resonance occurs below threshold for reaction (13b), while it is above threshold for (13a) due to the fact that the resonance is also coupled to the $K^+\Sigma^+$ channel with its high threshold. This is a particularly clear example of the large deviations from symmetry that can be produced by the mass differences.

The remaining discrepancy which has been reported[5] is in $p\bar{p}$ interactions at a center-of-mass energy of 2700 MeV. As it involves only a factor of about 2 in cross sections, and the energy is not particularly large compared to the masses involved, it is clear that this can easily be accounted for by the mass differences.

To summarize, it would appear that none of the reported discrepancies between $SU(3)$ and the results of scattering experiments are so large, considering the energies at which the experiments have been done, that they might not be due entirely to the effects of the mass differences within multiplets, with no other large symmetry breaking mechanism required. Conversely, the cases in which agreement has been found[4,6] are quite probably fortuitous. Because our results would indicate that there are uncertain, but probably quite large, effects due simply to the mass differences, it would seem that scattering experiments may not be a very fruitful way either of gaining evidence for $SU(3)$ or of studying the nature of its violations. In any event, data will be needed at considerably higher center-of-mass energies than those at which experiments have now been done.

ACKNOWLEDGMENTS

The author is grateful to Professor Reilly Atkinson for helpful conversations, and to David Sachs for calling his attention to some very pertinent questions during the early stages of the work. The numerical computations were done at the Tufts Computation Center.

PHYSICAL REVIEW VOLUME 139, NUMBER 5B 6 SEPTEMBER 1965

Quadrilocal Model of Baryons and Unitary Symmetry

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(Received 15 March 1965; revised manuscript received 3 May 1965)

A unified theory of baryons is proposed based on a spinor wave equation that depends on four space-time points or equivalently on the center of mass and three relative-coordinate vectors. The associated subsidiary condition and the structure of the mass operator are such that the four-point association is maintained within a small region of Minkowski space-time with characteristic length and that the theory has $U(9)$ symmetry in the full symmetry limit. By the couplings of internal motions this symmetry is reduced to the direct product of the usual unitary-spin group $U(3)$ and the other unitary group $U(3)'$ characteristic of spherical-oscillator-type motions, and then this latter is further reduced to simple rotational invariance. Baryonic states are assigned to the 165-dimensional irreducible representation (IR) of the $U(9)$ corresponding to the first excited shell with respect to the oscillatory motions of relative coordinates. These states are subgrouped according to the IR of the usual $SU(3)$ and to the eigenvalue of the relative angular momentum. Identifications with known levels are then made. The whole treatment is carried out covariantly, and minimum violation of causality is implied inside the particle.

INTRODUCTION

WE propose in this paper a unified theory of elementary particles, specifically of baryons, based on the hypothesis that a particle has a configuration represented by four space-time points $y_\mu{}^\alpha$ ($\alpha=1, \cdots, 4$). This just doubles the coordinates describing an elementary particle as compared with the bilocal model[1] of Yukawa.

The attractive feature of our theory lies in the fact that it represents the simplest possible model endowing an elementary particle with *full* and finite extension in space-time in conformity with relativistic covariance, that the usual $U(3)$ symmetry together with its breakdown is directly ascribed to this space-time nature of particles (rather than to the characteristics of meson-baryon interactions), and that internal attributes such as charge and hypercharge are reduced to quantized internal motions themselves,[2] in contradistinction with the viewpoint of the usual composite models.[3]

Furthermore, our model implies underlying broken $U(9)$ symmetry such that its irreducible representation (IR) $(3,0,0,\cdots,0)$ groups together, baryon supermultiplets belonging to different relative orbital angular-momentum states.

[1] H. Yukawa, Phys. Rev. **77**, 219 (1950); **80**, 1047 (1950); **91**, 415 (1953); Progr. Theoret. Phys. (Kyoto) **31**, 1167 (1964); M. Markov, Nuovo Cimento Suppl. 3, 760 (1956).

[2] T. Takabayasi, Nuovo Cimento **33**, 668 (1964).

[3] S. Sakata, Prog. Theoret. Phys. (Kyoto) **16**, 686 (1956); M. Gell-Mann, Phys. Letters **8**, 214 (1964).

THEORY OF QUADRILOCAL MODEL

The four-point system is equivalently described by the center-of-mass coordinate

$$X_\mu=\tfrac{1}{4}\sum_\alpha y_\mu{}^\alpha$$

and three relative coordinate vectors $x_\mu{}^r$ ($r=1, 2, 3$), which are obtained from $y_\mu{}^\alpha$ by a 4-dimensional orthogonal transformation[4]

$$x_\mu{}^\xi=C^{\xi\alpha}y_\mu{}^\alpha,\quad C^{\xi\alpha}C^{\zeta\alpha}=\delta_{\xi\zeta},\quad (\xi,\zeta=0,1,2,3),\quad (1)$$

where the constant coefficients $C^{\xi\alpha}$ must have the additional special property $C^{0\alpha}=\frac{1}{2}$; consequently,

$$C^{r\alpha}C^{s\alpha}=\delta_{rs},\quad \sum_\alpha C^{r\alpha}=0.$$

Then $\frac{1}{2}x_\mu{}^0=X_\mu$ represents the center of mass, while the three vectors $x_\mu{}^r$, which we call normal axes, describe the internal configuration of our deformable object extended in Minkowski space-time.

