If the coherence width Γ is calculated by comparing $R(\epsilon)$ with a Lorentzian form near $\epsilon=0$, the bias and variance of the value so obtained may be calculated. Near $\epsilon=0$, the assumption is made that the bias in $R(\epsilon)$ is independent of ϵ. This leads to the formulas

$$\Gamma=\Gamma_0\left[\frac{(n-1)(4n-4+N)}{4n^2}\right]^{1/2}$$

and

$$\mathrm{Var}(\Gamma)=(\Gamma_0^4/4\Gamma^2)\,\mathrm{Var}[NR(0)]$$

for the expected value Γ of the coherence width and the variance of the value obtained.

This paper is a summary of some of the results of a report[5] which contains plots of further Monte Carlo results as well as a fuller treatment of the questions considered here. Also treated are questions not considered here, such as the effect of finite sample size on the frequency distribution function.

[5] W. R. Gibbs, Los Alamos Scientific Laboratory Report LA 3266, 1965 (available from Clearing House for Federal Scientific and Technical Information, National Bureau of Standards, U. S. Department of Commerce, Springfield, Virginia).

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Remarks on Charged-Particle Scattering

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The scattering of charged particles from a shielded Coulomb potential is reviewed. The limit as the shielding radius becomes infinite is discussed. A method of determining reaction cross sections, recently introduced by the authors is treated in detail and applied to the scattering of protons from He^4 and H^2 at 40 MeV.

INTRODUCTION

THE study of forces between charged particles is complicated by the presence of the infinite-ranged Coulomb force. For a potential which falls off as slowly as r^{-1} we cannot apply the usual scattering boundary condition to the wave function, viz.,

$$\psi(\mathbf{r})\sim\exp(i\mathbf{k}\cdot\mathbf{r})+r^{-1}f(\theta)\exp(ikr) \tag{1}$$

since even at very large distances the incident wave is distorted. In a practical laboratory scattering experiment this problem does not arise, however, because the charges are shielded so that the potential vanishes beyond some shielding radius R. If the radiation source is placed well beyond the shielding radius, an initial state approximating a plane wave can be prepared. We shall consider the case in which the shielding radius is very large and both the source and detector are located very far outside that shielding radius. We idealize this to the case where $R\rightarrow\infty$, still maintaining the condition that the source and detector are located far beyond R.

I. THE SHIELDED COULOMB FIELD

Let us consider a point charge located at the origin shielded by a double layer of charge such that the potential energy is given by

$$\begin{aligned}V(r)&=zz'e^2/r\,, \quad r<R\\ &=0\,, \quad r>R.\end{aligned} \tag{1.1}$$

We may treat the scattering problem with the shielded Coulomb potential Eq. (1.1) by means of the angular-momentum expansion. We write

$$\psi(\mathbf{r})=(kr)^{-1}\sum_l a_l i^l(2l+1)F_l(k,r)P_l(\cos\theta)\,. \tag{1.2}$$

The radial function $F_l(k,r)$ is a solution of the radial equations

$$\frac{d^2F_l}{d\rho^2}+\left[1-\frac{l(l+1)}{\rho^2}-\frac{\eta}{\rho}\right]F_l=0\,, \quad r<R \tag{1.3}$$

$$\frac{d^2F_l}{d\rho^2}+\left[1-\frac{l(l+1)}{\rho^2}\right]F_l=0\,, \quad r>R. \tag{1.4}$$

Here $\rho=kr$ and $\eta=zz'e^2/\hbar v$. The regular solution of Eq. (1.3) takes the asymptotic form

$$F_l\approx A\sin[kr-l\pi/2+\sigma_l-\eta\ln(2kr)] \tag{1.5}$$

for $kr\gg l$, where $\sigma_l=\arg\Gamma(l+1+i\eta)$. The solutions of Eq. (1.4) take the asymptotic form

$$F_l\approx\sin(kr-l\pi/2+\delta_l) \tag{1.6}$$

for $kr\gg l$. Thus, if $R\gg l/k$, we may equate the logarithmic derivatives of the two solutions at the shielding radius R to obtain the result

$$\begin{aligned}&\cot(kR-l\pi/2+\delta_l)\\ &\quad=(1-\eta/kR)\cot[kR-l\pi/2+\sigma_l-\eta\ln(2kR)]\,.\end{aligned} \tag{1.7}$$

The phase shift δ_l given by Eq. (1.7) has the form,

$$\delta_l=\sigma_l-\eta\ln(2kR)+\gamma_l \tag{1.8}$$

* Work performed under the auspices of the U. S. Atomic Energy Commission.

where

$$\gamma_l=(\eta/2kR)\sin2[kR-l\pi/2+\sigma_l-\eta\ln(2kR)]+O[(\eta/2kR)^2]. \quad (1.9)$$

We note that γ_l approaches zero as η/kR goes to zero.

