Relativistic Effects in the Form Factors of He^3 and H^3 †

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Relativistic corrections to Schiff's nonrelativistic analysis of the three-body nuclei form factors are estimated. High-energy electron scattering from these nuclei is re-examined using an impulse approximation in which the intermediate nucleon states are taken to propagate and interact as free particles. We conclude that the corrections to the form factors are small ($\lesssim 5\%$) for $-q^2 \lesssim 8\ F^{-2}$. However, it is shown that they assume great importance if the experiments are used to investigate the neutron charge form factor. Various effects arising from the use of the impulse approximation are discussed. These include considerations of current conservation, of the static limit ($q^2 \to 0$), and of the extraction of nuclear form factors from the impulse-approximation nuclear current.

I. INTRODUCTION

INFORMATION on the structure of the three-body nuclei (H^3 and He^3—the trions or trinucleons) and an independent indication of the neutron charge form factor have been obtained from recent analyses of high-energy elastic electron-scattering experiments performed at Stanford.[1] Both nuclei have spin $\frac{1}{2}$, and the Rosenbluth formula[2] has been used in each case to extract two form factors. All radiative corrections have been included in the experimental analyses.

Schiff[3] has attempted to fit these measured form factors by using a nonrelativistic additive electromagnetic nucleon current to describe the nuclear-current operator and making various assumptions concerning the form of the nuclear wave function. Several attempts to improve upon his results by a more detailed examination of the nuclear states[4,5] and through considerations of general meson-exchange effects[5,6] have been made. In all these papers it has been assumed that relativistic effects are small ($<5\%$), the motivation for this coming from the fact that a Foldy-Wouthuysen (F.W.) reduction[7] of the nucleon-current operator yields an additional term of relative order $q^2/8M^2$, which is $<5\%$ (upper limits in this paper will always be defined by the present experimental upper limit of $-q^2 \leqq 8\ F^{-2}$).[8]

It is the purpose of the present paper to examine in more detail, although still not completely, these relativistic corrections. We first point out the physical origin of the parts of the problem from which these corrections arise:

(a) In the intermediate states, both the interacting particle and the two "spectator" particles must be reduced to nonrelativistic (N.R.) form;

(b) the nuclear wave function or, in momentum space, the vertex that connects three nucleons with a trion "the trion-3 nucleon vertex," has, in principle, a completely covariant form and this must be reduced to N.R. form;

(c) the nuclear wave function, referred to in (b), is normally described in the center-of-momentum system (C.M.) of the nucleus, while we shall require its form in a moving frame.

We shall consider (b) and (c) together in what follows. In Sec. II we set up the formalism so that the impulse approximation (I.A.) may be used in a way that we consider the most natural here, namely, that of using truncated sets of intermediate states. The calculation follows in Sec. III, where a partial-integration technique, explicitly illustrated in Appendix I, is employed to perform the integrations over the intermediate nucleon momentum variables. In Appendix II we indicate how nuclear form factors may be extracted from the impulse-approximated T matrix while in Appendix III we show that even in the I.A., current is conserved.

II. FORMALISM

As stated above, the method employed is one closely related to the I.A., which in turn is closely related to the formalism of the many-body theory. Just as there are several equivalent techniques for attacking the latter there are several for the former,[9] the choice of

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[1] H. Collard *et al.*, Phys. Rev. Letters **132**, (1963); Proceedings of the 12th International Conference on High Energy Physics, Dubna, 1964 (Moscow, 1965); Phys. Rev. **138**, B57 (1965).

[2] See, for example, S. D. Drell, and F. Zachariasen, *Electromagnetic Structure of Nucleons* (Oxford University Press, New York, 1961).

[3] L. I. Schiff, Phys. Rev. **133**, B802 (1964).

[4] T. A. Griffy, Phys. Letters **11**, 155 (1964); B. F. Gibson and L. I. Schiff, Phys. Rev. **138**, B26 (1965); J. S. Levinger and B. K. Srivastava, *ibid.* **137**, B246 (1965).

[5] David A. Krueger and A. Goldberg, Phys. Rev. **135**, B934 (1964).

[6] A. Q. Sarker, Phys. Rev. Letters **13**, 375 (1964).

[7] K. W. McVoy and L. Van Hove, Phys. Rev. **125**, 1034 (1962).

[8] The notation used for kinematical quantities is illustrated in Fig. 1 (see also Appendix III). M is the nucleon mass. We use the metric in which the scalar product of two 4-vectors a and b is given by

$$a\cdot b \equiv a_\mu b^\mu = a_0 b^0 - \mathbf{a}\cdot\mathbf{b};$$

and in particular $q^2 = (q^0)^2 - \mathbf{q}^2$. The usual conventions involving summations and indices are observed. For convenience, we put $\hbar = c = 1$, so that the fine structure constant α in these units becomes e^2, the square of the electron charge.

[9] See, for example, G. F. Chew and M. L. Goldberger, Phys. Rev. **87**, 778 (1952), who use the time-independent techniques of Lippmann and Schwinger.

any particular one depending very much upon the nature of the problem. Since high-energy electron scattering is more familiarly described in terms of a time-dependent formalism[2] and because relativistic effects are clearly bound up with the idea of covariance, we employ the time-dependent approach and use the Heisenberg representation. We shall often employ the language of field theory, although this is not necessary.

The complete Hamiltonian H will describe the free fields of electrons, nucleons, mesons, and photons as well as the specific interactions; that is, we take electrons interacting with the nucleon field only through intermediate photons, and nucleons with each other only through mesons. There may also be a photon-meson interaction. All renormalization effects are assumed to have been included.

