Generalization of Levinson's Theorem to Three-Body Systems

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Levinson's theorem is generalized to systems of three particles. The usual two-body result relates the number of bound states of given angular momentum to the corresponding eigenphase shifts of the S matrix. Because of disconnected diagrams the three-body S matrix has continuous eigenphase shifts in addition to any discrete ones; however, it is possible to define a unitary connected matrix that has only discrete eigenphase shifts. Levinson's theorem is given in terms of these phase shifts, and it is the same as the usual multichannel result, except that there are an infinite number of eigenphase shifts to be summed over for each value of the total angular momentum. The proof is carried out within the framework of the Faddeev equations by generalizing Jauch's proof for two-body systems.

1. INTRODUCTION

ONE of the important problems in the theory of elementary particles is the determination of whether or not a particle is elementary or composite. In a Lagrangian theory an elementary particle must be put in the Lagrangian. In a model based on dispersion theory there is the well-known ambiguity of Castillejo, Dalitz, and Dyson.[1] They showed that an infinite number of solutions exist in the charged scalar theory without recoil. In both kinds of theories, it has been suggested that Levinson's theorem[2] could be used as a means of selecting the proper Lagrangian or the proper solution to the dispersion relations. In its simplest form as first given by Levinson, the theorem says that in the scattering of a particle from a spherically symmetric central potential, the number of bound states of the particle in a given angular-momentum state is related to the phase shift by

$$N\pi=\delta(0)-\delta(\infty). \tag{1.1}$$

Jauch[3] generalized the proof to a larger class of potentials than that treated by Levinson, and also he showed that the relation (1.1) is a result of the completeness of the eigenfunctions of two operators H and H_0, provided that the interaction term tends to zero sufficiently rapidly at large distances. H is the full Hamiltonian for the system, and H_0 is the Hamiltonian in the absence of interactions. The result has been generalized to the case in which H_0 also has a discrete spectrum,[4,5]

$$(N_H-N_{H_0})\pi=\delta(0)-\delta(\infty); \tag{1.2}$$

N_H and N_{H_0} are the number of bound states of H and H_0, respectively. Since H_0 is the Hamiltonian operator for a noninteracting system, all points in its discrete spectrum represent elementary particles. Levinson's theorem has been further generalized to many-channel systems by Kazes.[6] In view of the possible application of Levinson's theorem to determining which equations—and which solutions to them—nature actually selects, it seems important to extend the theorem to systems of more than two particles. In this paper, we generalize the theorem to three-body systems.

The three-body problem has two important complications which are not present in two-body problems. One difference is in the number of variables in the system. In two-body scattering, the S matrix can be completely diagonalized by projecting out the total angular momentum, whereas in three-body scattering the S matrix depends upon additional energy and angular variables, and a further diagonalization is necessary. Unfortunately, it is not known how to do this. The second major difference is the connectedness structure; that is, in three-body scattering there exist situations in which two particles interact and the third particle is always beyond the range of the forces. As a result of this disconnectedness, the kernel of the Lippmann-Schwinger equation has a continuous spectrum.[7] Similarly, the S matrix will have a continuous spectrum, that is, it will not have only discrete eigenphase shifts which can be summed to give an equation such as (1.1). However, because of the simple origin of the continuous spectrum, it is possible to define a unitary operator closely related to the S matrix and having only a discrete spectrum. Unlike the two-body case, there are here an infinite number of eigenphase shifts even after the separation of angular momentum, and the expression for the number of bound states involves an infinite sum. In the special case in which there are no two-body bound states, the number of three-body bound states is shown to be

$$N=\frac{1}{\pi}\sum_n\{\delta_n(0)-\delta_n(\infty)\}, \tag{1.3}$$

where δ_n are the eigenphase shifts of the unitary operator mentioned above.

The proof of (1.3) is carried out within the framework of the set of three-body equations developed by

[1] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).
[2] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **25**, No. 9 (1949).
[3] J. M. Jauch, Helv. Phys. Acta **30**, 143 (1957).
[4] J. C. Polkinghorne, Proc. Cambridge Phil. Soc. **54**, 560 (1958).
[5] M. Ida, Progr. Theoret. Phys. (Kyoto) **21**, 625 (1959).
[6] E. Kazes, Nuovo Cimento **13**, 983 (1959).
[7] S. Weinberg, Phys. Rev. **133**, B232 (1964).

Faddeev,[8–11] and it is based upon the completeness relationships of the eigenfunctions of the operators H and H_0. If there are no two-body bound states, the eigenfunctions of H and H_0 are related to each other by one isometric operator, the Møller[12] wave matrix. In Secs. 2 and 3, we restrict ourselves to this situation, as it contains all the essential problems without the many algebraic complexities that arise when two-body bound states are permitted. In Sec. 2, we introduce the Faddeev[11] equations and the projection operator onto the three-body bound states. In Sec. 3, we derive Eq. (1.3).

In Secs. 4 and 5, we relax the restriction on two-body bound states to permit one bound state between each pair of particles. Section 4 contains the generalization of the Møller wave matrices to allow for this possibility, and Sec. 5 contains the generalization of Eq. (1 3). Finally, the more tedious calculations can be found in the Appendices.

2. THREE-BODY WAVE MATRICES

In this section, we outline the method of proof and introduce the Faddeev equations[8–11] and the isometric operators which are the generalization of the Møller wave matrices[12] to three-particle systems. A complete account of the operators and their properties can be found in Ref. 11.

The basis for the proof is the same as for Jauch's original proof[3] for two-body systems. All calculations are carried out for fixed total angular momentum l. The total Hamiltonian is split into two parts,

$$H=H_0+V\,, \tag{2.1}$$

where H_0 is the free-particle Hamiltonian and V is the interaction term. We assume that all the eigenstates ϕ_E of H_0 are continuum states with energy $E>0$,

$$H_0\phi_E=E\phi_E\,, \tag{2.2}$$

and that H has N points in the discrete spectrum with $E_n<0$ $(n=1, 2,\cdots, N)$. H is assumed to have the same continuous spectrum as H_0:

$$\begin{aligned} H\psi_E&=E\psi_E\,, \quad \text{with} \quad E>0\,,\\ H\psi_n&=E_n\psi_n\,, \quad \text{with} \quad E_n<0\,. \end{aligned} \tag{2.3}$$

The isometric operator that maps the continuum eigenstates of H_0 onto the continuum eigenstates of H is called the Møller wave operator[12] and is given by

$$\Omega=\int_0^\infty dE|\psi_E\rangle\langle\phi_E|\,. \tag{2.4}$$

The completeness of the eigenstates of H and H_0 gives the relationships

$$\Omega^\dagger\Omega=\int_0^\infty dE|\phi_E\rangle\langle\phi_E|=I \tag{2.5}$$

and

$$\Omega\Omega^\dagger=\int_0^\infty dE|\psi_E\rangle\langle\psi_E|=I-P_d\,. \tag{2.6}$$

Here I is the identity operator and P_d is the projection operator on the discrete spectrum of H. Combining Eqs. (2.5) and (2.6), we have

$$P_d=\Omega^\dagger\Omega-\Omega\Omega^\dagger\,. \tag{2.7}$$

Since the trace of a projection operator is the dimension of the space it projects onto, we have, for the number of bound states,

$$N=\mathrm{Tr}P_d=\mathrm{Tr}(\Omega^\dagger\Omega-\Omega\Omega^\dagger)\,. \tag{2.8}$$

It is convenient to use two sets of variables in the calculation of the trace in (2.8). The final answer is independent of the variables used, but the proofs are often simpler for a particular choice of variables. One set is the same as that used by Omnes,[13] which consists of the individual kinetic energies $(\omega_1,\omega_2,\omega_3)$ in the over-all center-of-mass system, a total angular momentum J, and its projections M on a space-fixed axis, and M' on a body-fixed axis.

The second set of variables is essentially an angular-momentum decomposition of Faddeev's. A pair of particles is denoted by the symbol α, for example, the 2-3 pair is denoted by $\alpha=1$. In the center of mass of pair α we introduce the kinetic energy ν_α and the relative angular-momentum variables l_α and m_α. These variables refer only to pair α. In the total center-of-mass system we let ω_α be the translational energy of the center of mass of pair α and the third particle. A third total-energy variable $E=\omega_\alpha+\nu_\alpha$ will often be used instead of ω_α. For simplicity, we denote the angular variables l_α and m_α by λ_α; sometimes λ_α is omitted entirely, as it is inessential to the calculations. Obviously, there are three sets of variables as there are three distinct pairs of particles, and we will often change from one description to another. The total angular momentum J and its projection M on a space-fixed axis complete the set of variables. We will always work in a system with J and M fixed, so they will be omitted.

Before discussing the three-body problem, it is necessary to have the solution to the two-body Lippmann-

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[8] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].
[9] L. D. Faddeev, Dokl. Akad. Nauk. SSSR **138**, 565 (1961) [English transl.: Soviet Phys.—Doklady **6**, 384 (1961)].
[10] L. D. Faddeev, Dokl. Akad. Nauk SSSR **145**, 301 (1962) [English transl.: Soviet Phys.—Doklady **7**, 600 (1963)].
[11] L. D. Faddeev, *Mathematical Problems of the Quantum Theory of Scattering for a Three-Particle System* (Steklov Mathematical Institute, Leningrad, 1963), No. 69 (English transl.: H. M. Stationery Office, Harwell, 1964).
[12] C. Møller, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **23**, No. 1 (1945).
[13] R. Omnes, Phys. Rev. **134**, B1358 (1964).