For $l\gg kR$, the angular momentum barrier dominates and δ_l vanishes. Thus we have

$$\begin{aligned}\delta_l&=\sigma_l-\eta\ln(2kR)+\gamma_l, \quad l\ll kR\\ &=0, \quad l\gg kR.\end{aligned} \quad (1.10)$$

In the range where $l\sim kR$, the interior solution F_l does not take on its asymptotic form at $r=R$ so that the phase shifts δ_l are not given so simply, although they may, of course, be obtained easily by standard methods. Once the phase shifts have been found the scattering amplitude is given by the familiar relation

$$f(\theta)=(2ik)^{-1}\sum_l(2l+1)[\exp(2i\delta_l)-1]P_l(\cos\theta). \quad (1.11)$$

At this point we note that the value of γ_l and the values of the phase shifts δ_l in the intermediate region, $l\sim kR$, depend upon the details of the shielding. Nonetheless any model consistent with our boundary conditions will yield phase shifts such as are given by Eq. (1.8), with γ_l going to zero as (η/kR) for large R. In order to keep the argument as simple as possible, therefore, we choose a cutoff of such a form that the phase shifts are given by

$$\begin{aligned}\delta_l&=\sigma_l-\Lambda, \quad l\leq L\\ &=0, \quad l>L,\end{aligned} \quad (1.12)$$

where $\Lambda=\eta\ln(2kR)$ and[1] $L=[bkR]$, with b a constant of the order of unity.

The scattering amplitude corresponding to the phase shifts of Eq. (1.12) is then given by Eq. (1.11). We will denote this scattering amplitude by $f_L(\theta)$. We proceed in the conventional manner, taking note that Λ is a constant independent of l, so that we can write the scattering amplitude, $f_L(\theta)$ as

$$f_L(\theta)=\frac{\exp(-2i\Lambda)}{2ik}\sum_{l=0}^{L}(2l+1)(e^{2i\sigma_l}-1)P_l(\cos\theta)+\frac{[\exp(-2i\Lambda)-1]}{2ik}\sum_{l=0}^{L}(2l+1)P_l(\cos\theta). \quad (1.13)$$

It may be noted that the shielding radius appears implicitly in L and in Λ.

We digress here briefly to note that the shielding implied by the phase shifts of Eq. (1.12) corresponds classically to an impact parameter cutoff. In the classical case the potential energy is given by

$$\begin{aligned}V(r)&=zz'e^2/r, \quad b<R\\ &=0, \quad v>R,\end{aligned} \quad (1.14)$$

where b is the impact parameter. The corresponding classical differential cross section is

$$\begin{aligned}\sigma(\theta)&=[\eta/2k\sin^2(\tfrac{1}{2}\theta)]^2, \quad \theta>2\eta/k\\ &=0, \quad \theta<2\eta/k.\end{aligned} \quad (1.15)$$

Here $(\eta/k)\equiv mzz'e^2/p^2$ and θ and b are related by $2\cot(\frac{1}{2}\theta)=(k/\eta)b$. The integrated cross section is πR^2.

II. THE LIMIT OF AN INFINITELY LARGE SHIELDING RADIUS

In the limit as R and hence L becomes infinite, more terms are added to the sums of Eq. (1.13) and the phase shift Λ increases without limit. Let us first consider the sum which appears in the second term in Eq. (1.13),

$$S_L^{(2)}(x)=\sum_{l=0}^{L}(2l+1)P_l(x), \quad (2.1)$$

where $x=\cos\theta$. A rearrangement of the terms in the sum permits us to use the recurrence relation for the Legendre polynomials,

$$(l+1)P_{l+1}(x)+lP_{l+1}(x)-(2l+1)xP_l(x)=0, \quad (2.2)$$

to sum $S_L^{(2)}(x)$. Straightforward algebraic manipulations immediately yield the result

$$\begin{aligned}S_L^{(2)}(x)&=\frac{1}{1-x}\{\sum_{l=0}^{L}[(l+1)P_l(x)+lP_{l-1}(x)\\ &\quad-(2l+1)xP_l(x)]+(L+1)[P_L(x)-P_{L+1}(x)]\}\\ &=\frac{L+1}{1-x}[P_L(x)-P_{L+1}(x)].\end{aligned} \quad (2.3)$$

We may obtain some insight into the behavior of Eq. (2.3) by noting that the Legendre polynomials can be approximated uniformly on the interval $\epsilon\leq\theta\leq\pi-\epsilon$ where ϵ is arbitrarily small by[2]

$$P_l(\cos\theta)=\sqrt{2}(\pi l\sin\theta)^{-1/2}\cos[(l+\tfrac{1}{2})\theta-\pi/4]+O(l^{-3/2}). \quad (2.4)$$