Let us first consider the electromagnetic (e.m.) interaction (the scattering) which is described by the term

$$\int j_\mu{}^{(n)}(x_n)A^\mu(x_n)d\mathbf{x}_n+\int j_\mu{}^{(\mathrm{el})}(x_{\mathrm{el}})A^\mu(x_{\mathrm{el}})d\mathbf{x}_{\mathrm{el}}\,, \qquad (1)$$

(n indicates the nucleon, el the electron). The electron current $j_\mu{}^{(\mathrm{el})}$ is well known[10]; it is how we treat $j_\mu{}^{(n)}$, the nucleon current, that defines the I.A. We do know that if we include all the mesonic effects in $j_\mu{}^{(n)}$ we can replace it by $j_\mu{}^{(N)}$, the current describing the nucleus (N) as a whole, and which, from general invariance principles,[10] can be written in terms of the nuclear form factors. Alternatively, if we include only part of the mesonic effects (only those contributing to the diagrams in which a meson line begins and ends on the same nucleon, or is coupled to a photon), $j_\mu{}^{(n)}$ can be written in terms of nucleon form factors. The remaining meson effects, in which the meson line connects different nucleons, give rise to the binding and may be treated in some higher order.[9] It is this latter approach that we pursue.

We shall always be working to order α^2 in the e.m. interaction, i.e., only one photon is exchanged between the electron and the nucleus. Higher orders have been discussed elsewhere in dealing with nucleon[2] and α-particle[11] targets, and there is no reason to believe that the contribution in our case exceeds these estimates.

The initial and final states, describing free nuclei and electrons, are denoted by $|\Phi\rangle$; these are eigenfunctions of the complete Hamiltonian H and are assumed to be separable into an electron and nuclear part.

We write the S matrix as

$$S_{ba}=\langle\Phi_b|S_c|\Phi_a\rangle\,, \qquad (2)$$

where the subscript c indicates that we focus our attention on the "bare" nucleons making up the nucleus.

[10] V. Glaser and B. Jaksic, Nuovo Cimento **5**, 197 (1957).

[11] A. Goldberg, Nuovo Cimento **20**, 1191 (1961).

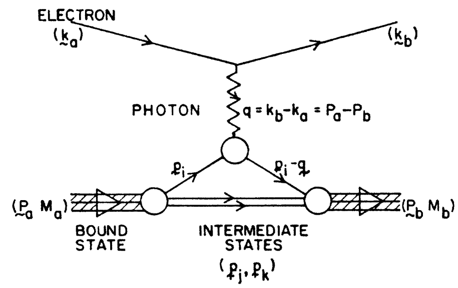

FIG. 1. The zeroth-order impulse-approximation term in electron-trinucleon scattering. The kinematical notation as used in text [see Eqs. (4)] is explicitly illustrated.

We now insert two complete sets of states which must span the space of H. The I.A. essentially limits this set of states to those containing no mesons; we call these $|\phi\rangle$. Hence we define the S matrix in the I.A. as

$$S_{ba}{}^I\equiv\sum_{m,n}\langle\Phi_b|\phi_m\rangle\langle\phi_m|S_c|\phi_n\rangle\langle\phi_n|\Phi_a\rangle. \qquad (3)$$

If we neglect antinucleon effects, the intermediate states will contain only three nucleons.

The electron sums are straightforward and we obtain the expected factor $\langle\mathbf{k}_a+\mathbf{q}|j_\mu{}^{(\mathrm{el})}|\mathbf{k}_a\rangle$. In calculating the nucleon contribution we put the intermediate states on the mass shell; this is equivalent to neglecting the principal value part of the nucleon propagator. This gives rise to a lack of energy conservation at the trion-3 nucleon vertex, normally associated with the I.A.[9]

In terms of diagrams, we are considering only Fig. 1 and rejecting all diagrams of the type of Fig. 2, the so-called ladder diagrams. As to actual computation there are evidently various methods of approach, e.g., one could write down the diagrams initially and use Feynman-type rules to calculate, or, as has been attempted with the deuteron,[12] use dispersion-theory techniques to describe the vertices. We choose the direct method of evaluating each factor separately.

III. CALCULATION

The matrix element $(S_c)_{mn}\equiv\langle\phi_m|S_c|\phi_n\rangle$ describes the scattering from a system of independent nucleons and can be easily evaluated. One obtains—(we suppress the electron factors)—

$$\begin{aligned}S_{ba}{}^I=3\sum_{ijkr}&\langle\mathbf{P}_bm_b|\mathbf{p}_i-\mathbf{q},\lambda_r;\mathbf{p}_j\lambda_j;\mathbf{p}_k\lambda_k\rangle\\&\times\langle\mathbf{p}_i-\mathbf{q},\lambda_r|j^{\mu(n)}|\mathbf{p}_i,\lambda_i\rangle\\&\times\langle\mathbf{p}_i,\lambda_i;\mathbf{p}_j,\lambda_j;\mathbf{p}_k,\lambda_k|\mathbf{P}_am_a\rangle\\&\times\delta[q^0-p_i{}^0+(\mathbf{q}^2+\mathbf{p}_i{}^2-2\mathbf{p}_i\cdot\mathbf{q})^{1/2}]\,,\end{aligned} \qquad (4)$$

where $|\mathbf{P}m\rangle$ represents the nuclear eigenstate of H with c.m. momentum $\mathbf{P}$ and internal quantum numbers m;

[12] See, for example, H. F. Jones, Nuovo Cimento **26**, 790 (1962).

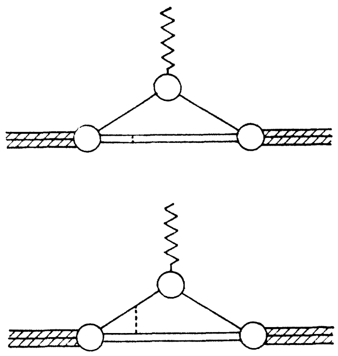

FIG. 2. Two graphs showing typical first-order corrections to Fig. 1 arising from the nuclear vertex. The electron contribution and notation are suppressed.