We denote the momentum vectors conjugate to $y_\mu{}^\alpha$ by $q_\mu{}^\alpha$; they satisfy the covariant commutation relations[5]

$$[y_\mu{}^\alpha,q_\nu{}^\beta]=i\delta_{\alpha\beta}\delta_{\mu\nu}.$$

According to (1), $q_\mu{}^\alpha$ transform to $p_\mu{}^\xi=C^{\xi\alpha}q_\mu{}^\alpha$ which satisfy

$$[x_\mu{}^\xi,p_\nu{}^\zeta]=i\delta_{\xi\zeta}\delta_{\mu\nu}.\quad (2)$$

Then the quantity $2p_\mu{}^0=P_\mu=\sum_\alpha q_\mu{}^\alpha$ represents the momentum-energy of the particle,[6] since (2) contains $[X_\mu,P_\nu]=[x_\mu{}^0,p_\nu{}^0]=i\delta_{\mu\nu}$. Equation (2) also indicates that $x_\mu{}^r$ and their conjugates $p_\mu{}^r$ are "internal variables"[2] which not only are translation-invariant but also commute with X_μ. Since (1) is an orthogonal transformation in the "α space," one has

$$\begin{aligned}y_\mu{}^\alpha y_\nu{}^\alpha&=x_\mu{}^\xi x_\nu{}^\xi=4X_\mu X_\nu+x_\mu{}^r x_\nu{}^r,\\ q_\mu{}^\alpha q_\nu{}^\alpha&=p_\mu{}^\xi p_\nu{}^\xi=\tfrac{1}{4}P_\mu P_\nu+p_\mu{}^r p_\nu{}^r.\end{aligned}\quad (3)$$

The choice of normal axes is not unique, but any possible set of them can be obtained from the "standard set"[7]

$$\begin{aligned}x_\mu{}^1&=(2)^{-1/2}(y_\mu{}^2-y_\mu{}^1),\quad x_\mu{}^2=(6)^{-1/2}(2y_\mu{}^3-y_\mu{}^1-y_\mu{}^2),\\ x_\mu{}^3&=(12)^{-1/2}(3y_\mu{}^4-y_\mu{}^1-y_\mu{}^2-y_\mu{}^3),\end{aligned}\quad (4)$$

through an orthogonal transformation in the figure space

$$x_\mu{}^{r\prime}=R^{rs}x_\mu{}^s,\quad RR^T=I,\quad (5)$$

which includes figure-space rotations and reflections. The former are generated by

$$L^r=\epsilon_{rst}x_\mu{}^s p_\mu{}^t,\quad [L^r,L^s]=i\epsilon_{rst}L^t.\quad (6)$$

The transformation (5) corresponds to such 4-dimensional orthogonal transformations of $y_\mu{}^\alpha$ with respect to the α index as leave X_μ invariant[8]:

$$y_\mu{}^{\alpha\prime}=\bar{R}^{\alpha\beta}y_\mu{}^\beta,\quad \bar{R}\bar{R}^T=I,\quad \sum_\alpha \bar{R}^{\alpha\beta}=1.$$

These include in particular the S_4 subgroup of permutations of the four points y^α, including transpositions (y^α,y^β) and cyclic permutations like (y^1,y^2,y^3), (y^1,y^3,y^2).

The binding mechanism keeping the four points within a small space-time region around X_μ to construct a particle may be supposed to be supplied by a direct "invariant potential" V working inside the particle in conjunction with the subsidiary condition stated below [see Eq. (17)]. Under the assumption that V is a scalar function constructed from

$$V_{\mu\nu}=\sum_{[\alpha,\beta]}(y_\mu{}^\alpha-y_\mu{}^\beta)(y_\nu{}^\alpha-y_\nu{}^\beta)=4x_\mu{}^r x_\nu{}^r,$$

all normal axes are mutually equivalent, resulting in the $O(3)$ symmetry independent of Lorentz transformations with the conservation of L^r, which constitute part of the unitary spins [see Eq. (16) below].

The simplest possibility for V is the relativistic Hooke potential $V=V_{\mu\mu}$, namely, the sum of 6 invariant squared distances among $y_\mu{}^\alpha$. The model then implies the free-particle wave equation

$$H\psi=0,\quad H=(2\mu)^{-1}q_\mu{}^\alpha q_\mu{}^\alpha+\tfrac{1}{2}KV_{\mu\mu},\quad (7)$$

[besides the subsidiary condition (17) below]. By the transformation to normal coordinates the center-of-mass degrees are separated [owing to (3)], and (7) is brought to the diagonal form $H=(P_\mu{}^2+M^2)/8\mu$, with

$$M^2=\tfrac{1}{2}\mu_0{}^2(l_0{}^2p_\mu{}^r p_\mu{}^r+l_0{}^{-2}x_\mu{}^r x_\mu{}^r),\quad (8)$$

$$l_0=(4\mu K)^{-1/4},\quad \mu_0 l_0=2\sqrt{2},\quad (9)$$

and the wave equation becomes

$$(P_\mu{}^2+M^2)\psi=0.\quad (10)$$

Evidently (8) represents the (mass)^2 operator for this model, for which the characteristic length l_0 and the scale of mass μ_0 are related by (9).

We now define the oscillator variables

$$a_\mu{}^r=2^{-1/2}(l_0{}^{-1}x_\mu{}^r+il_0p_\mu{}^r),\qquad a_\mu{}^{r\dagger}=2^{-1/2}(l_0{}^{-1}x_\mu{}^r-il_0p_\mu{}^r),$$

[4] In this paper suffixes $\alpha,\beta,\cdots$ usually range from 1 to 4, and $\xi,\zeta,\cdots$ from 0 to 3; Latin suffixes $r, s, t, u,\cdots$ run over 1, 2, 3. The summation convention is understood for any repeated suffixes *unless otherwise stated*.

[5] We set $\hbar=c=1$, and use the convention of imaginary Minkowskian fourth component for any real 4-vector; suffixes μ, ν, κ, $\cdots$ run over 1 to 4, while $i, j, k,\cdots$ over 1, 2, 3.

[6] If one has to treat a system of n points $y_\mu{}^1,\cdots,y_\mu{}^n$, one should perform an n-dimensional orthogonal transformation for the separation of center-of-mass motion; thus, one takes (1) with $\alpha=1,\cdots,n$, and $\xi=0,\cdots,n-1$, and puts $C^{0\alpha}=1/\sqrt{n}$. Then $X_\mu=\sum_\alpha y_\mu{}^\alpha/n=x_\mu{}^0/\sqrt{n}$ and $P_\mu=\sum_\alpha q_\mu{}^\alpha=(\sqrt{n})p_\mu{}^0$. The present quadrilocal model corresponds to the case $n=4$.

[7] A quadrilocal theory is considered also by H. Yukawa, Y. Katayama, and E. Yamada, with a different choice of relative coordinates and in a more generalized frame (to be published).