Under this approximation Eq. (2.3) becomes

$$S_L^{(2)}(x)=\left[\frac{2(L+1)}{\pi\sin\theta}\right]^{1/2}\frac{\sin[(L+1)\theta-\pi/4]}{\sin(\frac{1}{2}\theta)}, \quad \epsilon\leq\theta\leq\pi-\epsilon, \quad \epsilon>0. \quad (2.5)$$

To examine the behavior of Eq. (2.1) at small angles or for $x=\cos\theta\approx1$, we expand $P_l(x)$ about $x=1$, and obtain

$$P_l(x)=\sum_{n=0}^{l}\frac{(-1)^n(l+n)!(1-x)^n}{(n!)^2(l-n)!2^n}. \quad (2.6)$$

[1] The brackets [ ] stand for the "greatest-integer function," i.e., $[x]$ is the largest integer less than or equal to x for any real x.

[2] Gabor Szego, *Orthogonal Polynomials* (American Mathematical Society, New York, 1959), p. 192.

Thus, through terms of order θ^2 we obtain

$$S_L^{(2)}=\sum_{l=0}^{L}(2l+1)\left[1-\frac{l(l+1)}{2}(1-x)+\cdots\right]$$
$$=\sum_{l=0}^{L}(2l+1)-\frac{(1-x)}{2}\sum_{l=0}^{L}(2l+1)l(l+1)+\cdots$$
$$=(L+1)^2\left[1-\frac{L(L+2)}{4}(1-x)+\cdots\right]$$
$$\doteq(L+1)^2[1-\tfrac{1}{8}(L+1)^2\theta^2+\cdots]. \quad (2.7)$$

From this equation we see that $S_L^{(2)}$ is strongly peaked in the forward direction, rising to $(L+1)^2$ for angles $\theta\ll L^{-1}$ and from Eq. (2.5) we see that the sum oscillates rapidly at larger angles, the amplitude going as $L^{1/2}$ and the period as L^{-1}. If we were to look at a differential cross section where $S_L^{(2)}$ appears in interference with a slowly varying term, the individual peaks could not be resolved and the contributions from S_L in such a term would go as $L^{-1/2}$. In the limit as L tends to infinity, $S_L^{(2)}$ becomes

$$S_\infty^{(2)}(x)=\sum_{l=0}^{\infty}(2l+1)P_l(x)=\lim_{\epsilon\to0}2\delta(1-\epsilon-x), \quad (2.8)$$

where the limit $\epsilon\to0$ is taken after any integrations.

The sum $S_L^{(2)}(x)$ arises from the scattering of the incident waves from the shielding charge. The exact form depends on the shielding, but the basic properties of $S_L^{(2)}(x)$, peaking in the forward direction and oscillating rapidly at larger angles, should be independent of the details of the shielding.

Let us now consider the first sum of Eq. (1.13),

$$S_L^{(1)}=(2ik)^{-1}\sum_{l=0}^{L}(2l+1)(e^{2i\sigma_l}-1)P_l(\cos\theta) \quad (2.9)$$

and its limit as L tends to infinity. For large l the phase shifts σ_l vary slowly with l. The sum for $L=\infty$ converges only in the sense of a generalized function or distribution; that is, the infinite sum is well defined only when placed under an integral sign.[3] The integration is done term by term. Though it does not converge to a point function when placed under an integral sign, the infinite sum $S_\infty^{(1)}$ can be equivalent to the point function

$$f_c(\theta)=\frac{-\eta\exp[-2i\eta\ln\sin(\frac{1}{2}\theta)+2i\sigma_0]}{2k\sin^2(\frac{1}{2}\theta)}$$
$$=-(\eta/k)e^{2i\sigma_0}2^{i\eta}(1-x)^{-(1+i\eta)}, \quad (2.10)$$

where $\sigma_0=\arg\Gamma(1+i\eta)$ and $x=\cos\theta$, subject to certain other conditions which will be specified later. The function given in Eq. (2.10) has an essential singularity at $\theta=0$ or $x=1$, so we must establish rules for handling the function Eq. (2.10) in this region.

[3] See, for example, M. J. Lighthill, *Fourier Analysis and Generalized Functions* (Cambridge University Press, Cambridge, 1958).