$|\mathbf{p}_i,\lambda_i;\mathbf{p}_j,\lambda_j;\mathbf{p}_k,\lambda_k\rangle$ is a three-nucleon eigenstate of the noninteracting Hamiltonian (H with the mesonic part subtracted); $\mathbf{p}$ is a momentum and λ spin-isospin variables; p_i^0 is taken to be on the mass shell: $p_i^0=(\mathbf{p}_i^2+M^2)^{1/2}$ (similarly for j and k).

This may be transformed to a coordinate-space representation to give

$$S_{ba}{}^I=\sum_{\{ijk\}}\sum_r\int d\mathbf{p}_i d\mathbf{p}_j d\mathbf{p}_k$$
$$\times\{dx_i'dx_j'dx_k'\bar{U}(\mathbf{P}_b;x_i',x_j',x_k')u(\mathbf{p}_i-\mathbf{q})u(\mathbf{p}_j)u(\mathbf{p}_k)$$
$$\times e^{iP_b\cdot x'}e^{-i(p_i-q)\cdot x_i'}e^{-ip_j\cdot x_j'}e^{-ip_k\cdot x_k'}\eta_r\eta_j\eta_k\}$$
$$\times\langle\mathbf{p}_i-\mathbf{q},\lambda_r|j^{\mu(n)}|\mathbf{p}_i,\lambda_i\rangle$$
$$\times\delta[q^0-p_i^0+(\mathbf{q}^2+\mathbf{p}_i^2-2\mathbf{p}_i\cdot\mathbf{q})^{1/2}]$$
$$\times\left\{\int dx_i dx_j dx_k\bar{u}(\mathbf{p}_i)\bar{u}(\mathbf{p}_j)\bar{u}(\mathbf{p}_k)\right.$$
$$\times e^{ip_i\cdot x_i}e^{ip_j\cdot x_j}e^{ip_k\cdot x_k}\eta_i^\dagger\eta_j^\dagger\eta_k^\dagger$$
$$\left.\times U(\mathbf{P}_a;x_i,x_j,x_k)e^{-iP_a\cdot x}\right\}.\quad(5)$$

The notation $\sum_{\{ijk\}}$ implies a sum over the three cyclic permutations of $\{ijk\}$. The exact relativistic nuclear wave function $U(\mathbf{P};x_i,x_j,x_k)$ transforms like a spin-$\frac{1}{2}$ spinor in "nuclear space"; its dependence on the external coordinates of the nucleons (x_i,x_j,x_k) is explicitly indicated. Likewise, $u(\mathbf{p})$ is a nucleon spinor and transforms like a spin-$\frac{1}{2}$ spinor in nucleon space. We use invariant normalization for both U and u. Nucleon isospinors are represented by η. The c.m. coordinate x is related to the x_i.

We now attempt to make a consistent nonrelativistic reduction of this equation. Immediately one sees the difficulties associated with the time-like integrations and the subsequent extraction of energy conservation. In this examination we limit ourselves to the assumption that, from a proper treatment, these problems would be resolved and that the term $\delta(P_b^0-P_a^0+q^0)$ would factor from $S_{ba}{}^I$ leaving a completely time-independent T matrix. The motivation for this approach is that we are hoping to gain insight into the three-body wave function. The only forms of this that we can construct with any certainty are N.R. and necessarily refer to an "equal times" nucleus. In this case one can show that the relevant δ function does indeed factor out.

We now assume that the above do not introduce errors into the problem any greater than those to be neglected (one can give plausibility arguments to lend weight to this).[13] Further, we work in the laboratory frame (defined by $\mathbf{P}_a=0$) since at the moment, it does not appear that any one frame has any overwhelming advantage or deeper physical significance.

We transform to relative coordinates defined by

$$\mathbf{r}_i=\mathbf{x}_i-\mathbf{x}\quad(\text{similarly for } j,\,k)$$

and change integration variables from $(\mathbf{x}_i,\mathbf{x}_j,\mathbf{x}_k)$ to $(\mathbf{x},\mathbf{r},\boldsymbol{\varrho})$, where $(\mathbf{r},\boldsymbol{\varrho})$ have yet to be specified. The $\mathbf{x}$ and $\mathbf{x}'$ integrations are now straightforward and we can separate out the momentum δ functions

$$\delta[\mathbf{P}_b-(\mathbf{p}_i+\mathbf{p}_j+\mathbf{p}_k)+\mathbf{q}]\delta[\mathbf{p}_i+\mathbf{p}_j+\mathbf{p}_k-\mathbf{P}_a].$$

The $\mathbf{p}_k$ integration then gives the expected $\delta(\mathbf{P}_b-\mathbf{P}_a+\mathbf{q})$.

The relativistic spinors are replaced by their upper ("large") components; this is a consistent approximation so long as $q^2/36M^2\ll1\%$. To see this, consider the following plausibility argument: $U(\mathbf{P})$ contains nucleon spinors whose average momentum is $\sim\frac{1}{3}\mathbf{P}$; the intermediate-state spinors have momentum $\mathbf{p}$ and, as illustrated later, these give contributions $\sim\frac{1}{3}\mathbf{q}$. The lower spinor components, which we shall neglect, therefore give rise to contributions

$$\sim|\tfrac{1}{3}\mathbf{P}||\tfrac{1}{3}\mathbf{q}|/4M^2.$$

As the maximum value of $\mathbf{P}$ is $\mathbf{q}$, this factor becomes $q^2/36M^2\lesssim1\%$, which is comparable to terms already neglected.[13] We now have