Schwinger equation[14] for the t matrix,

$$t_\alpha(\nu_\alpha;\nu_\alpha';\lambda_\alpha;s)=V_\alpha(\nu_\alpha;\nu_\alpha';\lambda_\alpha)-\int_0^\infty d\nu_\alpha''\frac{V_\alpha(\nu_\alpha;\nu_\alpha'';\lambda_\alpha)t_\alpha(\nu_\alpha'';\nu_\alpha';\lambda_\alpha;s)}{\nu_\alpha''-s}. \quad (2.9)$$

We have assumed that the potential is of the form $V_{12}(|\mathbf{r}_1-\mathbf{r}_2|)$ in coordinate space so that V_α and t_α are diagonal in λ_α. The kernel of the three-body equations involves the operator $T_\alpha(s)$:

$$\langle\nu,\lambda,\omega|T_\alpha(s)|\nu',\lambda',\omega'\rangle=\delta(\omega_\alpha-\omega_\alpha')\times\delta(\lambda_\alpha,\lambda_\alpha')t_\alpha(\nu_\alpha;\nu_\alpha';\lambda_\alpha;s-\omega_\alpha). \quad (2.10)$$

Although the three-body transition operator satisfies an integral equation like (2.9), the kernel is not compact, because of the disconnected graphs. However, it is possible to define operators that satisfy a set of coupled integral equations in which the disconnected terms are explicitly summed. An iterate of the kernel of these equations has been shown by Faddeev to be compact.

Let $M_{\alpha\beta}(s)$ be the amplitude for an interaction where pair α is the first to interact and pair β is the last. These operators satisfy the equations

$$M_{\alpha\beta}(s)=\delta_{\alpha\beta}T_\alpha(s)-T_\alpha(s)\frac{1}{H_0-s}\sum_{\gamma\neq\alpha}M_{\gamma\beta}(s). \quad (2.11)$$

Here H_0 is the energy operator for all particles free and noninteracting. In our representation, it is just multiplication by $E=\nu_\alpha+\omega_\alpha$. The kernel of the operator will be written

$$\langle\nu,\omega,\lambda|M_{\alpha\beta}(s)|\nu',\omega',\lambda\rangle=M_{\alpha\beta}(\omega,\nu,\lambda;\omega',\nu',\lambda';s)$$

or

$$\langle\omega_1,\omega_2,\omega_3,M|M_{\alpha\beta}(s)|\omega_1',\omega_2',\omega_3',M'\rangle=M_{\alpha\beta}(\omega_1,\omega_2,\omega_3,M;\omega_1',\omega_2',\omega_3',M';s), \quad (2.12)$$

depending upon which variables we are using.

The generalization of the wave matrix is given by

$$\Omega_0=\delta(\omega-\omega')\delta(\nu-\nu')\delta(\lambda,\lambda')-\sum_{\alpha,\beta}\frac{M_{\alpha\beta}(\omega,\nu,\lambda;\omega',\nu',\lambda';s=\omega'+\nu'+i\epsilon)}{\omega+\nu-\omega'-\nu'-i\epsilon}. \quad (2.13)$$

If there are no two-particle bound states, the projection operator on the three-particle bound states is

$$\Lambda=\Omega_0^\dagger\Omega_0-\Omega_0\Omega_0^\dagger. \quad (2.14)$$

The operator Ω_0 is a sum of several terms which we write as

$$\Omega_0=1-W_1-W_2-W_3-W_0, \quad (2.15)$$

with W_1, W_2, W_3 being the disconnected terms,

$$\langle\omega,\nu,\lambda|W_\alpha|\omega',\nu',\lambda'\rangle=\delta(\omega_\alpha-\omega_\alpha')\delta(\lambda_\alpha,\lambda_\alpha')\times t_\alpha(\nu_\alpha;\nu_\alpha';\lambda_\alpha;\nu_\alpha'+i\epsilon)/(\nu_\alpha-\nu_\alpha'-i\epsilon). \quad (2.16)$$

[14] B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

The term W_0 is that part of (2.13) with no delta functions, that is, the connected part. Using Eq. (2.15) we have, for Λ,

$$\Lambda=\sum_{\alpha=\beta=0;\,\alpha\neq\beta}^{3}[W_\alpha^\dagger,W_\beta]+[W_0^\dagger,W_0]+\sum_{\alpha=1}[W_\alpha^\dagger,W_\alpha]. \quad (2.17)$$

The last term is the two-body expression equivalent to Eq. (2.7) and can be written

$$[W_\alpha^\dagger,W_\alpha]=\delta(\omega_\alpha-\omega_\alpha')P_\alpha, \quad (2.18)$$

where P_α is a projection operator on the two-particle bound states of pair α. Since we assume there are no two-particle bound states, $P_\alpha=0$. Later we include the possibility of these bound states.

Because $P_\alpha=0$, we need only take the trace of the terms in (2.17) that do not have an over-all delta function. The answer is given in terms of the three-to-three S matrix, which is defined by

$$S_{00}=\delta(\omega-\omega')\delta(\nu-\nu')\delta(\lambda,\lambda')-2\pi i\delta(\omega+\nu-\omega'-\nu')T_{00}, \quad (2.19)$$

with

$$T_{00}=\sum_{\alpha,\beta}M_{\alpha\beta}(\omega,\nu,\lambda;\omega',\nu',\lambda';s=\nu'+\omega'+i\epsilon). \quad (2.20)$$

The trace of Λ is evaluated in Appendix A, and the separation of eigenphase shifts is discussed in the next section.

3. THREE-BODY LEVINSON THEOREM

The number of three-body bound states of the system can now be obtained by taking the trace of both sides of (2.14). The result, as given in Appendix A, is

$$N=i\pi\int_0^{\infty\,\prime}dE\,\mathrm{Tr}\left\{T_{00}^\dagger\frac{\partial T_{00}}{dE}-T_{00}\frac{\partial T_{00}^\dagger}{\partial E}\right\}. \quad (3.1)$$

The prime on the integral means that terms with an over-all delta function are to be omitted from the trace. To obtain a result in terms of a sum over eigenphase shifts, it is necessary to have a compact operator. A connected T matrix is defined by

$$S_c=1-2\pi i\delta(E-E')T_c, \quad (3.2)$$

with

$$S_c=S_1^\dagger S_2^\dagger S_3^\dagger S_{00}. \quad (3.3)$$

Here S_1 is the two-body S matrix multiplied by $\delta(\omega_1-\omega_1')$:

$$S_1=1-2\pi i\delta(E-E')T_1; \quad (3.4)$$

S_c is a unitary operator, and it is easily verified that T_c has no delta functions in it. For fixed total energy, T_c is a square integrable operator, since its kernel is bounded for all values of the variables and the integration is over a finite range; that is,

$$\mathrm{Tr} T_c T_c^\dagger < \infty \,. \tag{3.5}$$

Because of unitarity, T_c is also a normal operator,

$$T_c^\dagger T_c = T_c T_c^\dagger \,, \tag{3.6}$$

and therefore it has a spectral expansion of the form

$$T_c = -\frac{1}{\pi} \sum_n e^{i\delta_n} \sin\delta_n |e_n\rangle\langle e_n| \,. \tag{3.7}$$

The eigenvalues depend upon the particular order of the S_α in (3.3) but the final result does not. For the total energy $E=0$, $\mathrm{Tr}(T_c^\dagger T_c) \equiv 0$, since the subenergy integrations are over an interval of zero length. Therefore the eigenvalues $\sin^2\delta(E=0)$ all vanish identically.