[8] These transformations, which we call bodily transformations, form in fact an $O(3)$ subgroup of $O(4)$; the meaning of such transformations is analyzed in detail in T. Takabayasi, NUDP-Report T-1, 1965 (unpublished) [Progr. Theoret. Phys. (Kyoto) (to be published)].

satisfying the covariant commutation relations

$$[a_\mu{}^r, a_\nu{}^{s\dagger}] = \delta_{rs}\delta_{\mu\nu}. \tag{11}$$

It is important to notice that since $x_4{}^r$ and $p_4{}^r$ are pure imaginary, the above definition implies that $a_k{}^{r\dagger}=a_k{}^{r*}$, $a_4{}^{r\dagger}=-a_4{}^{r*}$ (the asterisk designates Hermitian conjugate), whence (11) means that $[a_i{}^r,a_i{}^{r*}]=1$ (r and i not summed) while $[a_4{}^r,a_4{}^{r\dagger}]=-[a_4{}^r,a_4{}^{r*}]=[a_4{}^{r*},a_4{}^r]=1$ (r not summed). Thus, if we define $n_i{}^r=a_i{}^{r\dagger}a_i{}^r$ (r and i not summed) and $n_4{}^r=a_4{}^ra_4{}^{r*}=-a_4{}^ra_4{}^{r\dagger}=-(a_4{}^{r\dagger}a_4{}^r+1)$ (r not summed), each of $n_i{}^r$ and $n_4{}^r$ takes non-negative integer eigenvalues 0, 1, 2, $\cdots$, representing the number of vibration quanta for each normal coordinate. For those quanta $a_i{}^r$ are annihilation and $a_i{}^{r\dagger}$ creation operators, while $a_4{}^r$ are creation and $a_4{}^{r\dagger}$ annihilation operators. Further we write

$$a_\mu{}^{r\dagger}a_\mu{}^r+1=\sum_k n_k{}^r-n_4{}^r\equiv n^{(r)} \quad (r \text{ not summed}). \tag{12}$$

Then (8) is rewritten as

$$M^2=\mu_0{}^2(a_\mu{}^{r\dagger}a_\mu{}^r+6)=\mu_0{}^2(\sum_r n^{(r)}+3), \tag{13}$$

and clearly the wave equation is invariant not only under the $O(3)$ group (5) but under the wider $U(3)$ transformation in the figure space:

$$a_\mu{}^r \to U^{rs}a_\mu{}^s, \quad a_\mu{}^{r\dagger}\to a_\mu{}^{s\dagger}(U^*)^{sr}, \quad UU^*=I. \tag{14}$$

This indicates that $a_\mu{}^r$ and $a_\mu{}^{r\dagger}$ are contravariant and covariant vectors, respectively, with respect to the figure-space suffix. The generators of the $U(3)$ group are

$$A_s{}^r=a_\mu{}^{s\dagger}a_\mu{}^r+\delta_{rs}, \tag{15}$$

satisfying

$$[A_s{}^r,A_v{}^u]=\delta_v{}^rA_s{}^u-\delta_s{}^uA_v{}^r, \quad (A_s{}^r)^*=A_r{}^s,$$

and contain in particular $A_r{}^r=n^{(r)}$ (r not summed). Isospin components and hypercharge are to be identified as

$$T_+=T_1+iT_2=A_1{}^2, \quad T_3=\tfrac{1}{2}(n^{(1)}-n^{(2)}),$$
$$Y=\tfrac{1}{3}\sum_r n^{(r)}-n^{(3)},$$

and are thus created by the oscillatory motions of normal axes. They commute with both X_μ and P_μ. One has

$$L^1=i(A_3{}^2-A_2{}^3)=2F_7, \quad L^2=-2F_5, \quad L^3=2T_2=2F_2, \tag{16}$$

where F_i are unitary spins in Gell-Mann notation.[9]

The $U(3)$ symmetry contains the invariance under the internal reciprocity

$$x_\mu{}^r\to l_0{}^2p_\mu{}^r, \quad p_\mu{}^r\to -x_\mu{}^r/l_0{}^2.$$

Really observable quantities are not $x_\mu{}^r$ and $p_\mu{}^r$ themselves but are quantities such as unitary spin, spin, and mass, which are all self-reciprocal. The theory is also invariant under the "multiplicative triality" $a_\mu{}^r\to\omega a_\mu{}^r$ ($\omega=e^{2\pi i/3}$) induced by the unitary operator

$$U_t=\exp(2\pi i a_\mu{}^{r\dagger}a_\mu{}^r/3), \quad (U_t)^3=1.$$

The $U(3)$ breakdown is connected with the situation that the full equivalence among the four points is partly violated so that $y_\mu{}^4$, say, becomes inequivalent with the other $y_\mu{}^r$ ($r=1, 2, 3$). Then the $O(3)$ symmetry is reduced to $O(2)$ corresponding to the equivalence among the three $y_\mu{}^r$ alone, and along with it the $U(3)$ symmetry, for which the $O(3)$ is a specified subgroup, must also be reduced to isospin and hypercharge conservation, as can be verified by consideration in the standard coordinates (4).

To complete the theory (on the one-particle level) it is essential to impose the subsidiary condition[10] $(\partial/\partial X_\mu)\times a_\mu{}^{r\dagger}\psi=0$. This is rewritten as

$$\Lambda^r\psi=0, \quad \Lambda^r\equiv P_\mu a_\mu{}^{r\dagger}. \tag{17}$$

Clearly (17) is a $U(3)$ *vector* equation having $U(3)$-invariant meaning, and is compatible with the wave equation (10) [or (21) below]. The subsidiary condition effectively reduces the internal degrees of freedom from 12 to 9, and at the same time eliminates the difficulties of infinite degeneracy of mass levels and of negative squared mass, which otherwise would have occurred. Let us first assume that P_μ is time-like. Then one may take the center-of-mass rest frame in which all P_k have vanishing eigenvalues and (17) reduces to $a_4{}^{r\dagger}\psi=0$, so $n_4{}^r=0$ (recall that $a_4{}^{r\dagger}$ are annihilation operators), so that the relative-time motion is restricted to the zero-point oscillation, resulting in $n^{(r)}=\sum_k n_k{}^r\geqq 0$ [see Eq. (12)]. But since $n^{(r)}$ is a scalar quantum number it must be positive semidefinite in *any* frame. Thus the subsidiary condition ensures the positive-definite property of the operator M^2 [Eq. (13)], and the wave equation (10) assures in turn that the quantity

$$P\equiv -P_\mu{}^2$$

must be positive definite for any physical state, in accord with the original assumption of time-like P_μ. On the other hand if one assumes a space-like P_μ, then one can prove that there exists no normalizable solution satisfying the subsidiary condition (17). It is thus verified that P_μ must be time-like as a consequence of the wave equation and the subsidiary condition.

