor. Fiz. **44**, 2007 (1963) [English transl.: Soviet Phys.—JETP **17**, 1349 (1963)].

tion of the final group. Moreover, for the 2+1 group, the spinors with upper and lower indices are equivalent so that one can get only those representations of the unitary group which are obtained only from one kind of spinor, i.e., representations by symmetric tensors. For example, for SU_3 or NU_3^2, multispinors formed out of the three-dimensional fundamental representations give only symmetric "quark" compounds. These are precisely the representations with one of the λ equal to zero in Eq. (3.5). All representations of SU_3 (NU_3^2) can be obtained by using both of the fundamental representations (i.e., quark-antiquark compounds). Because we start with one kind of 2+1 spinors, we see therefore why only special representations by symmetric tensors of the unitary groups are realized for the oscillator. Thus, for $N=3$, the interesting octet representation, for example, does not occur in the case of oscillator. To obtain an octet of SU_n, the basic dynamical objects, out of which one forms the compounds, must have at least two fundamental representations.

VI. APPLICATIONS TO OTHER SYSTEMS

It should be remarked that the essential features of the procedure we have followed are quite generally valid. Suppose we have a quantum-mechanical system with some energy or mass spectrum. If we form "direct products" of such systems we will get, in the same way as was discussed in Sec. III, a unitary degeneracy group and a larger noncompact dynamical group to describe all the states of the composite system. For the two- and three-dimensional oscillator we interpret part of the degeneracy with the spin. The original basic system can already have a spin degree of freedom.[3] In fact, we wish to start with a basic group which is larger than the 2+1 group and which has at least two fundamental spinor representations (corresponding to quarks and antiquarks), in order to get also the octet representation of SU_3, for example. The spin degeneracy will combine with the SU_N degeneracy due to the direct product to form a noncompact SU_{2N} group containing both the spin quantum number and the quantum numbers of the composite system.

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Function in Quantum Mechanics Which Corresponds to a Given Function in Classical Mechanics

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A generalization of the procedures of Weyl and McCoy for making transition from a classical function to a quantum-mechanical operator function corresponding to it has been carried out, in order to include terms involving more than one pair of conjugate variables. An application of the method is made to the case of two spinless interacting charges—the Darwin interaction. This result is verified by solving directly for the quantum-mechanical Hamiltonian operator, starting from equations which were derived from the classical equations by replacing the classical Poisson brackets by commutators and by requiring the equality of the energy and the Hamiltonian operator. An alternative expression for a quantized Hamiltonian, derived in the Appendix, reduces to the form given originally by Born and Jordan for a single set of conjugate variables.

I. INTRODUCTION

THE straightforward prescription for obtaining a quantum-mechanical Hamiltonian from a classical Hamiltonian $h(q_i,p_i)$ is to replace the canonical variables q_i and p_i by the corresponding operators Q_i and P_i, respectively, where $P_i=-i\hbar\partial/\partial Q_i$, thus leading to the quantum-mechanical Hamiltonian $H(Q_i,P_i)$.

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