We now write (3.1) in terms of S_{00},

$$N = i\pi \int_0^{\infty\,\prime} dE \, \mathrm{Tr} \left\{ \frac{1}{4\pi^2} \left[S_{00}^\dagger \frac{\partial S_{00}}{\partial E} - S_{00} \frac{\partial S_{00}^\dagger}{\partial E} \right] - \frac{1}{2\pi i} \frac{\partial}{\partial E} (T_{00} + T_{00}^\dagger) \right\} , \tag{3.8}$$

then we use the fact that trace T_{00} vanishes at zero and infinite energy to eliminate all but the S matrix. Substituting (3.3) for S_{00}, we have

$$N = 4\pi \int_0^{\infty\,\prime} dE \, \mathrm{Tr} \left[(S_c^\dagger S_1^\dagger S_2^\dagger S_3^\dagger) \frac{\partial}{\partial E} (S_3 S_2 S_1 S_c) - (S_3 S_2 S_1 S_c) \frac{\partial}{\partial E} (S_c^\dagger S_1^\dagger S_2^\dagger S_3^\dagger) \right] . \tag{3.9}$$

Using the unitarity of the S matrices and the identity $\mathrm{Tr}AB = \mathrm{Tr}BA$, we have

$$N = \frac{i}{4\pi} \int_0^{\infty\,\prime} dE \, \mathrm{Tr} \left[S_c^\dagger \frac{\partial}{\partial E} S_c - S_c \frac{\partial}{\partial E} S_c^\dagger + S_1^\dagger \frac{\partial}{\partial E} S_1 + S_2^\dagger \frac{\partial}{\partial E} S_2 + S_3^\dagger \frac{\partial}{\partial E} S_3 - S_1 \frac{\partial}{\partial E} S_1^\dagger - S_2 \frac{\partial}{\partial E} S_2^\dagger - S_3 \frac{\partial}{\partial E} S_3^\dagger \right] . \tag{3.10}$$

The prime on the integral reminds us that all the terms with an over-all delta function are to be omitted. Finally, then, we have

$$N = \frac{i}{4\pi} \int_0^\infty dE \, \mathrm{Tr} \left[S_c^\dagger \frac{\partial}{\partial E} S_c - S_c \frac{\partial}{\partial E} S_c^\dagger \right] . \tag{3.11}$$

This can be rewritten with T_c rather than S_c,

$$N = i\pi \int_0^\infty dE \, \mathrm{Tr} \left\{ T_c^\dagger \frac{\partial}{\partial E} T_c - T_c \frac{\partial}{\partial E} T_c^\dagger \right\} . \tag{3.12}$$

To compute the trace, we use the eigenfunctions of T_c as a basis. The diagonal elements are easily computed to give

$$\langle e_n | T_c^\dagger \frac{\partial T_c}{\partial E} - T_c \frac{\partial}{\partial E} T_c^\dagger | e_n \rangle = e^{i\delta_n} \sin\delta_n \langle e_n | \frac{\partial T_e}{\partial E} | e_n \rangle - e^{i\delta_n} \sin\delta_n \langle e_n | \frac{\partial T_c^\dagger}{\partial E} | e_n \rangle . \tag{3.13}$$

The eigenvalues of T_c are given as a functional which is stationary with respect to variations of the wave functions,

$$(-e^{i\delta_n}/\pi) \sin\delta_n = \langle e_n | T_c | e_n \rangle / \langle e_n | e_n \rangle . \tag{3.14}$$

Taking the derivative of both sides with respect to E, we have

$$\frac{\langle e_n | \partial T_c / \partial E | e_n \rangle}{\langle e_n | e_n \rangle} = \frac{-\partial (e^{i\delta_n} \sin\delta n)}{\pi \partial E} , \tag{3.15}$$

since the derivative of the eigenfunction gives zero because of the stationary property. Finally, then, (3.13) becomes

$$\langle e_n | T_c^\dagger \frac{\partial T_c}{\partial E} - T_c \frac{\partial T_c^\dagger}{\partial E} | e_n \rangle = \frac{2i}{\pi^2} \sin^2\delta_n \frac{d\delta_n}{dE} . \tag{3.16}$$

To obtain the trace, the above expression is summed over n to yield

$$N = \frac{-2}{\pi} \int_0^\infty dE \sum_n \sin^2\delta_n \frac{d\delta_n}{dE} . \tag{3.17}$$

Interchanging integration and summation, we have

$$N = \frac{1}{\pi} \sum_n \left\{ \delta_n(0) - \delta_n(\infty) - \frac{\sin 2\delta_n(0)}{2} + \frac{\sin 2\delta_n(\infty)}{2} \right\} . \tag{3.18}$$

The integration and summation can be interchanged if the partial sums are bounded by an integrable function. The partial sums are bounded if only a finite number of phase shifts have arbitrarily large derivatives. We assume that this is the case. The bound is integrable provided that the T matrix falls to zero sufficiently rapidly as $E \to \infty$.

Since the amplitude vanishes at infinite energy, $\sin\delta(\infty) = 0$; we have already shown $\sin\delta(0) = 0$, therefore we have

$$N = \frac{1}{\pi} \sum \{ \delta_n(0) - \delta_n(\infty) \} . \tag{3.19}$$

4. THREE-BODY WAVE MATRICES IN THE PRESENCE OF TWO-BODY BOUND STATES

In this section, we extend the discussion of Sec. 2 to allow for the presence of two-body bound states. In that case, it is necessary to know the bound-state wave function ψ_α:

$$\psi_\alpha(\nu_\alpha;\lambda_\alpha)=\frac{1}{\nu_\alpha+B_\alpha}\times\int_0^\infty d\nu_\alpha' V_\alpha(\nu_\alpha;\nu_\alpha';\lambda_\alpha)\psi_\alpha(\nu_\alpha;\lambda_\alpha)\,,\quad(4.1)$$

where the binding energy is $-B_\alpha$. The wave functions are normalized to unity:

$$\int_0^\infty d\nu_\alpha|\psi_\alpha(\nu_\alpha;\lambda_\alpha)|^2=1\,.\quad(4.2)$$

We will assume that there is one s-wave bound state in each two-body system. This is not essential, but it simplifies the algebra considerably. In this case, $\lambda_\alpha=\{l,m\}_\alpha\equiv 0$ for the bound-state pair α.

The bound state causes the two-body t matrix to have a pole at $s=-B_\alpha$. The three-body amplitude $M_{\alpha\beta}$ will then have a pole at $s=\omega_\alpha-B_\alpha$. Similarly $M_{\alpha\beta}$ has a pole at $s=\omega_\beta'-B_\beta$. The residue at these poles and at the double pole $s=\omega_\beta'-B_\beta=\omega_\alpha-B_\alpha$ are closely related to the S matrices for bound-state scattering. To be more precise, it is not the residue of $M_{\alpha\beta}$ but rather the residue of $M_{\alpha\beta}$ with the two-body wave function projected out. We list these residues and their relationship to the S matrices and the Møller wave matrices in Eqs. (4.3) to (4.12). For a complete discussion of their properties the reader may consult Ref. 11. The residue at $s=\omega_\beta'-B_\beta$ with the wave function projected out is given by

$$L_{\alpha\beta}(\omega,\nu,\lambda;\omega_\beta';s)=(s+B_\beta-\omega_\beta')\times\int_0^\infty\frac{M_{\alpha\beta}(\omega,\nu,\lambda;\nu_\beta',\nu_\beta',0;s)}{\nu_\beta'+\omega_\beta'-s}\psi_\beta(\nu_\beta')d\nu_\beta'\,,\quad(4.3)$$

and the residue at $s=\omega_\alpha-B_\alpha$ by

$$\tilde{L}_{\alpha\beta}(\omega_\alpha;\omega',\nu',\lambda';s)=(s+B_\alpha-\omega_\alpha)\times\int_0^\infty\sum_{\lambda_\alpha}\frac{d\nu_\alpha M_{\alpha\beta}(\omega_\alpha,\nu_\alpha,0;\omega,\nu,\lambda;s)\psi_\alpha^*(\nu_\alpha)}{\nu_\alpha+\omega_\alpha-s}\,.\quad(4.4)$$

The $M_{\alpha\beta}$ satisfy the relation

$$M_{\alpha\beta}(\omega,\nu,\lambda;\omega',\nu',\lambda';s)=M_{\beta\alpha}^*(\omega',\nu',\lambda';\omega,\nu,\lambda;s^*)\,,\quad(4.5)$$

and the $L_{\alpha\beta}$ satisfy

$$L_{\alpha\beta}^*(\omega,\nu,\lambda;\omega_\beta';s^*)=\tilde{L}_{\beta\alpha}(\omega_\beta';\omega,\nu,\lambda;s)\,.\quad(4.6)$$

The $L_{\alpha\beta}$ operator has a unity term in it coming from the projection of the term $T_\alpha\delta_{\alpha\beta}$ in Eq. (4.3). Separating this term out, we define an operator $K_{\alpha\beta}$ and the corresponding operator $\tilde{K}_{\alpha\beta}$ by

$$L_{\alpha\beta}=(\nu_\alpha+B_\alpha)\psi_\alpha(\nu_\alpha)\delta(\omega_\alpha-\omega_\alpha')\delta_{\alpha\beta}+K_{\alpha\beta}(\omega,\nu,\lambda;\omega_\beta';s)\,.\quad(4.7)$$

The residue of $M_{\alpha\beta}$ at the double pole with both wave functions removed is denoted $F_{\alpha\beta}$, and it is obtained by separating $K_{\alpha\beta}$ into a term regular at $s=-B_\alpha+\omega_\alpha$ and a pole term:

$$K_{\alpha\beta}=G_{\alpha\beta}+[(\nu_\alpha+B_\alpha)\psi_\alpha(\nu_\alpha)/(s+B_\alpha-\omega_\alpha)]\times F_{\alpha\beta}(\omega_\alpha;\omega_\beta';s)\,,\quad(4.8)$$

$$\tilde{K}_{\alpha\beta}=\tilde{G}_{\alpha\beta}+F_{\alpha\beta}(\omega_\alpha;\omega_\beta';s)\times[(\nu_\beta'+B_\beta)\psi_\beta(\nu_\beta')/(s+B_\beta-\omega_\beta')]\,.\quad(4.9)$$