If one considered the limit $l_0\to 0$, the subsidiary condition (17), which is re-expressed as $P_\mu(x_\mu{}^r-il_0{}^2p_\mu{}^r)\psi=0$, would tend to $P_\mu x_\mu{}^r\psi=0$, meaning that in the center-of-mass rest frame all four "events" $y_\mu{}^\alpha$ occur simultaneously: $y_0{}^1=y_0{}^2=y_0{}^3=y_0{}^4$ ($y_0{}^\alpha\equiv y_4{}^\alpha/i$). In fact, however, because of the finiteness of l_0, the subsidiary condition (17) suppresses time-like extensions to a

[9] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).

[10] This condition is analogous to the Lorentz condition $(\partial A_\mu/\partial x_\mu)\times\Psi=0$ in the case of electrodynamics.

minimum but finite degree such that there always persist zero-point oscillations in relative times. Thus in our theory the unitary symmetry is necessarily related to the minimum violation of causality inside the particle.

The orbital angular-momentum tensor $y_{[\mu}{}^{\alpha}q_{\nu]}{}^{\alpha}$ of our system (the square brackets denote antisymmetrization with respect to the indices) is separated into that of the center of mass and the part due to relative motion:

$$y_{[\mu}{}^{\alpha}q_{\nu]}{}^{\alpha}=X_{[\mu}P_{\nu]}+L_{\mu\nu}\,,\quad L_{\mu\nu}=x_{[\mu}{}^{r}p_{\nu]}{}^{r}\,.$$

The covariant relative angular momentum[11] is defined by

$$\begin{aligned}W_{\mu}=P^{-1/2}\tilde{L}_{\mu\nu}P_{\nu}&=(-i/P^{1/2})\epsilon_{\mu\nu\kappa\lambda}x_{\kappa}{}^{r}p_{\lambda}{}^{r}P_{\nu}\\&=(-i/P^{1/2})\epsilon_{\mu\nu\kappa\lambda}y_{\kappa}{}^{\alpha}q_{\lambda}{}^{\alpha}P_{\nu}\,,\end{aligned}\tag{18}$$

which commutes with P_{μ} and is space-like because $W_{\mu}P_{\mu}=0$. This W_{μ} means the infinitesimal operators of the little group with respect to P_{μ}, and its magnitude is positive semidefinite, taking integer eigenvalues $W_{\mu}{}^{2}=W(W+1)$, $W=0, 1, 2, \cdots$. In the rest frame it is reduced to

$$W_{i}=-i\epsilon_{ijk}a_{j}{}^{r\dagger}a_{k}{}^{r}\,,\quad W_{1}=L_{23}\text{ etc.},\quad W_{4}=0\,.$$

Evidently, $L_{\mu\nu}$ is the antisymmetric part of the $U(3)$-invariant tensor

$$K_{\mu\nu}=a_{\mu}{}^{r\dagger}a_{\nu}{}^{r}\,,\quad [K_{\mu\nu},A_{s}{}^{r}]=0\,,$$

namely $L_{\mu\nu}=-i(K_{\mu\nu}-K_{\nu\mu})$. The three quantities $W_{\mu}{}^{2}$, W_{3}, and

$$\Theta=W_{\mu}K_{\mu\nu}W_{\nu}=W^{r*}W^{r}\,,\quad (W^{r}\equiv a_{\mu}{}^{r}W_{\mu})$$

are all $U(3)$-invariant and mutually commuting, and they correspond to the three degrees of freedom of rotation of our extended object with respect to an inertial frame.[12]

The $K_{\mu\nu}$ themselves do not commute with the subsidiary condition. However, we define the associated space-like tensor

$$K_{\mu\nu}'=O_{\mu\rho}O_{\nu\sigma}K_{\rho\sigma}\,,$$

with the aid of the projection operator $O_{\mu\nu}=\delta_{\mu\nu}+P_{\mu}P_{\nu}/P$, which has the properties

$$O_{\mu\nu}P_{\nu}=0\,,\quad O_{\lambda\mu}O_{\mu\nu}=O_{\lambda\nu}\,;\tag{19}$$

then $K_{\mu\nu}'$ satisfies $[K_{\mu\nu}',\Lambda^{r}]=[K_{\mu\nu}',A_{s}{}^{r}]=0$. Since W_{μ} and Θ can be written as $W_{\mu}=-\epsilon_{\mu\nu\kappa\lambda}K_{\kappa\lambda}'P_{\nu}/\sqrt{P}$ and $\Theta=W_{\mu}K_{\mu\nu}'W_{\nu}$, they also commute with Λ^{r}. Now $K_{\mu\nu}'$ constitute the generators of the group $U(3)'$. In particular, in the center-of-mass rest frame the space components satisfy $[K_{ij}',K_{kl}']=\delta_{jk}K_{il}'-\delta_{il}K_{jk}'$ and $(K_{ij}')^{*}=K_{ji}'$, while all time components vanish: $K_{\mu 4}'=K_{4\nu}'=0$. This $U(3)'$ is the symmetry characteristic of an oscillator-type model.