$$T_{ba}{}^{(I)}\cong\sum_{\{ijk\}}\sum_r\int d\mathbf{p}_i d\mathbf{p}_j\delta(\mathbf{P}_b-\mathbf{P}_a+\mathbf{q})$$
$$\times\left\{\int d\mathbf{r}'\int d\boldsymbol{\varrho}'H^*(\mathbf{r}'\boldsymbol{\varrho}')\chi_r\chi_j\chi_k\eta_r\eta_j\eta_k\right\}$$
$$\times e^{i(\mathbf{p}_i-\mathbf{q})\cdot\mathbf{r}_i'}e^{i\mathbf{p}_j\cdot\mathbf{r}_j'}e^{-i(\mathbf{p}_i+\mathbf{p}_j)\cdot\mathbf{r}_k'}$$
$$\times\left(\frac{E_j+M}{2M}\right)\left(\frac{E_k+M}{2M}\right)\langle\mathbf{p}_i-\mathbf{q},\lambda_r|j^{\mu(n)\prime}|\mathbf{p}_i,\lambda\rangle$$
$$\times\left\{\int d\mathbf{r}\int d\boldsymbol{\varrho}\chi_i^\dagger\chi_j^\dagger\chi_k^\dagger\eta_i^\dagger\eta_j^\dagger\eta_k^\dagger e^{-i\mathbf{p}_i\cdot\mathbf{r}_i}e^{-i\mathbf{p}_j\cdot\mathbf{r}_j}\right.$$
$$\left.\times e^{i(\mathbf{p}_i+\mathbf{p}_j)\cdot\mathbf{r}_k}H(\mathbf{r},\boldsymbol{\varrho})\right\},\quad(6)$$

[13] We are essentially always neglecting binding energy terms; a term like $\langle\mathbf{p}^2\rangle$, the average square of the nucleon momentum in the nucleus at rest, contributes $\sim\langle\mathbf{p}^2\rangle/M^2\sim1\%$.

where $H(\mathbf{r},\boldsymbol{\varrho})$ is the N.R. nuclear wave function in the nuclear c.m. system, and the χ's are two-component Pauli spinors.

The energy of one of the spectator nucleons E_k is to be taken as a function of $\mathbf{p}_i$ and $\mathbf{p}_j$; the normalization factor for the interacting particle $(E_i+M)/2M$ is to be taken as incorporated in the single-particle current matrix element, now indicated by a prime. Furthermore, the Jacobian for the transformation to the $(\mathbf{r},\boldsymbol{\varrho})$ system is taken to be one.

The $\mathbf{p}_j$ integration is easily performed to give derivatives of the δ function $\delta(\mathbf{r}'-\mathbf{r})$, where $\mathbf{r}$ is defined by $\mathbf{r}\equiv\mathbf{r}_k-\mathbf{r}_j$. We now complete the definition of our change in variables by the equation

$$\boldsymbol{\varrho}\equiv-\tfrac{3}{2}\mathbf{r}_i,$$

(the factor $-\frac{3}{2}$ ensures that the Jacobian is 1).

The $\mathbf{r}'$ integration is trivial, while the $\mathbf{r}$ integration can be reduced to a convenient form using a partial integration as illustrated in Appendix I. If we neglect all terms of $O(1/M^4)$ and terms like $\langle\mathbf{p}^2\rangle/4M^2$,[13] (this is consistent with all our previous assumptions), one obtains

$$T_{ba}{}^{I}\simeq\sum_{\{ijk\}}\sum_{r}\int d\mathbf{p}_i\delta(\mathbf{P}_b-\mathbf{P}_a+\mathbf{q})\int d\mathbf{r}\left(1+\frac{\mathbf{p}_i^2}{4M^2}\right)$$
$$\times\left\{\int d\boldsymbol{\varrho}'H^*(\mathbf{r},\boldsymbol{\varrho}')\chi_r\eta_r e^{-i\mathbf{p}_i\cdot\boldsymbol{\rho}'}e^{\frac{2}{3}i\mathbf{q}\cdot\boldsymbol{\rho}'}\right\}$$
$$\times\langle\mathbf{p}_i-\mathbf{q},\lambda_r|j^{\mu(n)\prime}|\mathbf{p}_i,\lambda_i\rangle$$
$$\times\left\{\int d\boldsymbol{\varrho}\chi_i{}^\dagger\eta_i{}^\dagger H(\mathbf{r},\boldsymbol{\varrho})e^{i\mathbf{p}_i\cdot\boldsymbol{\rho}}\right\}. \quad (7)$$

This contains the nuclear effects described under (b) and (c) in the Introduction as well as contributions from the free propagation of the spectator nucleons. We emphasize at this point, the role of the intermediate-state $\mathbf{p}$'s. They act as momentum operators between the initial and final states and hence can be of order $\mathbf{q}$. In general, therefore, care must be taken before dropping various orders of terms like $\mathbf{p}_i^2/M^2$ when compared to terms like q^2/M^2.

We have not yet said anything of the interacting particle current; from general invariance principles it can be written in terms of nucleon form factors[2,10] $(F_{1,2})$

$$\langle\mathbf{p}_i-\mathbf{q},\lambda_r|j^{\mu(n)}|\mathbf{p}_i,\lambda_i\rangle=\bar{u}(\mathbf{p}_i-\mathbf{q})\eta_r{}^\dagger[(F_1{}^s+\tau_3F_1{}^v)\gamma^\mu$$
$$+i(F_2{}^s+\tau_3F_2{}^v)\sigma^{\mu\nu}q_\nu]u(\mathbf{p}_i)\eta_i, \quad (8)$$

where τ_3 is the z component of the nucleon isospin operator, and S and V stand for isoscalar and isovector, respectively. The form factors $F_{1,2}$ are scalar functions of q^2 only (see, however, Appendix III). This may be reduced to two-component form by either a F.W. reduction[7] or, more simply, by direct computation.