We define three isometric operators by

$$\Omega_\alpha=\sum_\beta\frac{L_{\beta\alpha}(\omega,\nu,\lambda;\omega_\alpha';s=-B_\alpha+\omega_\alpha'+i\epsilon)}{(\omega+\beta+B_\alpha-\omega_\alpha'-i\epsilon)}\,.\quad(4.10)$$

The S matrix is given by

$$S_{\alpha\beta}=\delta_{\alpha\beta}\delta(\omega_\alpha-\omega_\alpha')-2\pi i(\omega_\alpha-B_\alpha-\omega_\beta'+B_\beta)T_{\alpha\beta}(\omega_\alpha;\omega_\beta')\quad(4.11)$$

and

$$S_{0\alpha}=-2\pi i\delta(\omega+\nu+B_\alpha-\omega_\alpha')T_{0\alpha}(\omega,\nu,\lambda;\omega_\alpha')\,,\quad(4.12)$$

where

$$T_{\alpha\beta}(\omega_\alpha;\omega_\beta')\equiv F_{\alpha\beta}(\omega_\alpha;\omega_\beta';s=\omega_\beta'-B_\beta+i\epsilon)$$

and

$$T_{0\alpha}(\omega,\nu,\lambda;\omega_\alpha')\equiv\sum_\beta K_{\beta\alpha}(\omega,\nu,\lambda;\omega_\alpha';s=-B_\alpha+\omega_\alpha'+i\epsilon)\,.$$

The subscript zero denotes a state with all particles free, for example, S_{01} is the S matrix for particle-1 scattering on a bound state of particles 2 and 3 with all final particles free. S_{12} is the S matrix for a rearrangement collision with particle 2 free initially and particle 1 free in the final state.

The Ω operators are formally defined by Faddeev[11] to be a mapping of one Hilbert space onto another. Define the space $\hat{h}$ by the orthogonal sum

$$\hat{h}=h_0\oplus h_1\oplus h_2\oplus h_3\,,\quad(4.13)$$

where h_0 is the space of functions of the variables ω, ν, λ that satisfy

$$\sum_\lambda\int_0^\infty d\omega\int_0^\infty d\nu|f_0(\omega,\nu,\lambda)|^2<\infty\,,\quad(4.14)$$

and h_α is the space of square integrable functions of ω_α

$$\int_0^\infty d\omega_\alpha|f_\alpha(\omega_\alpha)|^2<\infty\,.\quad(4.15)$$

The subspaces h_0, h_α reduce the total energy operator $\hat{H}$

defined on $\hat{h}$ as follows:

$$\text{if } f_0 \epsilon h_0, \quad \text{then} \quad \hat{H} f_0 = (\omega+\nu) f_0; \\ \text{if } f_\alpha \epsilon h_\alpha, \quad \text{then} \quad \hat{H} f_\alpha = (-B_\alpha + \omega_\alpha) f_\alpha. \tag{4.16}$$

Here $\hat{H}$ is the total energy of a free or "asymptotic" system, either a bound state plus a free particle or all particles free. The total Hamiltonian H acts on a space h which is formally identical to h_0. We now define an isometric operator $\hat{\Omega}$ which maps $\hat{h}$ onto h. It is reduced by the subspaces h_0, h_α with

$$\hat{\Omega} f_0 = \Omega_0 f_0$$

and

$$\hat{\Omega} f_\alpha = \Omega_\alpha f_\alpha. \tag{4.17}$$

The states f_0 and f_α are continuum states, and they are mapped only onto continuum states of H in h. Hence if f_d is a discrete eigenstate of H, then $\hat{\Omega}^\dagger f_d = 0$. The orthogonality relations

$$\Omega_\beta{}^\dagger \Omega_\alpha = I_\alpha \delta_{\alpha\beta} = \delta(\omega_\alpha - \omega_\alpha{}') \delta_{\alpha\beta},$$

$$\Omega_0{}^\dagger \Omega_\alpha = 0,$$

and

$$\Omega_0{}^\dagger \Omega_0 = I_0 = \delta(\omega-\omega')\delta(\nu-\nu')\delta(\lambda,\lambda') \tag{4.18}$$

also hold where I_0 and I_α are the identity operators on h_0 and h_α, respectively. Finally, then, we have

$$\Omega_0 \Omega_0{}^\dagger + \sum_\alpha \Omega_\alpha \Omega_\alpha{}^\dagger = I - P_d, \tag{4.19}$$

where I is the identity on h and P_d is the projection operator on the space spanned by the discrete eigenstates of H. Since h is formally the same space as h_0, I is the same as I_0, and (4.19) becomes

$$\Lambda_d = \Omega_0{}^\dagger \Omega_0 - \Omega_0 \Omega_0{}^\dagger - \sum_\alpha \Omega_\alpha \Omega_\alpha{}^\dagger. \tag{4.20}$$

By taking the trace of (4.20) we have an expression for the number of bound states of H.

5. THREE-BODY LEVINSON'S THEOREM IN THE PRESENCE OF TWO-BODY BOUND STATES

The trace of the first two terms of the Eq. (4.20) has already been evaluated with the exception of the parts having an over-all delta function. That part was given in Eq. (2.18),

$$[W_\alpha{}^\dagger, W_\alpha] = \delta(\omega_\alpha - \omega_\alpha{}') P_\alpha = I_\alpha P_\alpha.$$

When there were no two-body bound states, P_α was zero, but now it must be included. The identity operator is replaced by $\Omega_\alpha{}^\dagger \Omega_\alpha$, since they are equal, and then Λ_d becomes

$$\Lambda_d = \sum_{\alpha,\beta=0;\,\alpha\neq\beta}^{3} [W_\alpha{}^\dagger, W_\beta] + [W_0{}^\dagger, W_0] + \sum_{\alpha=1}^{3} \{P_\alpha \Omega_\alpha{}^\dagger \Omega_\alpha - \Omega_\alpha \Omega_\alpha{}^\dagger\}. \tag{5.1}$$

The W operators are given in Eq. (2.16), and the actual calculation of the traces is done in Appendices A and B. The number of three-body bound states is given by

$$N = i\pi \Bigg\{ \int_0^{\infty\,\prime} dE \, \mathrm{Tr}\left[T_{00}{}^\dagger \frac{\partial T_{00}}{\partial E} - T_{00} \frac{\partial T_{00}{}^\dagger}{\partial E} \right] + \sum_\alpha \int_0^\infty dE \, \mathrm{Tr}\left[T_{0\alpha}{}^\dagger \frac{\partial}{\partial E} T_{0\alpha} - T_{0\alpha} \frac{\partial}{\partial E} T_{0\alpha}{}^\dagger \right] + \sum_{\alpha,\beta} \int_{-\min(B_\alpha,B_\beta)}^{\infty} \left[T_{\alpha\beta}{}^\dagger \frac{\partial}{\partial E} T_{\alpha\beta} - T_{\alpha\beta} \frac{\partial}{\partial E} T_{\alpha\beta}{}^\dagger \right] \Bigg\}. \tag{5.2}$$

If we write the T matrix in block form,

$$T = \begin{bmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{bmatrix}, \tag{5.3}$$

then

$$N = i\pi \int' dE \, \mathrm{Tr}\left[T^\dagger \frac{\partial}{\partial E} T - T \frac{\partial}{\partial E} T^\dagger \right]. \tag{5.4}$$

The S matrix can also be written in block form,

$$S = \hat{I} - 2\pi i \delta(E-E') T, \tag{5.5}$$

with

$$\hat{I} = \begin{bmatrix} I_0 & 0 & 0 & 0 \\ 0 & I_1 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & I_3 \end{bmatrix},$$

and I_0, I_α are defined in Eq. (4.18). With the use of the above relation, the expression for N can be rewritten in a form similar to Eq. (3.8):

$$N = \frac{i}{4\pi} \int' dE \, \mathrm{Tr}\left[S^\dagger \frac{\partial S}{\partial E} - S \frac{\partial S^\dagger}{\partial E} + \frac{\partial}{\partial E}(S^\dagger - S) \right]. \tag{5.6}$$

Since the trace of each amplitude $T_{\alpha\beta}$ is assumed to vanish at its threshold and at infinite energy, and since the amplitudes are continuous through other thresholds, the term

$$\int' dE \, \mathrm{Tr}\left[\frac{\partial}{\partial E}(S^\dagger - S) \right] \equiv 0.$$

Define a unitary operator U by

$$U = \begin{bmatrix} S_1{}^\dagger S_2{}^\dagger S_3{}^\dagger & 0 & 0 & 0 \\ 0 & I_1 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & I_3 \end{bmatrix}. \tag{5.7}$$

The operator $S_1{}^\dagger S_2{}^\dagger S_3{}^\dagger$ was discussed in Sec. 3. A unitary connected S matrix can now be defined by

$$S_c = US, \tag{5.8}$$

and a connected T matrix by

$$S_c=\hat{I}-2\pi i\delta(E-E')T_c. \tag{5.9}$$

Substituting $U^\dagger S_c$ for S in Eq. (5.6), we obtain

$$N=\frac{i}{4\pi}\int' dE\,\mathrm{Tr}\left[(S_c{}^\dagger U)\frac{\partial}{\partial E}(U^\dagger S_c)-U^\dagger S_c\frac{\partial}{\partial E}(S_c{}^\dagger U)\right].$$