Moreover both the wave equation (10) and the subsidiary condition (17) are invariant under the $U(9)$ group containing $U(3)\otimes U(3)'$ as subgroup. The $U(9)$ generators are

$$A_{s}{}^{r}{}_{,\mu\nu}=O_{\mu\rho}O_{\nu\sigma}a_{\rho}{}^{s\dagger}a_{\sigma}{}^{r}$$

with the properties $P_{\mu}A_{s}{}^{r}{}_{,\mu\nu}=P_{\nu}A_{s}{}^{r}{}_{,\mu\nu}=0$, $[A_{s}{}^{r}{}_{,\mu\nu},\Lambda^{u}]=0$, and

$$[A_{s}{}^{r}{}_{,\mu\nu},A_{v}{}^{u}{}_{,\rho\sigma}]=\delta_{rv}O_{\rho\nu}A_{s}{}^{u}{}_{,\mu\sigma}-\delta_{su}O_{\mu\sigma}A_{v}{}^{r}{}_{,\rho\nu}\,.$$

This $A_{s}{}^{r}{}_{,\mu\nu}$ really contains the $U(3)$ and $U(3)'$ generators because

$$A_{s}{}^{r}{}_{,\mu\mu}=O_{\rho\sigma}a_{\rho}{}^{s\dagger}a_{\sigma}{}^{r}\,,\quad A_{r}{}^{r}{}_{,\mu\nu}=K_{\mu\nu}'\,,$$

where the former is equivalent to $A_{s}{}^{r}$ of (15) in so far as it operates on any state satisfying the subsidiary condition (17):

$$O_{\rho\sigma}a_{\rho}{}^{s\dagger}a_{\sigma}{}^{r}\psi=A_{s}{}^{r}\psi\,.$$

BARYONIC STATES

To deal with baryons more closely we assume that all (free) baryonic states are described by the fundamental spinor wave equation

$$(i\gamma_{\mu}\sum_{\alpha}q_{\mu}{}^{\alpha}+M)\psi(y^{1},y^{2},y^{3},y^{4})=0\,,\tag{20}$$

where ψ is a Dirac spinor depending on four points, and M is a certain operator invariant under inhomogeneous Lorentz transformations, depending on $y_{\mu}{}^{\alpha}-y_{\mu}{}^{\beta}$ and $q_{\mu}{}^{\alpha}=-i\partial/(\partial y_{\mu}{}^{\alpha})$. By the transformation (1), Eq. (20) is rewritten as[2]

$$(i\gamma_{\mu}P_{\mu}+M)\psi(X,x^{1},x^{2},x^{3})=0\,,\tag{21}$$

where M now depends on $x_{\mu}{}^{r}$, $p_{\mu}{}^{r}$, and possibly on P_{μ}, and means the mass operator.

The ground state (of internal motion) ψ_{0} is specified by imposing the additional condition

$$O_{\mu\nu}a_{\nu}{}^{r}\psi_{0}=0\,.\tag{22}$$

This is compatible with (17), and also has $U(9)$-invariant meaning, since (22) implies

$$[A_{s}{}^{r}{}_{,\mu\nu},O_{\kappa\lambda}a_{\lambda}{}^{u}]\psi_{0}=\delta_{su}O_{\mu\kappa}(O_{\nu\sigma}a_{\sigma}{}^{r})\psi_{0}=0\,.$$

In the center-of-mass rest frame Eq. (22) reduces to $a_{k}{}^{r}\psi_{0}=0$ so that for ψ_{0} all relative motions are in their oscillator ground states.

We now define

$$R_{\mu\nu}=\delta_{\mu\nu}+2P_{\mu}P_{\nu}/P\,,$$

which is the operator reflecting an arbitrary vector with respect to the hyperplane normal to P_{μ}, and accordingly has the property of an (improper) Lorentz transformation: $R_{\lambda\nu}R_{\mu\nu}=\delta_{\lambda\mu}$. In terms of $R_{\mu\nu}$ Eqs. (17) and (22)

[11] Cf., e.g., W. Pauli, lecture note, CERN, 1956 (unpublished).

[12] Precisely speaking Θ is a quantity related to couplings between oscillations and rotation and not one related to rotation only. Clearly Θ is also positive semidefinite.

are put in a unified equation

$$(x_\mu{}^r+il_0{}^2R_{\mu\nu}p_\nu{}^r)\psi_0=0.$$

The plane-wave solution is given by the internal minimum wave packet with space-time extension of order l_0:

$$\psi_0=(\pi l_0{}^2)^{-3}u(P_\mu{}')\exp(iP_\mu{}'X_\mu-\tfrac{1}{2}l_0{}^{-2}R_{\mu\nu}{}'x_\mu{}^rx_\nu{}^r). \quad (23)$$

Here $P_\mu{}'$ and $R_{\mu\nu}{}'$ denote respective eigenvalues, satisfying $P_\mu{}'^2=-m_0{}^2$, where m_0 is the (lowest) eigenvalue of the mass operator M; and $u(P_\mu{}')$ is the constant spinor satisfying $(i\gamma_\mu P_\mu{}'+m_0)u(P_\mu{}')=0$. From (22) one immediately obtains $A_s{}^r{}_{,\mu\nu}\psi_0=0$, so $A_s{}^r\psi_0=W_\mu\psi_0=\Theta\psi_0=0$. In the center-of-mass rest frame, ψ_0 damps in a Gaussian manner with respect to relative-time coordinates $x_0{}^r\equiv x_4{}^r/i$ as well as relative-space coordinates.[13] This ψ_0 does not vanish in the region where $x_\mu{}^r$ is time-like, but it is normalizable with respect to integrations over internal coordinates including relative times, and, in fact, expression (23) is exactly normalized to unity. Also each $x_\mu{}^r$ is space-like in its expectation value, with $\langle(x_\mu{}^r)^2\rangle_0=l_0{}^2$ (r not summed).

If one considered the limit $l_0\to 0$, then ψ_0 would tend, aside from the external factor, to a δ function for space- *and* time-relative coordinates, and one should approach the local theory.

We classify baryonic states under the additional restriction[14] $U_t\psi=\psi$, meaning $A_r{}^r=\sum_r n^{(r)}=3\nu$ (ν = integer). For simplicity we consider the more restrictive condition

$$K_{\mu\mu}{}'=A_r{}^r=\sum_r n^{(r)}=3, \quad (24)$$

to allow the "first" excited shell only. This condition is $U(9)$-invariant and is rewritten as

$$a_\mu{}^{r\dagger}a_\mu{}^r\psi=0, \quad \text{i.e.,} \quad K_{\mu\mu}\psi=0.$$

Then a wave function consistent with subsidiary conditions is generally written as

$$\Sigma a_\mu{}^{r\dagger}a_\nu{}^{s\dagger}a_\kappa{}^{t\dagger}\psi_0(m), \quad (25)$$

where $\psi_0(m)$ is a "generalized ground-state function"[15] and the summation is to be made over r, s, t and μ, ν, κ with appropriate coefficients such that (25) becomes an eigenstate of the mass operator M with the eigenvalue m. Note that each individual term in (25), denoted $\psi_{\mu\nu\kappa}{}^{rst}(m)=a_\mu{}^{r\dagger}a_\nu{}^{s\dagger}a_\kappa{}^{t\dagger}\psi_0(m)$, satisfies $P_\mu\psi_{\mu\nu\kappa}{}^{rst}\times(m)=P_\nu\psi_{\mu\nu\kappa}{}^{rst}(m)=P_\kappa\psi_{\mu\nu\kappa}{}^{rst}(m)=0$ as a consequence of the subsidiary condition (17).