Let us formally expand Eq. (2.10) in the usual way giving

$$f_c(\theta)=k^{-1}\sum_l(2l+1)a_lP_l(x)$$

with

$$a_l=(k/2)\int_{-1}^{1}f_c(x)P_l(x)dx. \quad (2.11)$$

Because of the singularity in $f_c(x)$, the limit as ϵ tends to zero of the integral

$$\int_{-1}^{1-\epsilon}f_c(x)P_l(x)dx$$

does not exist as a uniform limit. For the moment, however, we will formally define

$$a_l\equiv-B\int_{-1}^{1}(1-x)^{-(1+i\eta)}P_l(x)dx, \quad (2.12)$$

where $B=\eta e^{2i\sigma_0}2^{i\eta-1}$. Now we use the recurrence relation for the Legendre polynomials, which is valid for $l\geq2$,

$$l[P_l(x)-P_{l-1}(x)]-(l-1)[P_{l-1}(x)-P_{l-2}(x)]$$
$$+(1-x)\{[P_l'(x)-P_{l-1}'(x)]$$
$$+[P_{l-1}'(x)-P_{l-2}'(x)]\}=0. \quad (2.13)$$

Multiplying Eq. (2.13) by $-B(1-x)^{-(1+i\eta)}$ and integrating over x and using the definition Eq. (2.12), we get

$$l\Delta a_l-(l-1)\Delta a_{l-1}-B\int_{-1}^{1}(1-x)^{-i\eta}\{[P_l'(x)-P_{l-1}'(x)]$$
$$+[P_{l-1}'(x)-P_{l-2}'(x)]\}dx=0, \quad (2.14)$$

where $\Delta a_l\equiv a_l-a_{l-1}$. If we define I_l to be

$$I_l=\int_{-1}^{1}(1-x)^{-i\eta}[P_l'(x)-P_{l-1}'(x)]dx, \quad (2.15)$$

and by integrating Eq. (2.15) by parts, we find

$$I_l=\{(1-x)^{-i\eta}[P_l(x)-P_{l-1}(x)]\}_{-1}^{1}$$
$$-i\eta\int_{-1}^{1}(1-x)^{-(1+i\eta)}[P_l(x)-P_{l-1}(x)]dx$$
$$=2^{1-i\eta}(-1)^l+(i\eta/B)\Delta a_l. \quad (2.16)$$

With this result, Eq. (2.14) becomes

$$l\Delta a_l-(l-1)\Delta a_{l-1}-i\eta(\Delta a_l+\Delta a_{l-1})=0,$$

or

$$\Delta a_l=\frac{l-1+i\eta}{l-i\eta}\Delta a_{l-1}\quad l\geq2. \quad (2.17)$$

This homogeneous recursion relation defines the Δa_l to within a constant. If one is known, say $\Delta a_1 = a_1 - a_0$, the constant is fixed. But Δa_1 can easily be found without any additional assumptions. We note that $P_1(x) - P_0(x) = -(1-x)$ so that

$$\Delta a_1 = -B\int_{-1}^{1}(1-x)^{-(1+i\eta)}[P_1(x)-P_0(x)]dx = B\int_{-1}^{1}(1-x)^{-i\eta}dx$$

$$= -B\left[\frac{(1-x)^{1-i\eta}}{1-i\eta}\right]_{-1}^{1} = \frac{\eta e^{2i\sigma_0}}{1-i\eta}. \quad (2.18)$$

This can be rewritten as

$$\Delta a_1 = \frac{\eta}{1-i\eta}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}. \quad (2.19)$$

The Δa_l are now defined uniquely. We iterate Eq. (2.17) and use Eq. (2.19) to get

$$\Delta a_l = \frac{l-1+i\eta}{l-i\eta}\,\frac{l-2+i\eta}{l-1-i\eta}\cdots\frac{1+i\eta}{2-i\eta}\Delta a_1$$

$$= \frac{\eta}{l-i\eta}\frac{\Gamma(l+i\eta)}{\Gamma(l-i\eta)}, \quad (2.20)$$

or

$$2i\Delta a_l = \left[\frac{l+i\eta}{l-i\eta}-1\right]\frac{\Gamma(l+i\eta)}{\Gamma(l-i\eta)}. \quad (2.21)$$

Since the Δa_l are known, the a_l are defined to within an additive constant by

$$a_l = \frac{e^{2i\sigma_l}}{2i} + C = \frac{e^{2i\sigma_l}-1}{2i} + \frac{C'}{2i}, \quad (2.22)$$

where $\sigma_l = \arg\Gamma(l+1+i\eta)$. We have chosen to write the constant on the right in a rather suggestive manner.