With the use of $S_cS_c{}^\dagger=\hat{I}$, $U^\dagger U=\hat{I}$, and $\mathrm{Tr}(AB-BA)=0$, the above expression simplifies to

$$N=\frac{i}{4\pi}\int' dE\,\mathrm{Tr}\left[U\frac{\partial U^\dagger}{\partial E}-U^\dagger\frac{\partial U}{\partial E}+S_c{}^\dagger\frac{\partial S_c}{\partial E}-S_c\frac{\partial S_c{}^\dagger}{\partial E}\right]. \tag{5.10}$$

The prime on the integral requires that the terms with an over-all delta function be omitted, that is, the U terms. Finally, then, we have

$$N=\frac{i}{4\pi}\int dE\,\mathrm{Tr}\left[S_c{}^\dagger\frac{\partial S_c}{\partial E}-S_c\frac{\partial S_c{}^\dagger}{\partial E}\right]. \tag{5.11}$$

The eigenfunctions of T_c are used to compute the trace. For fixed energy T_c is a normal operator, since unitarity requires

$$T_c{}^\dagger T_c=T_cT_c{}^\dagger,$$

and it is square integrable, since all integrations are over a finite range and there are no singularities in T_c. Hence it has a spectral decomposition

$$T_c=\frac{1}{\pi}\sum_n e^{i\delta_n}\sin\delta_n|\phi_n\rangle\langle\phi_n|, \tag{5.12}$$

where ϕ_n form an orthonormal set not necessarily complete. To make the set complete, an orthonormal set of functions spanning the null space of T_c is added. The trace in Eq. (5.11) is computed with the ϕ's as a basis. The diagonal elements are given by

$$\langle\phi_n|S_c{}^\dagger\frac{\partial S_c}{\partial E}-S_c\frac{\partial S_c{}^\dagger}{\partial E}|\phi_n\rangle=4i\frac{d\delta_n}{dE}. \tag{5.13}$$

Suppose the thresholds are ordered in the following way:

$$0>-B_1>-B_2>-B_3;$$

then the answer for the number of three-body bound states is

$$N=\sum_n[\delta_n(0)-\delta_n(\infty)]+\sum_n[\delta_n(-B_1)-\delta_n(0)]+\sum_n[\delta_n(-B_2)-\delta_n(-B_1)]+\sum_n[\delta_n(-B_3)-\delta_n(-B_2)]. \tag{5.14}$$

The phase shifts are determined only modulo π, and since they must be a multiple of π at infinite energy, we are free to choose them to be zero. We can further require them to be continuous across the thresholds of newly opening channels. Rather than require the phase shifts at infinite energy to be zero, we will specify that only a finite number can be nonzero. The sum of the phase shifts will converge at any energy, and the only contribution will be from the elastic phase shifts at their thresholds

$$N\pi=\sum_\alpha\sum_{n_\alpha}\delta_{n_\alpha}{}^{\mathrm{el}}(-B_\alpha)+\sum_n\delta_n{}^{\mathrm{el}}(0)-\sum_n\delta_n(\infty). \tag{5.15}$$

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APPENDIX A

In this Appendix we evaluate the trace of the right-hand side of Eq. (2.17),

$$\Lambda=\sum_{\alpha,\beta=0;\,\alpha\neq\beta}^{3}[W_\alpha{}^\dagger,W_\beta]+[W_0{}^\dagger,W_0].$$

We have left out the term $[W_\alpha{}^\dagger,W_\alpha]$, since it is given by Eq. (2.18). Although there are a great many terms to evaluate, only three of them are different, so it is sufficient to calculate

$$\Lambda_{12}=\mathrm{Tr}[W_1{}^\dagger,W_2],$$

$$\Lambda_{01}=\mathrm{Tr}[W_1{}^\dagger,W_0],$$

and

$$\Lambda_{00}=\mathrm{Tr}[W_0{}^\dagger,W_0]. \tag{A1}$$

It is convenient to use the set of variables used by Omnes[13] and discussed in Sec. 2. We add one redundant variable, the total energy $E=\omega_1+\omega_2+\omega_3$. With this choice of variables, the operators W_α become

$$\langle\omega_1,\omega_2,\omega_3,M|W_\alpha|\omega_1',\omega_2',\omega_3',M'\rangle=\delta(\omega_\alpha-\omega_\alpha')\frac{t_\alpha(\omega_1,\,\omega_2,\,\omega_3;\,\omega_1',\,\omega_2',\,\omega_3';\,s=\omega_1'+\omega_2'+\omega_3'-\omega_\alpha'+i\epsilon;\,M,\,M')}{E-E'-i\epsilon}, \tag{A2}$$

$$\langle\omega_1,\omega_2,\omega_3,M|W_0|\omega_1,\omega_2,\omega_3,M'\rangle=T_0(\omega_1,\,\omega_2,\,\omega_3;\,\omega_1'\,\omega_2',\,\omega_3';\,s=\omega_1'+\omega_2'+\omega_3'+i\epsilon;\,M,\,M'). \tag{A3}$$

The total T matrix as given in Eq. (2.20) is just the sum

$$T_{00}=\sum_{\alpha}\delta(\omega_\alpha-\omega_\alpha')t_\alpha+T_0. \tag{A4}$$

The variables M, M', and s are omitted, as the M, M' variables are always involved in finite sums which present no problem. The arguments ω_α are always positive, so if one of them is replaced by $E-\omega_1-\omega_2$, for example, the entire expression is to be multiplied by a step function $\theta(E-\omega_1-\omega_2)$. This is also omitted, but implicitly understood to be present. To further simplify the notation, the set of variables ω_1, ω_2, ω_3 is denoted by ω whenever there can be no misunderstanding.

In this notation, the expression for Λ_{12} becomes

$$\Lambda_{12}=\mathrm{Tr}\int_0^\infty d\omega''\left\{\frac{t_1^\dagger(\omega,\omega'')t_2(\omega'',\omega')\delta(\omega_1-\omega_1'')\delta(\omega_2''-\omega_2')}{(E-E''-i\epsilon)(E'-E''+i\epsilon)}-\frac{t_2(\omega,\omega'')t_1^\dagger(\omega'',\omega')\delta(\omega_1''-\omega_1')\delta(\omega_2-\omega_2'')}{(E-E''-i\epsilon)(E'-E''+i\epsilon)}\right\}. \tag{A5}$$

To evaluate this expression, we separate the singular denominators into principal parts and delta functions. We assume that all integrals converge absolutely and uniformly at infinity so that it is permissible to interchange orders of integration except at the point where the denominators both vanish. For simplicity of notation, we let A be the contribution from the product of the two delta functions, C be from the product of the principal parts, and B be from the cross terms. A and C are shown to be zero. A_{12} is easy to evaluate because of the delta functions

$$\begin{aligned}A_{12}=\pi^2\,\mathrm{Tr}\delta(E-E')\{&t_1^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2',E-\omega_1-\omega_2')\\&\times t_2(\omega_1,\omega_2',E'-\omega_1-\omega_2';\omega_1',\omega_2',E'-\omega_1'-\omega_2')-t_1^\dagger(\omega_1',\omega_2,E-\omega_1'-\omega_2;\omega_1',\omega_2',E'-\omega_1'-\omega_2')\\&\times t_2(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1',\omega_2,E-\omega_1'-\omega_2)\}.\end{aligned} \tag{A6}$$

The diagonal elements of the term in the brackets vanish identically, and, since $x\delta(x)=0$, $A_{12}\equiv 0$.

The trace in C_{12} is written out explicitly:

$$\begin{aligned}C_{12}=P\int_0^\infty dE\int_0^\infty dE'\int_0^\infty d\omega_1\int_0^\infty \frac{d\omega_2}{(E-E')^2}\{&t_1^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E'-\omega_1-\omega_2)\\&\times t_2(\omega_1,\omega_2,E'-\omega_1-\omega_2;\omega_1,\omega_2,E-\omega_1-\omega_2)-t_1^\dagger(\omega_1,\omega_2,E'-\omega_1-\omega_2;\omega_1,\omega_2,E-\omega_1-\omega_2)\\&\times t_2(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E'-\omega_1-\omega_2)\}.\end{aligned} \tag{A7}$$

Since the numerator vanishes at $E=E'$ and the integrals converge absolutely and uniformly at infinity, it is permissible to interchange the orders of integration. Since the integrand is antisymmetric in E and E', $C_{12}\equiv 0$. The only nonvanishing contribution to Λ_{12} is B_{12},

$$\begin{aligned}B_{12}=i\pi\,\mathrm{Tr}\frac{P}{E'-E}\int d\omega_1 d\omega_2\{&t_1^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E-\omega_1-\omega_2)\\&\times t_2(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E'-\omega_1-\omega_2)-t_1^\dagger(\omega_1,\omega_2,E'-\omega_1-\omega_2;\omega_1,\omega_2,E'-\omega_1-\omega_2)\\&\times t_2(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E'-\omega_1-\omega_2)+t_1^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E'-\omega_1-\omega_2)\\&\times t_2(\omega_1,\omega_2,E'-\omega_1-\omega_2;\omega_1,\omega_2,E'-\omega_1-\omega_2)-t_1^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E'-\omega_1-\omega_2)\\&\times t_2(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E-\omega_1-\omega_2)\}.\end{aligned} \tag{A8}$$