Owing to the Dirac spinor character of ψ, the total angular-momentum tensor is

$$M_{\mu\nu}=X_{[\mu}P_{\nu]}+L_{\mu\nu}+\tfrac{1}{2}\sigma_{\mu\nu}, \quad (\sigma_{\mu\nu}=\gamma_{[\mu}\gamma_{\nu]}/2i)$$

so that the covariant particle spin is $J_\mu=\tilde{M}_{\mu\nu}P_\nu/\sqrt{P}=W_\mu+\Sigma_\mu$ with

$$\Sigma_\mu=-\epsilon_{\mu\nu\kappa\lambda}\gamma_\nu\gamma_\kappa P_\lambda/(4\sqrt{P}), \quad \Sigma_\mu{}^2=\tfrac{3}{4}.$$

Its magnitude,

$$J_\mu{}^2=W_\mu{}^2+\tfrac{3}{4}+2W_\mu\Sigma_\mu, \quad (26)$$

has eigenvalues $J(J+1)$ with $J=W\pm\frac{1}{2}$. The last term in (26) means covariant "spin-orbit coupling" and is written also as

$$2W_\mu\Sigma_\mu=-(i/2P^{1/2})\epsilon_{\mu\nu\kappa\lambda}\sigma_{\mu\nu}W_\kappa P_\lambda=(2i)^{-1}W_\mu W_\nu\sigma_{\mu\nu},$$

and in the rest frame $W_\mu\Sigma_\mu=\mathbf{W}\cdot\mathbf{\Sigma}=-(i/2)\epsilon_{ijk}a_i{}^{r\dagger}a_j{}^r\sigma_k$. The magnitudes of relative angular momentum and of the total J spin are good quantum numbers, while W_3 is not, and in its place one may employ the helicity $J_0\equiv J_4/i$.

In our model the rest mass should originate in general from the excitation of oscillatory motions of normal coordinates and their couplings, as illustrated in (8). For the mass operator of baryons the first approach will be to require that the wave equation (10) should still be fulfilled, to obtain $M=\mu_0(a_\mu{}^{r\dagger}a_\mu{}^r+6)^{1/2}$. However, it is more reasonable for baryons to assume the oscillator formula for M itself

$$M=\mu_0(a_\mu{}^{r\dagger}a_\mu{}^r+6)=\mu_0(A_r{}^r+3) \quad (27)$$

to begin with, where the theory has $U(9)$ symmetry. The additive constant 3 in (27) corresponds to zero-point oscillations. The $U(9)$ IR $(3,0,0,\cdots,0)$ with the dimensionality $D=165$ exactly represents the "first shell" defined by (24), with completely degenerate mass. Such full symmetry will be reduced successively. Here we consider the problem merely in a formal manner.

First we introduce couplings among different normal axes by $\langle AA\rangle\equiv A_s{}^rA_r{}^s$ and $\langle AAA\rangle\equiv\frac{1}{2}A_s{}^r\{A_u{}^s,A_r{}^u\}$. These $U(3)$ Casimir operators and the respective ones of $U(3)'$ are equal for any physical ψ satisfying (17):

$$\langle AA\rangle\psi=K_{\mu\nu}{}'K_{\nu\mu}{}'\psi, \quad \langle AAA\rangle\psi=\tfrac{1}{2}K_{\lambda\mu}{}'\{K_{\mu\nu}{}',K_{\nu\lambda}{}'\}\psi.$$

In place of them one may employ

$$\langle BB\rangle\equiv B_s{}^rB_r{}^s, \quad \langle BBB\rangle\equiv\tfrac{1}{2}B_s{}^r\{B_u{}^s,B_r{}^u\}, \quad (28)$$

in view of (24), where $B_s{}^r=A_s{}^r-\frac{1}{3}\delta_{rs}A_u{}^u$ are generators of the $SU(3)$ subgroup. By the above couplings the full $U(9)$ symmetry is reduced to $U(3)\otimes U(3)'$. The quantities (28) are related to the signature of $SU(3)$ IR (λ_1,λ_2) by $\langle BB\rangle=\frac{2}{3}\{(\lambda_1{}^2+\lambda_2{}^2)+\lambda_1\lambda_2+3(\lambda_1+\lambda_2)\}$, $\langle BBB\rangle=\frac{1}{9}(\lambda_1-\lambda_2)(2\lambda_1+\lambda_2+3)(\lambda_1+2\lambda_2+3)$.

Next the $U(3)'$ symmetry is reduced to the usual

[13] This point was emphasized by H. Yukawa, Progr. Theoret. Phys. (Kyoto) **31**, 1167 (1964), for the case of bilocal model.

[14] A possibility can be suggested of interpreting this restriction in terms of parastatistics regarding permutations among the four points. [Cf. Ref. 8].

[15] One generalizes ψ_0 given by (23) *mathematically* by dropping the condition that m_0 should be an eigenvalue of M; namely, one replaces m_0 by an arbitrary parameter m and denotes it by $\psi_0(m)$. This "generalized ground-state function" (or "core function," say) does not itself satisfy the wave equation but continues to satisfy (17) and (22).

rotational invariance. Without going into the possible mechanism of such reduction, we simply assume that it effectively brings to the mass a new contribution represented as the couplings of angular momenta, in conformity with the requirements that it should be compatible with the subsidiary condition (17) and preserve the usual $U(3)$ symmetry as well as the relativistic invariance. Then such a contribution must depend on W_μ^2, $W_\mu\Sigma_\mu$, and Θ alone [or equivalently on W_μ^2, J_μ^2, and Θ alone, in view of (26)]. The simplest mass formula under the above assumptions is $M=M_1+M_2$ with

$$\begin{aligned} M_1&=\mu_0(3+A_r{}^r+a_1\langle BB\rangle)\,,\\ M_2&=\mu_0(a_2W_\mu{}^2+2a_3W_\mu\Sigma_\mu)\,, \end{aligned} \tag{29}$$

where in particular we have neglected terms depending on $\langle BBB\rangle$ and Θ.