If we knew one of the a_l, say a_0, the constant and hence all of the a_l would be completely defined. To this end, let us examine the integral

$$A(\alpha) = -B\int_{-1}^{\alpha}(1-x)^{-(1+i\eta)}dx = -\frac{B}{i\eta}(1-x)^{-i\eta}\Big|_{-1}^{\alpha}$$

$$= \frac{e^{2i\sigma_0}-1}{2i} - \frac{1}{2i}\{\exp[-i\eta \ln\tfrac{1}{2}(1-\alpha)]-1\}. \quad (2.23)$$

The limit of $A(\alpha)$ as α tends to 1 does not exist as a uniform limit. It is possible to choose a discrete sequence $\{\alpha_n\}$ such that in the limit as n tends to infinity, $\alpha_n \sim 1$ and such that as n tends to infinity, the limit of $A(\alpha_n)$ does exist. With such a sequence $\exp[-i\eta \ln\frac{1}{2}(1-\alpha_n) + 2i\sigma_0] \sim \exp(2i\gamma)$ which is of unit amplitude. In this event we find

$$a_l = \frac{e^{2i\sigma_l}-1}{2i} - \frac{e^{2i\gamma}-1}{2i}. \quad (2.24)$$

Substituting Eq. (2.24) into the expansion Eq. (2.11), we find

$$f_c(x) = -(\eta/k)e^{2i\sigma_0}2^{i\eta}(1-x)^{-(1+i\eta)}$$

$$= (2ik)^{-1}\sum_{l=0}^{\infty}(2l+1)(e^{2i\sigma_l}-1)P_l(x) + ik^{-1}(e^{2i\gamma}-1)\delta(1-x), \quad (2.25)$$

where the delta function is defined in the sense of Eq. (2.8). We shall choose the sequence which defines γ in such a way that the term with the delta function vanishes. A sequence $\{\alpha_n\}$ which achieves this end is defined by

$$x_n = \alpha_n = 1 - 2\exp[-(2n\pi - 2\lambda_0)/\eta]. \quad (2.26)$$

We could have written the variable of integration in Eq. (2.12) as θ instead of $x=\cos\theta$. In this case a sequence is defined by

$$\theta_n = \epsilon_n = 2\exp[(\sigma_0 - n\pi)/\eta]. \quad (2.27)$$

We see then that the infinite sum Eq. (2.9) is equivalent to the point function Eq. (2.10) in an integrand provided that the limits of the integral are handled in the manner of Eq. (2.26) or Eq. (2.27).

We now return to a consideration of the phase Λ in Eq. (1.13). When we take the infinite limit we must do so in a nonuniform way so that in this limit $\exp(-2i\Lambda)$ is mathematically defined. In practice, however, the factor $\exp(-2i\Lambda)$ is common to all terms in the scattering amplitude except the shielding term and so is undetectable except in a narrow cone in the forward direction. This cone shrinks to zero in the limit as $R\sim\infty$, i.e.,

$$f_L \sim f_\infty = e^{-2i\Lambda_\infty}f_c(x) + (ik)^{-1}(e^{-2i\Lambda_\infty}-1)\delta(1-x). \quad (2.28)$$

Thus, in an actual measurement the extra phase and the forward delta function are undetectable.

III. THE OPTICAL THEOREM

Total reaction cross sections for cases where many exit channels are open are often difficult to measure. We here outline a method for determining reaction cross sections from elastic scattering data.[4] The term "total reaction cross section" is taken to include all nonelastic processes.

The optical theorem states

$$4\pi k^{-1}\,\mathrm{Im}f(0) = \sigma^T, \quad (3.1)$$

where σ^T is the total cross section and includes both

[4] J. T. Holdeman and R. M. Thaler, Phys. Rev. Letters 14, 81 (1965).

elastic and inelastic processes. From the partial-wave expansion of the scattering amplitude

$$f(\theta)=(2ik)^{-1}\sum_l(2l+1)(e^{2i\delta_l}-1)P_l(\cos\theta), \quad (3.2)$$

we obtain the partial-wave expansion of the optical theorem, viz.,

$$4\pi k^{-2}\sum_l(2l+1)\,\mathrm{Im}\left(\frac{e^{2i\delta_l}-1}{2i}\right)P_l(1)$$
$$=4\pi k^{-2}\sum_l(2l+1)\left|\frac{e^{2i\delta_l}-1}{2i}\right|^2$$
$$+\pi k^{-2}\sum_l(2l+1)(1-|e^{2i\delta_l}|^2), \quad (3.3)$$

where the phase shifts δ_l are in general complex.

We have already considered the scattering of charged particles from a shielded Coulomb potential. The shielded Coulomb scattering amplitude $f_L^{sc}(\theta)$ is given by Eq. (1.13). Let us now consider the scattering from a nuclear potential plus the shielded Coulomb potential above. Later, when we take the limit as the shielding radius becomes infinite, we shall take the limit in the same way as in Sec. II. In the second case above it will be convenient to introduce the residual amplitude $f'(\theta)$ defined by $f'(\theta)\equiv f(\theta)-f_L^{sc}(\theta)$. The amplitude $f(\theta)$ is given by Eq. (1.11) with $\delta_l\equiv\sigma_l-\Lambda+\hat{\delta}_l$, where $\hat{\delta}_l$ is defined to be the additional phase shift due to the nuclear potential. The residual amplitude can be expanded as

$$f'(\theta)\equiv e^{-2i\Lambda}\bar{f}'(\theta)=(2ik)^{-1}e^{-2i\Lambda}\sum_l(2l+1)e^{2i\sigma_l}$$
$$\times[\exp(2i\hat{\delta}_l)-1]P_l(\cos\theta). \quad (3.4)$$