Hence we now can obtain the T matrix in a consistent N.R. reduced form containing relativistic corrections correct to $O(q^2/M^2)$. Following the procedure Gourdin has used for the deuteron,[14] we equate this with the invariant form of the T matrix $(\equiv T_{ba}{}^{(E)})$ written in terms of nuclear form factors. From this equation,

$$T_{ba}{}^{(I)}=T_{ba}{}^{(E)}, \quad (9)$$

individual nuclear form factors may be deduced.

Each T matrix, $T_{ba}{}^{(E)}$ or $T_{ba}{}^{(I)}$, is of the form

$$T_{ba}\propto\langle\mathbf{k}_a+\mathbf{q}|j^{\mu(\mathrm{el})}|\mathbf{k}_a\rangle(1/q^2)\langle\Gamma^\mu\rangle$$
$$\times\delta(\mathbf{P}_b-\mathbf{P}_a+\mathbf{k}_a-\mathbf{k}_b), \quad (10)$$

where $\langle\Gamma^\mu\rangle$ represents the total nuclear current. It is clear that we cannot simply cancel out the electron part on each side of Eq. (9) and say that the nuclear currents, written in their respective forms, may be equated. We discuss this point further in Appendix II; in the following, we shall employ the results proved there.

The time-like components of $\langle\Gamma^\mu\rangle$ are easiest to deal with and we briefly indicate the steps. To $O(1/M^2)$ one easily obtains (in the nuclear laboratory frame)

$$\langle\mathbf{p}_i-\mathbf{q},\lambda_r|j^{0(n)}|\mathbf{p}_i,\lambda_i\rangle=\eta_r{}^\dagger\chi_r{}^\dagger\Bigg[\tilde{F}_{\mathrm{ch}}{}^\tau$$
$$\times\left\{1-\frac{q^2}{4M^2}+\frac{3}{4M^2}\mathbf{p}_i\cdot(\mathbf{p}_i-\mathbf{q})-\frac{i\boldsymbol{\sigma}\cdot(\mathbf{p}_i\times\mathbf{q})}{4M^2}\right\}$$
$$+\tilde{F}_{\mathrm{mag}}{}^\tau\frac{i\boldsymbol{\sigma}\cdot(\mathbf{p}_i\times\mathbf{q})}{4M^2}\Bigg]\chi_i\eta_i, \quad (11)$$

where the isospin operator τ_3 has for convenience been incorporated into our form factors $\tilde{F}^\tau$:

$$\tilde{F}_{\mathrm{ch(mag)}}{}^\tau\rightarrow\tilde{F}_{\mathrm{ch(mag)}}{}^s+\tau_3\tilde{F}_{\mathrm{ch(mag)}}{}^v$$
$$=\tfrac{1}{2}(1+\tau_3)\tilde{F}_{\mathrm{ch(mag)}}{}^p+\tfrac{1}{2}(1-\tau_3)\tilde{F}_{\mathrm{ch(mag)}}{}^n \quad (12)$$

(here p indicates the proton and n the neutron). $\tilde{F}_{\mathrm{ch}}$ is exactly the same as F_{ch} (or G_{ch}) used in Refs. 1–6. $\tilde{F}_{\mathrm{mag}}$ is related to their corresponding magnetic form factors by

$$\tilde{F}_{\mathrm{mag}}/2\mathfrak{M}=\mu F_{\mathrm{mag}}=\mu G_{\mathrm{mag}} \quad (13)$$

(μ is the magnetic moment of the particle in terms of particle magnetons, i.e., in terms of $Ze\hbar/2\mathfrak{M}$, where Ze is the particle charge and $\mathfrak{M}$ its mass). Equation (11) is now substituted into (7) and the spin-isospin sums are performed. The $\mathbf{p}_i$ integration gives rise to various derivatives of $\delta(\boldsymbol{\varrho}-\boldsymbol{\varrho}')$, making the $\boldsymbol{\varrho}'$ integration trivial. The partial integration technique used previously in the $\mathbf{r}$ integration is now employed to rearrange the form of the integrand (see Appendix I).

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[14] See, for example, M. Gourdin, Nuovo Cimento 28, 533 (1963).

We obtain

$$\langle\Gamma^0\rangle^I=\sum_{\{ijk\}}\sum_{\tau}\int d\mathbf{r}\int d\boldsymbol{\rho}\, H^*(\mathbf{r},\boldsymbol{\rho})e^{\frac{2}{3}i\mathbf{q}\cdot\boldsymbol{\rho}}(F_{ch}{}^s+\tau_3F_{ch}{}^v)\times(1+D_1)H(\mathbf{r},\boldsymbol{\rho}).\quad(14)$$

D_1 is an operator defined to be $D_1=-\nabla_\rho{}^2/M^2$, where ∇_ρ represents the gradient operator with respect to the coordinate $\boldsymbol{\rho}$. $H(\mathbf{r},\boldsymbol{\rho})$ is now understood to contain the nucleon spin-isospin variables, and $\sum_{\{ijk\}}$ to represent the sum of the contributions from each particle, the index i occurring in the guise of $\boldsymbol{\rho}\equiv\frac{3}{2}\mathbf{r}_i$. Note that, to this order, spin-dependent terms do not contribute.

It is an easy matter to show that the time-like components of the nuclear current, expressed in terms of nuclear form factors, may be written as

$$\langle\Gamma^0\rangle^{(E)}=(1-q^2/4M_N{}^2)^{1/2}\chi_b{}^\dagger F_{ch}{}^N\chi_a.\quad(15)$$

The Pauli spinors χ_a and χ_b here refer to the nucleus. Note that $q^2/4M_N{}^2\simeq q^2/36M^2\lesssim1\%$ and may be consistently neglected.[13]

Equating (14) and (15) [see Eq. (A10)], we obtain

$$F_{ch}{}^N=\sum_{\{ijk\}}\sum_{\tau}\int d\mathbf{r}\int d\boldsymbol{\rho}\, H^*(\mathbf{r},\boldsymbol{\rho})e^{\frac{2}{3}i\mathbf{q}\cdot\boldsymbol{\rho}}(F_{ch}{}^s+\tau_3F_{ch}{}^v)\times(1+D_1)H(\mathbf{r},\boldsymbol{\rho}).\quad(16)$$

The nuclear spin states are both taken to be "up" in this representation. We can easily compare our results with the N.R. work, by taking $D_1=0$.[3] Note that the $\nabla_\rho{}^2$ term is still a "relativistic" effect, due to the presence of the phase factor $e^{\frac{2}{3}i\mathbf{q}\cdot\boldsymbol{\rho}}$.