On account of the M_2 term the wave equation now involves, via W_μ, explicit couplings between the internal $x_{[\mu}{}^r p_{\nu]}{}^r$ and the external P_μ, and it further implies a higher order differential equation in X_μ. (Another way of looking at the equation is to regard it as simply applying for each eigenstate of W_μ^2 and J_μ^2.) By analogy with the case of the rotator model[16] it is natural to assume that $a_2>0$ [see (35) below]. The formally introduced "spin-orbit coupling" should be of different origin from similar ones such as occur in atomic physics. It explicitly contains γ matrices, but still M formally maintains the meaning of mass operator. In fact, from (21), one gets by iteration $\{P_\mu{}^2+M^2-iP_\mu[\gamma_\mu,M]\}\psi=0$, which is the Klein-Gordon equation (10), provided that $P_\mu[\gamma_\mu,M]=0$. But this condition is really satisfied by the spin-orbit coupling term since

$$P_\rho[\gamma_\rho,W_\mu\Sigma_\mu]=(i/4P^{1/2})\epsilon_{\mu\nu\kappa\lambda}P_\rho[\gamma_\rho,\sigma_{\mu\nu}]W_\kappa P_\lambda=0\,.$$

Finally the $U(3)$ symmetry is broken in a fashion analogous to that of Gell-Mann and Okubo (GMO),[9,17] to obtain $M=M_1+M_2+M_3$ with

$$M_3=-\kappa_1 Y+\kappa_2\{T(T+1)-\tfrac{1}{4}Y^2-\tfrac{1}{6}\langle BB\rangle\}\,. \tag{30}$$

The average of M_3 over any $SU(3)$ IR vanishes.

Baryonic states (25) under the shell condition (24) are now classified according to simultaneous eigenstates of (28) and[18]

$$(Y,\mathbf{T}^2,T_3,W_\mu{}^2,J_\mu{}^2,\Theta,J_0)\,, \tag{31}$$

as shown in Table I, where identifications with known levels are indicated.

TABLE I. Baryonic states in the first shell.

d	(λ_1,λ_2)	$\langle BB\rangle$	$\langle BBB\rangle$	W	Θ	J^P	Baryon
1	(0,0)	0	0	0	0	$\frac{1}{2}^+$	$Y_0^*(1405)$
8	(1,1)	6	0	1	3	$\frac{1}{2}^+$, $\frac{3}{2}^+$	(N,Λ,Σ,Ξ)
				2	10	$\frac{3}{2}^+$, $\frac{5}{2}^+$	$N_{1/2}^*(1688)$, $Y_0^*(1815)$, ⋯
10	(3,0)	12	18	1	0	$\frac{1}{2}^+$, $\frac{3}{2}^+$	$(N_{3/2}^*,Y_1^*,\Xi_{1/2}^*,\Omega^-)$
				3	0	$\frac{5}{2}^+$, $\frac{7}{2}^+$	$N_{3/2}^*(1920)$, ⋯

For example, the totally antisymmetric eigenfunction

$$\psi^{(0)}=(i/\sqrt{P})\epsilon_{\mu\nu\kappa\lambda}P_\mu a_\nu{}^{1\dagger}a_\kappa{}^{2\dagger}a_\lambda{}^{3\dagger}\psi_0(m)$$

yields the unitary singlet, which is identified with the known $Y_0^*(1405)$. According to the mass formula (29) and (30) its mass is given by

$$m(Y_0^*)=6\mu_0\,, \tag{32}$$

from which one may fix the unit μ_0 as $\mu_0=1405/6 \approx 234.2$ MeV. It is remarkable that this exactly agrees with the basic mass unit which we previously introduced from empirical mass systematics of baryons and mesons.[19]

We assign by convention odd intrinsic parity ($J^P=\frac{1}{2}^-$) to the "ground state" ψ_0; then all baryonic states in the "first" shell should have even parity.[20] The feature of the "first shell" is that once an $SU(3)$ IR and relative angular momentum are specified, Θ takes on a unique eigenvalue, and that for every $SU(3)$ IR its dimensionality d equals the multiplicity due to W to be accommodated therein: $d=\Sigma_W(2W+1)$. In fact,

$$\mathbf{165}=(1,1)+(8,8)+(10,10)\,.$$

Although experimental identifications are not yet complete,[21] it is the characteristic feature of our model that our super-supermultiplet **165** accommodates the usual octet and decuplet together with their supposed (first) Regge recurrence supermultiplets, by grouping different relative orbital angular-momentum states into a single $U(9)$ representation.

The usual baryon octet and the $\frac{3}{2}^+$ decuplet are comprised in an "18-plet," or, say, 56-plet, according to the p-state nature of the internal motion, $W=1$. This positive-pairty 18-plet has further properties

$$J(J+1)-\tfrac{3}{4}=3(J-\tfrac{1}{2})=2+\mathbf{W}\boldsymbol{\sigma}\,, \tag{33}$$

$$\tfrac{1}{2}\langle BB\rangle=3(J+\tfrac{1}{2})=J(J+1)+9/4\,, \tag{34}$$

[16] This model corresponds to one for which $x_\mu{}^r$ are no longer independent variables but are subjected to the constraints $(x_\mu{}^r x_\mu{}^s-\frac{1}{2}s_0\delta_{rs})\psi=0$, which are equivalent to the restriction that all the six invariant squared distances among four points y^α are equal and constant: $(y_\mu{}^\alpha-y_\mu{}^\beta)^2=s_0(1-\delta_{\alpha\beta})$ [α, β not summed]. For this model the $U(3)$ symmetry reduces to the $O(3)$ symmetry (5) with the conservation of L^r of (16) only.

[17] S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).

[18] Note that on account of the conditions (17) and (24) on the one hand, and of the additional degree of ψ polarization on the other, one needs the set of 9 commuting quantities for a complete classification of internal eigenstates.