This is of the form

$$\mathrm{Tr}[1/(E-E')]\{f(E,E')g(E,E)-f(E,E')g(E',E')\},$$

which, when we take the limit $E\to E'$ and integrate, becomes

$$\int dE\, f(E,E)\frac{\partial g(E,E)}{\partial E}.$$

Finally, then, we have

$$B_{12}=i\pi\int_0^\infty dE\int_0^\infty d\omega_1 d\omega_2\{t_1^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E-\omega_1-\omega_2)$$
$$\times\frac{\partial}{\partial E}t_2(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E-\omega_1-\omega_2)-t_2(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E-\omega_1-\omega_2)$$
$$\times\frac{\partial}{\partial E}t_1^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2,E-\omega_1-\omega_2)\}\,. \quad \text{(A9)}$$

This can be put in a compact form by using T_1 and T_2 [see Eq. (2.10)], where T_1 is now an on-the-energy-shell T matrix,

$$B_{12}=i\pi\int_0^\infty dE\,\mathrm{Tr}\left\{T_1^\dagger\frac{\partial T_2}{\partial E}-T_2\frac{\partial T_1^\dagger}{\partial E}\right\}\,. \quad \text{(A10)}$$

The analysis for Λ_{01} is quite similar:

$$\Lambda_{01}=\mathrm{Tr}\int_0^\infty d\omega''\left\{\frac{t_1^\dagger(\omega,\omega'')T_0(\omega'',\omega')\delta(\omega_1-\omega_1'')-T_0(\omega,\omega'')t_1^\dagger(\omega'',\omega')\delta(\omega_1'-\omega_1'')}{(E-E''-i\epsilon)(E'-E''+i\epsilon)}\right\}\,. \quad \text{(A11)}$$

Proceeding as before and doing all the traces except the E trace, we have

$$A_{01}=\pi^2\,\mathrm{Tr}\delta(E-E')\int d\omega_2''d\omega_1 d\omega_2\{t_1^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2'',E-\omega_1-\omega_2'')$$
$$\times T_0(\omega_1,\omega_2'',E'-\omega_1-\omega_2'';\omega_1,\omega_2,E'-\omega_1-\omega_2)-t_1^\dagger(\omega_1,\omega_2'',E'-\omega_1-\omega_2'';\omega_1,\omega_2,E'-\omega_1-\omega_2)$$
$$\times T_0(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2'',E-\omega_1-\omega_2'')\}\,. \quad \text{(A12)}$$

Interchange ω_2 and ω_2'' in the second term, and then the expression is explicitly antisymmetric in E and E' and hence vanishes. In the expression for C_{01}, we do all the traces explicitly:

$$C_{01}=\int_0^\infty dE\int_0^\infty dE'\int_0^\infty\frac{d\omega_2'd\omega_1 d\omega_2}{(E-E')^2}\{t_1^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2',E'-\omega_1-\omega_2')$$
$$\times T_0(\omega_1,\omega_2',E'-\omega_1-\omega_2';\omega_1,\omega_2,E-\omega_1-\omega_2)$$
$$-t_1^\dagger(\omega_1,\omega_2',E'-\omega_1-\omega_2';\omega_1,\omega_2,E-\omega_1-\omega_2)T_0(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2',E'-\omega_1-\omega_2')\}\,. \quad \text{(A13)}$$

Again we interchange ω_2 and ω_2' in the second term. The expression is then explicitly antisymmetric in E and E', and the integral therefore vanishes. As before, the entire contribution comes from B:

$$B_{01}=i\pi\,\mathrm{Tr}\frac{P}{E'-E}\int d\omega_1 d\omega_2 d\omega_2'\{t_1^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2',E-\omega_1-\omega_2')$$
$$\times T_0(\omega_1,\omega_2',E-\omega_1-\omega_2';\omega_1,\omega_2,E'-\omega_1-\omega_2)-t_1^\dagger(\omega_1,\omega_2',E'-\omega_1-\omega_2';\omega_1,\omega_2,E'-\omega_1-\omega_2)$$
$$\times T_0(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2',E'-\omega_1-\omega_2')+t_1^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2',E'-\omega_1-\omega_2')$$
$$\times T_0(\omega_1,\omega_2',E'-\omega_1-\omega_2';\omega_1,\omega_2,E'-\omega_1-\omega_2)-t_1^\dagger(\omega_1,\omega_2',E-\omega_1-\omega_2';\omega_1,\omega_2,E'-\omega_1-\omega_2)$$
$$\times T_0(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1,\omega_2',E-\omega_1-\omega_2')\}\,. \quad \text{(A14)}$$

After interchanging ω_2 and ω_2' in the appropriate places and taking the limit $E\to E'$, we obtain

$$B_{01}=i\pi\int_0^\infty dE\,\mathrm{Tr}\left\{T_1^\dagger\frac{\partial T_0}{\partial E}-T_0\frac{\partial T_1}{\partial E}\right\}\,. \quad \text{(A15)}$$

The final term we have to calculate is Λ_{00}:

$$\Lambda_{00}=\mathrm{Tr}\int d\omega''\left\{\frac{T_0^\dagger(\omega,\omega'')T_0(\omega'',\omega')-T_0(\omega,\omega'')T_0^\dagger(\omega'',\omega')}{(E-E''-i\epsilon)(E'-E''+i\epsilon)}\right\}. \quad \text{(A16)}$$

For A_{00}, we have,

$$A_{00}=\pi^2\,\mathrm{Tr}\delta(E-E')\int_0^\infty d\omega_1 d\omega_2 d\omega_1' d\omega_2'\{T_0^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1',\omega_2',E-\omega_1'-\omega_2')$$
$$\times T_0(\omega_1',\omega_2',E-\omega_1'-\omega_2';\omega_1,\omega_2,E-\omega_1-\omega_2)-T_0(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1',\omega_2',E-\omega_1'-\omega_2')$$
$$\times T_0^\dagger(\omega_1',\omega_2',E-\omega_1'-\omega_2';\omega_1,\omega_2,E-\omega_1-\omega_2)\}. \quad \text{(A17)}$$

Upon interchange of ω_1', ω_2' and ω_1, ω_2, the expression inside the brackets vanishes identically, and therefore $A_{00}=0$. For C_{00} we have,

$$C_{00}=\mathrm{Tr}\int_0^\infty dE\int_0^\infty dE'\int_0^\infty \frac{d\omega_1 d\omega_2 d\omega_1' d\omega_2'}{(E-E')^2}\{T_0^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1',\omega_2',E'-\omega_1'-\omega_2')$$
$$\times T_0(\omega_1',\omega_2',E'-\omega_1'-\omega_2';\omega_1,\omega_2,E-\omega_1-\omega_2)-T_0^\dagger(\omega_1',\omega_2',E'-\omega_1'-\omega_2';\omega_1,\omega_2,E-\omega_1-\omega_2)$$
$$\times T_0(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1',\omega_2',E'-\omega_1'-\omega_2')\}. \quad \text{(A18)}$$

The integrand is antisymmetric upon interchange of all variables, and hence C_{00} vanishes. The final contribution comes from B_{00}:

$$B_{00}=i\pi\,\mathrm{Tr}\frac{P}{E'-E}\int d\omega_1 d\omega_2 d\omega_1' d\omega_2'\{T_0^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1',\omega_2',E-\omega_1'-\omega_2')$$
$$\times T_0(\omega_1',\omega_2',E-\omega_1'-\omega_2';\omega_1,\omega_2,E'-\omega_1-\omega_2)-T_0^\dagger(\omega_1',\omega_2',E'-\omega_1'-\omega_2';\omega_1,\omega_2,E'-\omega_1-\omega_2)$$
$$\times T_0(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1',\omega_2',E'-\omega_1'-\omega_2')+T_0^\dagger(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1',\omega_2',E'-\omega_1'-\omega_2')$$
$$\times T_0(\omega_1',\omega_2',E'-\omega_1'-\omega_2';\omega_1,\omega_2,E'-\omega_1-\omega_2)-T_0^\dagger(\omega_1',\omega_2',E-\omega_1'-\omega_2';\omega_1,\omega_2,E'-\omega_1-\omega_2)$$
$$\times T_0(\omega_1,\omega_2,E-\omega_1-\omega_2;\omega_1',\omega_2',E-\omega_1'-\omega_2')\}. \quad \text{(A19)}$$

This reduces in the usual way to

$$B_{00}=i\pi\int_0^\infty dE\,\mathrm{Tr}\left\{T_0\frac{\partial T_0^\dagger}{\partial E}-T_0^\dagger\frac{\partial T_0}{\partial E}\right\}. \quad \text{(A20)}$$

From Eqs. (A4) and (2.20) the three-body T matrix associated with S_{00} is given by

$$T_{00}=T_1+T_2+T_3+T_0,$$
$$S_{00}=1-2\pi i\delta(E-E')T_{00}. \quad \text{(A21)}$$

Combining all the results of this Appendix, we have

$$\Lambda_0=i\pi\int_0^{\infty\,\prime} dE\,\mathrm{Tr}\left\{T_{00}^\dagger\frac{\partial T_{00}}{\partial E}-T_{00}\frac{\partial T_{00}}{E}\right\}, \quad \text{(A22)}$$

where the prime on the integral means that the disconnected parts—that is, the terms with an over-all delta function—are to be left out.