This sum contains angular momenta only up to the order of k times the nuclear radius. Since only a few partial waves contribute to $f'(\theta)$, the residual amplitude is slowly varying at small angles and can be approximated by

$$f'(\theta)=\sum_l a_lP_l(x)\sim\sum_l a_l+(\tfrac{1}{4})[\sum_l a_ll(l+1)]$$
$$\times\sin^2(\tfrac{1}{2}\theta)+\cdots. \quad (3.5)$$

Applying the optical theorem to the shielded Coulomb amplitude, Eq. (1.13), we obtain

$$4\pi k^{-1}\,\mathrm{Im}f_L^{sc}(0)=\sigma_L^{sc}. \quad (3.6)$$

Applying the optical theorem to the entire shielded scattering amplitude, Eq. (3.2), we obtain

$$4\pi k^{-1}\,\mathrm{Im}f(0)=\sigma^T. \quad (3.7)$$

Taking the difference of Eq. (3.6) and Eq. (3.7), we find

$$4\pi k^{-1}\,\mathrm{Im}f'(0)=\sigma', \quad (3.8)$$

where σ' is defined by

$$\sigma'=\sigma^T-\sigma_L^{sc} \quad (3.9)$$

and $f'(0)$ is the residual amplitude evaluated at $\theta=0$. The quantity σ' contains both elastic and inelastic terms. We now write

$$\sigma'=\sigma_r+\sigma_{el}', \quad (3.10)$$

where σ_r is the total reaction cross section and σ_{el}' is the residual elastic cross section. The quantity σ_{el}' is given formally by

$$\sigma_{el}'=\int\sigma'(\theta)d\Omega=\int[\sigma(\theta)-\sigma_L^{sc}(\theta)]d\Omega, \quad (3.11)$$

where $\sigma(\theta)$ is the experimentally measurable differential elastic cross section with the nuclear potential present. Note that we are using the convention that when σ is written with an argument as $\sigma(\theta)$ it is a differential cross section. When written otherwise, it is an integrated cross section. Using Eqs. (3.8)–(3.11), we find

$$\sigma_r=4\pi k^{-1}\,\mathrm{Im}f'(0)-\int[\sigma(\theta)-\sigma_L^{sc}(\theta)]d\Omega,$$
$$\mathrm{I}=4\pi k^{-1}\,\mathrm{m}f'(0)-\int\sigma_{el}'(\theta)d\Omega. \quad (3.12)$$

We can write $\sigma_{el}'(\theta)$ in terms of the scattering amplitudes as

$$\sigma_{el}'(\theta)\equiv\sigma(\theta)-\sigma_L^{sc}(\theta)=|f_L^{sc}(\theta)+f'(\theta)|^2-|f_L^{sc}(\theta)|^2$$
$$=|f'(\theta)|^2+2\,\mathrm{Re}[f'(\theta)f_L^{sc*}(\theta)]. \quad (3.13)$$

Since $\sigma_{el}'(\theta)$ appears in an integrand, if we take the limit as R approaches infinity in the manner previously indicated, we find

$$\sigma_{el}'(\theta)=|f'(\theta)|^2+2|f'(\theta)||f_c(\theta)|\cos[\alpha_c(\theta)-\phi(\theta)]$$
$$-4k^{-1}\,\mathrm{Re}[e^{i\Lambda_\infty}\sin\Lambda_\infty f'(\theta)\delta(1-x)], \quad (3.14)$$

where $\alpha_c(\theta)$ is the phase of the Coulomb amplitude and $\phi(\theta)$ is the phase of the residual amplitude. We note that Λ_∞ appears only in the last term of Eq. (3.14). Because of its form, this last term cannot be evaluated from data at finite angles. It may be formally integrated to give $-(8\pi/k)\,\mathrm{Re}[\exp(i\Lambda_\infty)\sin\Lambda_\infty f'(0)]$. From Eq. (3.4) we may easily obtain the result that

$$\frac{4\pi}{k}\mathrm{Im}f'(0)+\frac{8\pi}{k}\mathrm{Re}[e^{i\Lambda}\sin\Lambda f'(0)]=\frac{4\pi}{k}\mathrm{Im}\bar{f}'(0), \quad (3.15)$$

where, of course, $\bar{f}'(0)$ is the residual amplitude for $\Lambda=0$. Thus we see that the forward delta function and the phase Λ_∞ make no physical difference in any measurable quantity. The phase Λ_∞ may take on any value and so for convenience we will take it to be zero in the discussion that follows. With this choice the phase of $f'(\theta)$ corresponds to the choice of the Coulomb phase $\alpha_c(\theta)=-2\eta\ln\sin(\tfrac{1}{2}\theta)+2\sigma_0$.