We may perform an exactly analogous procedure with the three-vector current to extract the magnetic form factor. Besides being plagued by lengthy algebra, the manipulation here is complicated by the fact that we must pay due regard to certain restrictions in the choice of axes, as discussed in Appendix II. Using the notation established there it is easy to obtain

$$\langle\mathbf{B}\rangle=-(1-q^2/4M_N{}^2)^{-\frac{1}{2}}\chi_b{}^\dagger(\tilde{F}_{mag}{}^N/2M_N)\boldsymbol{\sigma}_N\chi_a$$

or

$$\langle\mathbf{B}\rangle_x\simeq-\tilde{F}_{mag}{}^N/2M_N,\quad(17)$$

where we have chosen a representation in which the only nonzero contributions arise when the initial and final nuclear states differ by a spin flip. We now extract $\langle\mathbf{B}\rangle_x$ from $\langle\boldsymbol{\Gamma}\rangle_I$. A lengthy calculation then leads to

$$\frac{\tilde{F}_{mag}{}^N}{2M_N}\simeq\sum_{\{ijk\}}\sum_{\tau}\int d\mathbf{r}\int d\boldsymbol{\rho}\, H^*(\mathbf{r},\boldsymbol{\rho})e^{\frac{2}{3}i\mathbf{q}\cdot\boldsymbol{\rho}}\left(\frac{\sigma_x}{2M}\right)\times[(\tilde{F}_{mag}{}^s+\tau_3F_{mag}{}^v)(1+D_2)+(F_{ch}{}^s+\tau_3F_{ch}{}^s)D_3]H(\mathbf{r},\boldsymbol{\rho}),\quad(18)$$

where

$$D_2\equiv-\frac{q^2}{12M^2}-\frac{3}{4M^2}\frac{\partial^2}{\partial\rho_x{}^2}$$

and

$$D_3\equiv(1/2M^2)\partial^2/\partial\rho_x{}^2.$$

The z axis, "the axis of quantization," is chosen along $\mathbf{q}$. Note that the representations used here are arbitrary, as may be checked by direct computation using a different choice of axes and a different spin representation.

IV. DISCUSSION

In order to discuss the correction terms D_1, D_2, and D_3 we must make a choice of wave function. Using the notation of Ref. 3 we take, for simplicity, the Gaussian form of the totally symmetric S state:

$$u=A\exp[-\alpha^2(\boldsymbol{\rho}^2+\tfrac{3}{4}\mathbf{r}^2)];\quad\text{Eq. (24) of Ref. 3.}$$

Other states mix in with only a small probability and our percentage correction to their contribution is not expected to be very different from corrections to the S state. Furthermore, other choices for the wave function, such as the Irving form, do not seem to give vastly differing fits to the data from those of the Gaussian form, but are far less tractable. We feel, therefore, that our general conclusions will not be very sensitive to a choice in wave function.

With the stated form of the wave function one easily calculates that terms like $(1/M^2)\partial^2/\partial\rho_x{}^2\to\sim\alpha^2/M^2<1\%$. We have, of course, implicitly neglected terms like this when neglecting binding effects.[13] This implies that effects due to the motion of the nucleon charge distribution are small. It should be noted that this type of effect is expected to be maximized in higher angular-momentum states, although as mentioned previously, because of the small admixtures of these states, the extra contributions will still be very small.

Calculating D_1 and D_2 under the above assumptions, one obtains

$$D_1\to\sim q^2/9M^2\lesssim5\%\quad\text{and}\quad D_2\to\sim q^2/12M^2\lesssim3\%,$$

implying that, indeed, the relativistic corrections are small. As far as these effects are concerned, therefore, one may feel confident that the present work does not require gross modification.[15]

We can ask, of course, the inverse question to that considered above, viz., at what q^2 value do these relativistic effects become important. In order to have a 15% correction one would require $-q^2\sim30\text{ F}^{-2}$. Note that the terms we previously neglected and which are $\sim q^2/36M^2$, only give $\sim3\%$ even at this energy.

If one wishes to obtain the neutron charge form factor from these experiments, then the relativistic corrections must be accurately calculated. To see how sensitive the neutron charge form factor is to the form of the D's, consider only the charge equation (16). By analogy

[15] Note, however, that other effects $\sim5\%$ have been considered—see Refs. 4, 5, 6, and B. F. Gibson (private communication).

with Schiff, define

$$F_1 \equiv \int d\mathbf{r} \int d\boldsymbol{\varrho}\, e^{\frac{1}{3}i\mathbf{q}\cdot\boldsymbol{\rho}} u(1+D_1)u$$

and

$$F_2 \equiv \int d\mathbf{r} \int d\boldsymbol{\varrho}\, e^{\frac{1}{3}i\mathbf{q}\cdot\boldsymbol{\rho}} [u(1+D_1)v_1 + v_1(1+D_1)u].$$

A typical relationship is then

$$\mathfrak{F}_1 \equiv 2F_{\mathrm{ch}}(\mathrm{He}^3) + F_{\mathrm{ch}}(\mathrm{H}^3) = 6F_{\mathrm{ch}}{}^s F_1.$$

We evaluate F_1, again using the Gaussian form, and obtain

$$F_1 \simeq (1 - q^2/9M^2) e^{q^2/18\alpha^2},$$

or

$$\ln_e(\mathfrak{F}_1/6F_{\mathrm{ch}}{}^s) \simeq (q^2/18\alpha^2) - (q^2/9M^2).$$

If we take $\alpha^2(\sim 0.16)$ as the only unknown in this equation, we may clearly neglect $q^2/9M^2$ on the right-hand side. However, suppose $F_{\mathrm{ch}}{}^n$ is taken to be the only unknown. We may take its q^2 dependence in this range as linear, i.e., $\propto q^2/cM^2$, where c is some constant to be determined. Our equations will now contain the term $(q^2/M^2)(1/c - \frac{1}{9})$. We know from previous experiments[2] that $c \gtrsim 3$ which immediately indicates that the determination of c by this method will be very sensitive to the D's.