APPENDIX B

In this Appendix we derive in detail the trace of the third term in Eq. (5.1), which we call Λ_B.

$$\Lambda_B=\sum_{\alpha=1}^{3}\mathrm{Tr}\{P_\alpha\Omega_\alpha^\dagger\Omega_\alpha-\Omega_\alpha\Omega_\alpha^\dagger\}. \quad \text{(B1)}$$

The Ω_α operators were defined in (4.10),

$$\Omega_\alpha=\sum_\beta\frac{L_{\alpha\beta}(\omega,\nu,\lambda;\omega_\alpha')}{\omega+\nu-\omega_\alpha'+B_\alpha-i\epsilon},$$

and the $L_{\alpha\beta}$ and $K_{\alpha\beta}$ in (4.7) and (4.8),

$$L_{\alpha\beta}=(\nu_\alpha+B_\alpha)\psi_\alpha(\nu_\alpha)\delta(\omega_\alpha-\omega_\alpha')\delta_{\alpha\beta}+K_{\alpha\beta},$$
$$K_{\alpha\beta}=G_{\alpha\beta}+\frac{(\nu_\alpha+B_\alpha)\psi_\alpha(\nu_\alpha)F_{\alpha\beta}(\omega_\alpha;\omega_\beta')}{\omega_\alpha-B_\alpha-\omega_\beta'+B_\beta-i\epsilon}.$$

Here P_α is a projection operator on the two-body bound state,

$$P_\alpha=|\psi_\alpha(\nu_\alpha)\rangle\langle\psi_\alpha(\nu_\alpha')|. \quad \text{(B2)}$$

The first term factors into $P_\alpha(\nu_\alpha,\nu_\alpha')\delta(\omega_\alpha-\omega_\alpha')$, and the trace of P_α is unity; $L_{\alpha\beta}$ has a term which is essentially a unit operator and commutes with the other terms to give zero. Therefore Λ_β can be written in terms of K:

$$\Lambda_B=\sum_{\alpha,\beta,\gamma}\mathrm{Tr}\left\{\sum_\lambda\int_0^\infty d\omega''d\nu''\frac{\tilde{K}_{\alpha\gamma}(\omega_\alpha;\omega'',\nu'',\lambda)K_{\beta\alpha}(\omega'',\nu'',\lambda;\omega_\alpha')}{(\omega_\alpha-B_\alpha-\omega''-\nu''-i\epsilon)(\omega_\alpha'-B_\alpha-\omega''-\nu''+i\epsilon)}\right.$$
$$\left.-\int_0^\infty d\omega_\alpha''\frac{K_{\beta\alpha}(\omega,\nu,\lambda;\omega_\alpha'')\tilde{K}_{\alpha\gamma}(\omega_\alpha'';\omega',\nu',\lambda')}{(\omega+\nu-\omega_\alpha''+B_\alpha-i\epsilon)(\omega'+\nu'-\omega_\alpha''+B_\alpha+i\epsilon)}\right\}. \quad \text{(B3)}$$

The usual assumption is made that the orders of integration could be interchanged except for the singularities from the denominators. There are two sources of singularities which occur when $\omega_\alpha'=\omega_\alpha$ and $\omega'+\nu'=\omega+\nu$. The first is exhibited explicitly above and is located at $\omega''+\nu''=\omega_\alpha-B_\alpha$. The second is hidden in the K term itself, and can be seen in Eq. (4.8). It is located at $\omega_\alpha-B_\alpha=\omega_\beta''-B_\beta$ and will occur only when $\beta=\gamma$ in Eq. (B3). The two singularities occur at different points, so they can be discussed separately.

In the proofs there will be many changes of variable of an essentially trivial nature. As in Appendix A, we omit all explicit reference to changes in the integration region. If an argument of a K or an F function is negative, the function is taken to be zero; that is, a step function of all arguments is implied. With this restriction, the integration on all variables is taken over the region of positive arguments of the functions K and F. The variable E is used for the total energy, either $\omega+\nu$ or $\omega_\alpha-B_\alpha$. Hereafter the operation "trace" will refer only to E. All other traces will be done explicitly.

First the singularity at $\omega''+\nu''=\omega_\alpha-B_\alpha$ is discussed as though the one from the F term did not exist. Then, presuming that the first singularity is absent, we treat the F term. The evaluation follows the procedure, that is by now familiar, of splitting the denominators into principal parts and delta functions. The term from the product of delta functions is called A_K or A_F, depending upon which singularity is being discussed. The term from the product of principal parts is called C_K or C_F. The cross terms are B_K and B_F. The first contribution to be evaluated is A_K:

$$A_K=\sum_{\alpha,\beta,\gamma}\mathrm{Tr}\,\delta(E-E'')\int\sum_\lambda d\nu[K_{\beta\alpha}(E-\nu,\nu,\lambda;E+B_\alpha)\tilde{K}_{\alpha\gamma}(E+B_\alpha;E''-\nu,\nu,\lambda)$$
$$-K_{\beta\alpha}(E-\nu,\nu,\lambda;E''+B_\alpha)\tilde{K}_{\alpha\gamma}(E+B_\alpha;E-\nu,\nu,\lambda)]. \quad \text{(B4)}$$

The term in brackets vanishes at $E=E''$, so $A_K=0$. Hereafter the variable λ will be omitted, as it adds nothing to the proof. The evaluation of C_K is straightforward:

$$C_K=\sum_{\alpha,\beta,\gamma}P\int_0^\infty dE\int_0^\infty\frac{dE'}{(E-E')^2}\int_0^\infty d\nu[\tilde{K}_{\alpha\gamma}(E+B_\alpha;E'-\nu,\nu)K_{\beta\alpha}(E'-\nu,\nu;E+B_\alpha)$$
$$-\tilde{K}_{\alpha\gamma}(E'+B_\alpha;E-\nu,\nu)K_{\beta\alpha}(E-\nu,\nu;E'+B_\alpha)]. \quad \text{(B5)}$$

The term in brackets vanishes at $E=E'$, so the principal part integration is well defined. Therefore the orders of integration may be interchanged, and $C_K=0$, because of the antisymmetry of the integrand.

We now consider the contribution from the term involving only one delta function.

$$B_K=i\pi\sum_{\alpha,\beta,\gamma}\mathrm{Tr}\frac{1}{E'-E}\int_0^\infty d\nu\{\tilde{K}_{\alpha\gamma}(E+B_\alpha;E-\nu,\nu)K_{\beta\alpha}(E-\nu,\nu;E'+B_\alpha)$$
$$-\tilde{K}_{\alpha\gamma}(E+B_\alpha;E'-\nu,\nu)K_{\beta\alpha}(E'-\nu,\nu;E'+B_\alpha)+\tilde{K}_{\alpha\gamma}(E+B_\alpha;E'-\nu,\nu)K_{\beta\alpha}(E-\nu,\nu;E+B_\alpha)$$
$$-\tilde{K}_{\alpha\gamma}(E'+B_\alpha;E'-\nu,\nu)K_{\beta\alpha}(E-\nu,\nu;E'+B_\alpha)\}. \quad \text{(B6)}$$

To evaluate B_K, take the limit $E\to E'$, which gives a derivative, and then integrate over E. The final result including the angular variable λ is

$$B_K=i\pi\sum_{\alpha,\beta,\gamma}\sum_\lambda\int_0^\infty dE\int_0^\infty d\nu\left\{\tilde{K}_{\alpha\gamma}(E+B_\alpha;E-\nu,\nu,\lambda)\frac{\partial}{\partial E}K_{\beta\alpha}(E-\nu,\nu,\lambda;E+B_\alpha)\right.$$
$$\left.-K_{B\alpha}(E-\nu,\nu,\lambda;E+B_\alpha)\frac{\partial}{\partial E}\tilde{K}_{\alpha\gamma}(E+B_\alpha;E-\nu,\nu,\lambda)\right\}. \quad \text{(B7)}$$

In terms of the transition operators $T_{0\alpha}$ defined in Eq. (5.12), the result is

$$B_K = i\pi \sum_{\alpha} \int_0^\infty dE\, \mathrm{Tr}\left\{ T_{0\alpha}^\dagger \frac{\partial T_{0\alpha}}{\partial E} - T_{0\alpha} \frac{\partial T_{0\alpha}^\dagger}{\partial E} \right\}. \tag{B8}$$