Using the phase convention of the previous paragraph

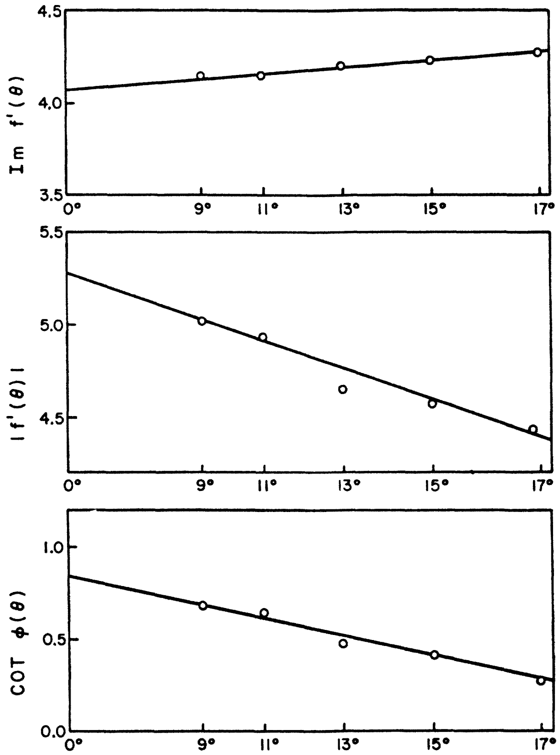

FIG. 1. p-α elastic scattering at 40 MeV. Plots of the imaginary part, magnitude, and phase of the residual amplitude $f'(\theta)$ versus $\sin^2(\frac{1}{2}\theta)$ at several angles θ in the center-of-mass system for the scattering of protons from He^4. The residual amplitude is given in fermis. The solid lines show the extrapolation to the forward direction.

we find

$$\sigma_{el}'(\theta)=|f'(\theta)|^2+2|f'(\theta)||f_c(\theta)|\times\cos[\alpha_c(\theta)-\phi(\theta)]. \quad (3.16)$$

At small angles $|f'(\theta)|$ and $\phi(\theta)$ are slowly varying functions of θ, whereas $|f_c(\theta)|$ and $\alpha_c(\theta)$ vary rapidly with angle. Thus, in the Coulomb interference region, experimental values of $\sigma'(\theta)\equiv\sigma(\theta)-\sigma_c(\theta)$ at neighboring angles may be used to obtain $f'(\theta)$. Values of $f'(\theta)$ so obtained may be plotted against $\sin^2(\frac{1}{2}\theta)$ and extrapolated to $\theta=0$ as in Fig. 1 and Fig. 2 to find $f'(0)$.

To find σ_{el}' one formally carries out the integration in Eq. (3.11). In practice no experimental data is available for angles smaller than some θ_0 and this small-angle region may contribute significantly to the integral. This causes no difficulty, however, because we have a theoretical expression given by Eq. (3.16), where $f'(\theta)$ is obtained in this region by the extrapolation mentioned above. Thus Eq. (3.11) becomes

$$\sigma_{el}'=2\pi\int_{\theta_0}^{\pi}\sigma_{expt}'(\theta)\sin\theta\,d\theta+2\pi\int_0^{\theta_0}|f'(\theta)|^2\sin\theta\,d\theta + \lim_{\epsilon_n\to 0}2\pi\,\mathrm{Re}\int_{\epsilon_n}^{\theta_0}f_c^*(\theta)f'(\theta)\sin\theta\,d\theta. \quad (3.17)$$

Since we are using the functional form of Eq. (2.10) for $f_c(\theta)$, we must use the corresponding limiting process (2.27) in evaluating the integral on the right in the equation above. If θ_0 is small enough that $f'(\theta)$ may be approximated by $f'(0)$, then Eq. (3.17) becomes

$$\sigma_{el}'=2\pi\int_{\theta_0}^{\pi}\sigma_{expt}'(\theta)\sin\theta\,d\theta +4\pi|f'(0)|^2\sin^2(\tfrac{1}{2}\theta_0)+4\pi k^{-1}\,\mathrm{Im}\{f'(0)\times[1-\exp(2i\eta\ln\sin(\tfrac{1}{2}\theta_0)-2i\sigma_0)]\}. \quad (3.18)$$