We conclude with a remark concerning the $q^2 \to 0$ limit. This limit has often been used as a check on the high-energy results. Conversely some authors have attempted to make remarks about the static limit from the high-energy behavior. The I.A. is an approximation that is good only for high energies and cannot be expected to give exact results at very low energies. Indeed, upon examining the magnetic moment operator in Eq. (18), it is obvious that no angular-momentum operator explicitly appears in the low-energy limit, which disagrees with the standard definition of the static nuclear magnetic moment operator.[16] The reason for this discrepancy is, once again, to be found in the replacement of all intermediate state nucleons by nucleons on the mass shell and without any orbital angular momentum. That the static limit does make some sense is simply a reflection of the fact that the intermediate states are actually only just off of the mass shell. However, as already noted, Eq. (18) does contain some angular-momentum dependence, arising from the $\partial^2/\partial\rho_x^2$ terms. This is because we have taken our free operators between "real" nuclear states which contain states of given angular momentum.

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APPENDIX I: A USEFUL PARTIAL-INTEGRATION TECHNIQUE

Frequently in this work we are confronted with integrals of the following type

$$I_\alpha \equiv \int [\partial_\alpha\{f^\dagger(\mathbf{y}) e^{ia\mathbf{q}\cdot\mathbf{y}}\}] O f(\mathbf{y}) d\mathbf{y}, \tag{A1}$$

where ∂_α means a differentiation along an arbitrary direction in coordinate space, a is a c number, O is an Hermitian operator independent of $\mathbf{y}$ (i.e., it commutes with any function of $\mathbf{y}$), and f is an arbitrary function of $\mathbf{y}$ except that it represents a bound state and may be restricted to only real values.

$$I_\alpha = \int e^{ia\mathbf{q}\cdot\mathbf{y}} \{iaq_\alpha f^\dagger + \partial_\alpha f^\dagger\} O f d\mathbf{y}$$

$$= iaq_\alpha \int e^{ia\mathbf{q}\cdot\mathbf{y}} f^\dagger O f d\mathbf{y} - \int f^\dagger \partial_\alpha \{O f e^{ia\mathbf{q}\cdot\mathbf{y}}\} d\mathbf{y}, \tag{A2}$$

where we have performed a partial integration on the second term and assumed the surface integral does not contribute. Using the Hermitian property of O and the reality of f we see that the second term is just I_α again. Hence

$$I_\alpha = \tfrac{1}{2} iaq_\alpha \int e^{ia\mathbf{q}\cdot\mathbf{y}} f^\dagger O f d\mathbf{y} \tag{A3}$$

and in this way, one may reduce integrals of this type containing any odd number of differentiations to those containing an even number.

APPENDIX II: EXTRACTION OF NUCLEAR FORM FACTORS

We wish to discuss the steps leading from Eq. (9) to the identification of the nuclear form factors. We rewrite Eq. (9) as

$$\langle j_\mu^{(\mathrm{el})}\rangle\langle\Gamma^\mu\rangle_I = \langle j_\mu^{(\mathrm{el})}\rangle\langle\Gamma^\mu\rangle_E. \tag{A4}$$

The subscript I indicates that the I.A. has been used and $\langle\ \rangle_I$ indicates various sums and integrals as explicitly written in Eq. (7), say; E indicates that the "exact" covariant form, in terms of nuclear form factors is used, and $\langle\ \rangle_E$ now indicates a matrix element between definite nuclear spin states. We stress here that we are dealing with only one scattering at a time, in which the electron and nucleus pass from one definite spin state to another. Whatever spin states are used on the left of (A4) must be used on the right. For the electron $\langle\ \rangle$ always represents spin-state matrix elements.

[16] See, for example, M. A. Preston, *Physics of the Nucleus* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962), Eqs. (4)–(8), p. 70.

Any current in (A4) must be conserved in the sense

$$q^\mu j_\mu = 0 \quad \text{or} \quad \mathbf{q}\cdot\mathbf{j} = q^0 j_0. \tag{A5}$$

We use this in (A4) to obtain, (provided $q^0 \neq 0$),

$$(1/q^{0^2})\langle \mathbf{q}\cdot\mathbf{j}^{(\text{el})}\rangle\langle \mathbf{q}\cdot\mathbf{\Gamma}\rangle_I - \langle \mathbf{j}^{(\text{el})}\rangle\cdot\langle\mathbf{\Gamma}\rangle_I = (1/q^{0^2})\langle \mathbf{q}\cdot\mathbf{j}^{(\text{el})}\rangle\langle \mathbf{q}\cdot\mathbf{\Gamma}\rangle_E - \langle \mathbf{j}^{(\text{el})}\rangle\cdot\langle\mathbf{\Gamma}\rangle_E. \tag{A6}$$

We can always decompose $\langle\mathbf{\Gamma}\rangle$ into two parts, one perpendicular and one parallel to $\mathbf{q}$:

$$\mathbf{\Gamma} = A\mathbf{q} + i\mathbf{B}\times\mathbf{q}, \quad \text{say}. \tag{A7}$$

In the laboratory frame, A is related only to $F_{\text{ch}}^{(N)}$ while $\mathbf{B}$ is related only to $F_{\text{mag}}^{(N)}$ (and the spin operator $\boldsymbol{\sigma}^N$). Each side of (A6) has the form

$$(q^2/q^{0^2})\langle A\rangle[\mathbf{q}\cdot\langle\mathbf{j}^{(\text{el})}\rangle] + i\langle\mathbf{B}\rangle\cdot[\mathbf{q}\times\langle\mathbf{j}^{(\text{el})}\rangle]. \tag{A8}$$

This implies that, with respect to the angle between $\mathbf{q}$ and $\langle\mathbf{j}^{(\text{el})}\rangle$, the T matrix has split up into two independent portions, one involving only $F_{\text{ch}}^{(N)}$ and the other only $F_{\text{mag}}^{(N)}$.