In addition to the singularity from the three-to-two amplitude, there is a term from the two-to-two amplitude. It comes from the singularity of the F term implicit in (B3). Referring to Eq. (4.8) and substituting the F term for the K term, we see that the only new singularity will come when $\beta=\gamma$. Therefore we have

$$\begin{aligned} A_F = \sum_{\alpha,\beta,\gamma} \delta_{\beta\gamma}\, \mathrm{Tr} \int_0^\infty d\nu_\beta & \\ \times \int_0^\infty & \frac{d\omega'' \tilde{F}_{\alpha\beta}(\omega_\alpha;\omega_\beta'') F_{\beta\alpha}(\omega_\beta'';\omega_\alpha') (\nu_\beta+B_\beta)^2 |\psi_\beta(\nu_\beta)|^2}{[(\omega_\alpha-B_\alpha-\omega''-\nu'')(\omega_\alpha'-B_\alpha-\omega''-\nu'')][(\omega_\beta''-B_\beta-\omega_\alpha'+B_\alpha-i\epsilon)(\omega_\gamma''-B_\gamma-\omega_\alpha+B_\alpha+i\epsilon)]} \\ - \int_0^\infty & \frac{d\omega_\alpha'' F_{\beta\alpha}(\omega_\beta;\omega_\alpha'') \tilde{F}_{\alpha\beta}(\omega_\alpha'';\omega_\beta') (\nu_\beta+B_\beta)^2 |\psi_\beta(\nu_\beta)|^2}{[(\omega+\nu-\omega_\alpha''+B_\alpha)(\omega'+\nu'-\omega_\alpha''+B_\alpha)][(\omega_\beta-B_\beta+B_\alpha-\omega_\alpha''-i\epsilon)(\omega_\gamma'-B_\gamma-\omega_\alpha''+B_\alpha+i\epsilon)]}. \end{aligned} \tag{B9}$$

The $i\epsilon$ has been left out of the denominators already treated, as they are presumed to be nonsingular. The γ dependence has been indicated in the denominators to make it clear that only $\beta=\gamma$ terms are singular. The expression is now evaluated in the usual way in terms of principal parts and delta functions. For A_F we have

$$\begin{aligned} A_F = -\pi^2 \sum_{\alpha,\beta} \mathrm{Tr} \int_0^\infty d\nu |\psi_\beta(\nu)|^2 \{ & \delta(\omega_\alpha-\omega_\alpha') \tilde{F}_{\alpha\beta}(\omega_\alpha; B_\beta+\omega_\alpha-B_\alpha) F_{\beta\alpha}(B_\beta+\omega_\alpha-B_\alpha;\omega_\alpha) \\ & - \delta(\omega_\beta-\omega_\beta') \tilde{F}_{\alpha\beta}(\omega_\beta-B_\beta+B_\alpha;\omega_\beta) F_{\beta\alpha}(\omega_\beta;\omega_\beta-B_\beta+B_\alpha) \}. \end{aligned} \tag{B10}$$

A simple change of variables,

$$\omega_\alpha', \omega_\alpha \to E'+B_\alpha, E+B_\alpha$$

and

$$\omega_\beta', \omega_\beta \to E'+B_\beta, E+B_\beta$$

puts the expression in a form in which it is explicitly antisymmetric in E and E'. It therefore vanishes, since it multiplies $\delta(E-E')$. The calculation for C_F proceeds along similar lines. We change variables to $E=\omega_\gamma-B_\gamma$ and obtain

$$\begin{aligned} C_F = \sum_{\alpha,\beta} P \int_{-\infty}^\infty dE \int_{-\infty}^\infty dE' \int_0^\infty d\nu \frac{(\nu+B_\beta)^2 |\psi_\beta(\nu)|^2}{(E-E')^2} & \\ \times \left\{ \frac{\tilde{F}_{\alpha\beta}(E+B_\alpha; E'+B_\beta) F_{\beta\alpha}(E'+B_\beta; E+B_\alpha)}{(E-\nu-E'-B_\beta)^2} - \frac{F_{\beta\alpha}(E+B_\beta; E'+B_\alpha) \tilde{F}_{\alpha\beta}(E'+B_\alpha; E+B_\beta)}{(E+B_\beta+\nu-E')^2} \right\} &. \end{aligned} \tag{B11}$$

The integrand vanishes at $E=E'$, and therefore the principal part integration is well defined. Interchanging orders of integration and using the antisymmetry of the integrand, we obtain $C_F=0$.

The final term to be evaluated is B_F,

$$\begin{aligned} B_F = i\pi \sum_{\alpha,\beta} \mathrm{Tr} \int_0^\infty d\nu \frac{(\nu+B_\beta)^2 |\psi(\nu)|^2}{(E'-E)} & \\ \times \left\{ \frac{\tilde{F}_{\alpha\beta}(E+B_\alpha; E'+B_\beta) F_{\beta\alpha}(E'+B_\beta; E'+B_\alpha)}{(E'+\nu+B_\beta-E)(\nu+B_\beta)} - \frac{\tilde{F}_{\alpha\beta}(E+B_\alpha; E'+B_\beta) F_{\beta\alpha}(E+B_\beta; E+B_\alpha)}{(\nu+B_\beta)(E'+B_\beta+\nu-E)} \right. & \\ \left. + \frac{\tilde{F}_{\alpha\beta}(E+B_\alpha; E+B_\beta) F_{\beta\alpha}(E+B_\beta; E'+B_\alpha)}{(B_\beta+\nu)(E+\nu+B_\beta-E')} - \frac{\tilde{F}_{\alpha\beta}(E'+B_\alpha; E'+B_\beta) F_{\beta\alpha}(E+B_\beta; E'+B_\alpha)}{(E+\nu+B_\beta-E')(\nu+B_\beta)} \right\} &. \end{aligned} \tag{B12}$$

Upon expansion of the term $(E'+\nu+B_\beta-E)^{-1}$ in powers of $(E-E')$, it is easily seen that the only term that need be kept in the expansion is the constant term, as all others cancel in the limit $E\to E'$. From Eq. (4.2), we have the normalization integral

$$\int_0^\infty d\nu |\psi_\beta(\nu)|^2 = 1.$$

In the limit $E \to E'$, the remaining terms give a derivative to yield

$$B_F = i\pi \sum_{\alpha,\beta} \int_{-\min(B_\alpha,B_\beta)}^\infty \Big\{ dE\, \tilde{F}_{\alpha\beta}(E+B_\alpha; E+B_\beta) \frac{\partial F_{\beta\alpha}}{\partial E}(E+B_\beta; E+B_\alpha) - F_{\beta\alpha}(E+B_\beta; E+B_\alpha) \frac{\partial}{\partial E} \tilde{F}_{\alpha\beta}(E+B_\alpha; E+B_\beta) \Big\} . \quad \text{(B13)}$$

This expression can be rewritten in terms of $T_{\alpha\beta}$ defined in Eq. (5.11),

$$B_F = i\pi \sum_{\alpha,\beta} \int_{-\min(B_\alpha,B_\beta)}^\infty dE \Big\{ T_{\beta\alpha}{}^\dagger \frac{\partial T_{\beta\alpha}}{\partial E} - T_{\beta\alpha} \frac{\partial}{\partial E} T_{\beta\alpha}{}^\dagger \Big\} . \quad \text{(B14)}$$

Combining Eqs. (B14) and (B8), we have

$$\Lambda_B = i\pi \Big[\sum_\alpha \int_0^\infty dE\, \mathrm{Tr} \Big\{ T_{0\alpha}{}^\dagger \frac{\partial}{\partial E} T_{0\alpha} - T_{0\alpha} \frac{\partial}{\partial E} T_{0\alpha}{}^\dagger \Big\} + \sum_{\alpha,\beta} \int_{-\min(B_\alpha,B_\beta)}^\infty dE \Big\{ T_{\alpha\beta}{}^\dagger \frac{\partial}{\partial E} T_{\alpha\beta} - T_{\alpha\beta} \frac{\partial}{\partial E} T_{\alpha\beta}{}^\dagger \Big\} \Big] . \quad \text{(B15)}$$

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Restrictions Implied by Lorentz and Spin Invariance for Scattering Amplitudes

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A combination of Lorentz invariance and spin independence may restrict scattering amplitudes so severely that no interesting reaction can be described. This is demonstrated for the scattering of two spin-$\frac{1}{2}$ particles with spin independence defined as in a definition of $SU(6)$ for quarks.

APPLICATIONS of $SU(3)$ and $SU(6)$ symmetries to strongly interacting particles are achieved in coexistence with a number of theorems[1] delineating a growing class of situations in which a nontrivial combination of the symmetry group with the Poincaré group is impossible. Situations in which these group-theoretic theorems are not applicable are characterized most noticeably by commutators of generators of the symmetry group with generators of the Poincaré group failing to be linear combinations of generators of the two groups. Here a different kind of theorem is to be expected. If a generator of the symmetry group and a generator of the Poincaré group both commute with the scattering operator, then their commutator also commutes with the scattering operator. If this commutator is not a linear combination of the generators of the symmetry and Poincaré groups, it represents an additional symmetry which may put entirely unwanted restrictions on the scattering amplitudes. A particularly simple example of this is demonstrated in the following:

Consider two particles each with positive mass and spin $\frac{1}{2}$. We describe the nth particle ($n=1,2$) by Hermitian position and momentum operators $\mathbf{Q}^{(n)}$ and $\mathbf{P}^{(n)}$ (which satisfy canonical commutation relations) and Hermitian spin operators $\mathbf{S}^{(n)}$ (which commute with $\mathbf{Q}^{(n)}$ and $\mathbf{P}^{(n)}$ and satisfy angular-momentum commutation relations) in terms of which the generators of the Poincaré group for two noninteracting particles have

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