[19] T. Takabayasi and Y. Ohnuki, Progr. Theoret. Phys. (Kyoto) **30**, 272 (1963); T. Takabayasi, Nuovo Cimento **30**, 1500 (1963). In this latter paper the relation (32) was pointed out.

[20] This implies that $Y_0^*(1405)$ should have $J^P=\frac{1}{2}^+$, which assignment was also adopted by J. Schwinger, Phys. Rev. Letters **12**, 237 (1964). The "second shell" with $A_r{}^r=6$ yields odd-parity states.

[21] Considerations about the predicted states (i.e., those left unidentified in Table I), for which some evidences exist, will be given elsewhere.

which allow us to reduce our baryon mass formula to a simpler form for the 18-plet.

CONCLUDING REMARKS

(i) Within the limits of the preceding arguments the parameters occurring in the mass formula (29) and (30) are not calculated, but one may estimate their values by comparison with observations. In fact, if one takes the choice

$$a_1=-\tfrac{1}{6},\quad a_2=a_3=\tfrac{2}{3},\qquad \kappa_1=6\kappa_2,\quad \mu_0=\kappa_1+\kappa_2, \tag{35}$$

and notes (33) and (34), one obtains the mass formula for the 18-plet[22]

$$M=\kappa_2\{28+6(J+\tfrac{1}{2}-Y)+[T(T+1)-\tfrac{1}{4}Y^2]\} \tag{36}$$

which is in good agreement with observations. We note that this semi-empirical formula contains, besides the GMO relations,[9,17] the following simple relations: (a) The central mass of the octet and that of the decuplet stand in the ratio $\langle m_B\rangle/\langle m_{B*}\rangle=\frac{5}{6}$. (b) The isosinglet masses belonging to the octet and to the decuplet are in the ratio $m_\Lambda/m_{\Omega^-}=\frac{2}{3}$. Both of these relations are in excellent agreement with observations.[23] (c) The common spacing within the decuplet is related to the octet spacing by $\delta_{10}=\frac{3}{8}(m_\Xi-m_N)$, which yields $\delta_{10}=143$ MeV as compared with the experimental value $\delta_{10}\approx 145$ MeV.

(ii) In the present multilocal scheme, leptons will be assigned to a trilocal configuration; for mesons an appropriate configuration is one which consists of eight points subjected to special constraints so that they again have nine internal degrees of freedom.

(iii) This paper is mainly devoted to establishing a concrete model verifying that elementary-particle symmetry should follow naturally from the hypothesis of a space-time configuration of the particle, such as described by a nonlocal framework. Clearly, our theory is still quite restricted in its scope, as we have established it merely on a one-particle level. However, this is the important step, since the quadrilocal theory means a rather drastic theoretical extension, and since our treatment already manifests characteristic features of the theory, including the qualitative prediction of the existence of broken $U(9)$ symmetry with its definite irreducible representation to be realized.

The treatment of interactions is the main problem to be pursued in a further study, in which one must perform second quantization. The interesting point then will be to see how the assumed finite extension of the particle should modify the interactions.

The process of symmetry-breaking, which was regarded as being of internal origin, should be reconsidered from the standpoint of interactions; the electromagnetic interaction is related to this problem also.

(iv) It is interesting to note that our broken $U(9)$ symmetry has a character similar to the $SU(6)$ recently discussed,[24] although the original ideas and standpoints are quite different.

Our theory is based on the set of three internal vector variables $a_\mu{}^r$ together with their adjoints $a_\mu{}^{r\dagger}$ derived from the assumed quadrilocal structure. If one postulates instead that the particle has internal structure represented by three 2-component spinors $a_\alpha{}^r$ ($r=1, 2, 3; \alpha=1, 2$), one can construct a model embodying $U(6)$ symmetry with the $U(6)$ generators $A_s{}^r{}_{,\alpha\beta}=a_\alpha{}^{s*}a_\beta{}^r$. This contains the spin represented by $J_i=\frac{1}{2}a_\alpha{}^{r*}(\sigma_i)_{\alpha\beta}a_\beta{}^r$ and unitary spins corresponding to $A_s{}^r=a_\alpha{}^{s*}a_\alpha{}^r$. Since this model has six internal variables only, it is much more restrictive than the quadrilocal model, and it indeed gives the relation (34) for the 18-plet. This scheme [like the usual $SU(6)$ theory] is not Lorentz-covariant. To remedy this deficiency we start with the internal variables consisting of the set of three 4-component spinors $\zeta_\rho{}^r$ ($r=1, 2, 3$) and impose the commutation relations $[\zeta_\rho{}^r,\bar{\zeta}_\sigma{}^s]=\delta_{rs}\delta_{\rho\sigma}$, $(\bar{\zeta}^r\equiv\zeta^{r*}\gamma_4)$. The model then embodies $U(6)$ symmetry in a covariant way with the generators $A_s{}^r{}_{,\mu\nu}=O_{\mu\rho}O_{\nu\sigma}(\bar{\zeta}^s\gamma_\rho\gamma_\sigma\zeta^r)+2\delta_{rs}O_{\mu\nu}$. This model resembles a composite system of three rotators rather than a multilocal one.

All these points, including a more detailed treatment of the mass-formula problem, will be discussed further elsewhere.

Note added in proof. The investigation of the quadrilocal model referred to in Ref. 7 is presented by Y. Katayama, E. Yamada, and H. Yukawa, Progr. Theoret. Phys. (Kyoto) **33**, 541 (1965). The unified model of baryons and mesons based on three spinor internal variables $\zeta_\rho{}^r$ (which are equivalent to three triads), referred to at the end of the text, is given in detail in T. Takabayasi, NUDP-Report, T-8, 1965 [Progr. Theoret. Phys. Suppl. (Kyoto) (to be published)].

ACKNOWLEDGMENTS

The author would like to thank Professor H. Yukawa and Professor Y. Ohnuki for valuable discussions.

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[22] This formula is essentially the one presented in T. Takabayasi, Phys. Letters **5**, 73 (1963).

[23] T. Takabayasi, Progr. Theoret. Phys. (Kyoto) **32**, 981 (1964); Nuovo Cimento **35**, 666 (1965).

[24] F. Gürsey and L. Radicati, Phys. Rev. Letters **13**, 173 (1964); B. Sakita, Phys. Rev. **136**, B1756 (1964).