Hence if we decompose $\mathbf{\Gamma}_I$:

$$\mathbf{\Gamma}_I = \alpha\mathbf{q} + i\boldsymbol{\beta}\times\mathbf{q}, \tag{A9}$$

we may immediately put

$$\langle\alpha\rangle = \langle A\rangle \propto F_{\text{ch}}^{(N)}$$

and

$$\langle\boldsymbol{\beta}\rangle_x = \langle\mathbf{B}\rangle_x \propto F_{\text{mag}}^{(N)}.$$

x here indicates the direction $\mathbf{q}\times\langle\mathbf{j}^{(\text{el})}\rangle$. Note that, since $(\mathbf{\Gamma}\cdot\mathbf{q})/\mathbf{q}^2 = A$ (or α), we may use (A5) to give us

$$\langle\Gamma^0\rangle_I = \langle\Gamma^0\rangle_E \tag{A10}$$

and we may obtain $F_{\text{ch}}^{(N)}$ directly from the zeroth current components.

To obtain $F_{\text{mag}}^{(N)}$ we see that, for an unpolarized nucleus, we may assign an arbitrary direction to $\mathbf{q}$ (z, say), after which we may assign either x or y, but not z to $\langle\mathbf{B}\rangle$ or $\langle\boldsymbol{\beta}\rangle$. It is in this choice of axes that the no-spin-flip amplitude is associated with the electric transition, while the spin-flip amplitude is associated with the magnetic transition.

There are clear advantages in dealing with the matrix elements rather than with the cross section, as Krueger and Goldberg[5] have done. First, we avoid the squaring and averaging procedure, which for $\langle\Gamma\rangle_I$ might prove an immense task. Secondly, we retain equations that contain all polarization effects and which may be useful for future references.

One last point of interest is the previously precluded case of $q^0=0$, defining the Breit frame. In this case $\Gamma_E{}^0$ depends only upon $F_{\text{ch}}^{(N)}$ and $\mathbf{\Gamma}_E$ only upon $F_{\text{mag}}^{(N)}$ and there is no need to decompose the 3-current.

APPENDIX III: KINEMATICS, CURRENT CONSERVATION, AND CORRECTIONS TO THE NUCLEON FORM FACTORS

For the nucleus in the laboratory frame, we have $q=P_a-P_b$ where $P_a=(M_N,\mathbf{0})$, $P_b=(E_b,-\mathbf{q})$, and $P_a{}^2=P_b{}^2=M_N{}^2$. Hence $q^0=q^2/2M_N$. For the nucleons we have taken $p_i{}^2=p_j{}^2=p_k{}^2=M^2$ and $\mathbf{p}_i+\mathbf{p}_j+\mathbf{p}_k=\mathbf{P}_a$. Note that although $\mathbf{p}_i-\mathbf{p}_r=\mathbf{q}$, $E_i-E_r\neq q^0$. In making our N.R. reduction we have expanded E_i and E_j in powers of $1/M^2$.

The ambiguity arising when considering current conservation comes from the definition of q^0. There is no question that for the over-all process $q^0=q^2/2M_N$, so that the nuclear current must obey (A5) in the form

$$\mathbf{q}\cdot\langle\mathbf{\Gamma}\rangle_E = (q^2/2M_N)\langle\Gamma_0\rangle_E. \tag{A11}$$

In terms of the I.A., we should have the analogous equation: $\mathbf{q}\cdot\langle\mathbf{\Gamma}\rangle_I = (q^2/2M_N)\langle\Gamma_0\rangle_I$. Now, we can take the scalar product $\mathbf{q}\cdot$ on the left-hand side through the $\langle\ \rangle_I$ brackets to operate on the individual nucleon currents, in which case one has the relationship

$$\mathbf{q}\cdot\langle\mathbf{\Gamma}\rangle_I = \langle(E_i-E_r)\Gamma_0\rangle_I. \tag{A12}$$

The question is, are the relationships (A11) and (A12) consistent. Expanding out E_i-E_r to $O[(q^2/2M)\times(q^2/M^2)]$ one has

$$E_i-E_r \equiv q'^0 = \frac{(2\mathbf{p}_i\cdot\mathbf{q}-\mathbf{q}^2)}{2M^2}\left[1-\frac{(2\mathbf{p}_i{}^2-2\mathbf{p}_i\cdot\mathbf{q}+\mathbf{q}^2)}{4M^2}\right].$$

Substituting this into (A12) and performing the implied integrations, [cf. the steps in going from Eq. (7) to Eq. (14)], one may verify that, to within the approximations used,[13] current is conserved.

There is one other effect, connected to the above, so far neglected. Since the q^2 for the over-all process is different from the q^2 associated with the virtual process [call this $q'^2 \equiv (q'^0)^2-\mathbf{q}^2$], we should evaluate the nucleon form factors at $q^2=q'^2$ before performing the intermediate state integrations. That the contribution from this effect is completely negligible may be seen by expanding $F(q'^2)$ about $q^2=q'^2$ using a Taylor's series.