Using this result in Eq. (3.12) gives us the final result for the reaction cross section[5]

$$\sigma_r=4\pi k^{-1}\,\mathrm{Im}[f'(0)\exp(2i\eta\ln\sin(\tfrac{1}{2}\theta_0)-2i\sigma_0)] -4\pi|f'(0)|^2\sin^2(\tfrac{1}{2}\theta_0) -2\pi\int_{\theta_0}^{\pi}[\sigma(\theta)-\sigma_c(\theta)]\sin\theta\,d\theta. \quad (3.19)$$

In the limit as the charge goes to zero, Eq. (3.19) yields the familiar result that the total reaction cross

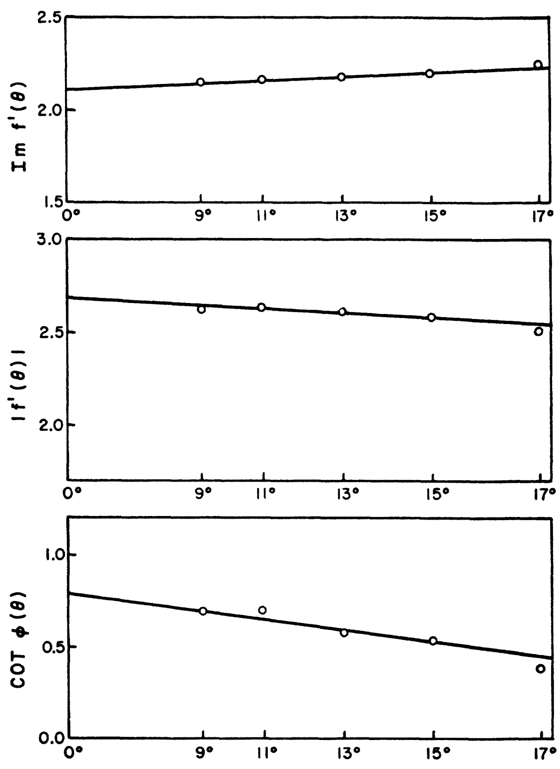

FIG. 2. p-D elastic scattering at 40 MeV. Plots of the imaginary part, magnitude, and phase of the residual amplitude $f'(\theta)$ versus $\sin^2(\frac{1}{2}\theta)$ at several angles θ in center of mass for the scattering of protons from H^2. The residual amplitude is given in fermis. The solid lines show the extrapolation to the forward direction.

[5] Equation (16) of Ref. 4 contains a typographical error. The result is given correctly here in Eq. (3.19).

section is the total cross section, as given by the optical theorem, minus the total elastic cross section. However, for the scattering of uncharged particles, the amplitude $f'(0)$ cannot, in general, be determined from the elastic-scattering data. For charged-particle scattering, $f'(0)$ can be determined, and hence Eq. (3.19) can be used to obtain the reaction cross section from the elastic-scattering data.

The derivations presented have been for spinless charged particles. These results, however, also apply to particles with spin.

IV. DETERMINATION OF REACTION CROSS SECTIONS FROM DATA

The results of Sec. III provide us with a way of determining reaction cross sections provided the prescriptions given there can be carried out. We have determined the reaction cross section in two cases in which high-precision small-angle data were available. The differential elastic scattering cross sections for protons scattered from He^4 and from H^2 at 40 MeV have been measured by Brussel and Williams[6,7] in the angular region $4° < \theta\,\mathrm{cm} < 140°$. Particular attention was given by Brussel and Williams to the angular region between 4 and 25°. This data has been analyzed to find the residual amplitude as a function of angle. The results are plotted in Fig. 1 and Fig. 2 with extrapolations to $\theta = 0$. These extrapolations were used in Eq. (3.16) to calculate the reaction cross sections. The results are summed up in Table I. The rather large uncertainties quoted ($\sim 25\%$ and 10%, respectively) are the result of two large quantities almost canceling. This cancellation is not always expected to occur so that the large uncertainties are not necessarily characteristic of the method.

TABLE I. Results of data analysis at 40 MeV. Lengths are given in fermis and angles in degrees.

Experiment	$\|f'(0)\|$ (F)	$\phi(0)$	$\mathrm{Im}\,f'(0)$ (F)	σ_{el}' (F^2)	σ_r (F^2)
$He^4(p,p)He^4$	5.28	49.6°	4.07	38.9	7.7±2
$H^2(p,p)H^2$	2.69	51.3°	2.11	17.8	10.8±1

[6] M. K. Brussel and J. H. Williams, Phys. Rev. **106**, 286 (1957).
[7] J. H. Williams and M. K. Brussel, Phys. Rev. **110**, 136 (1958).

TABLE II. Comparison of present work with complex-phase-shift analyses for p-α scattering at 40 MeV.

Author	$\|f'(0)\|$ (F)	$\phi(0)$	$\mathrm{Im}\,f'(0)$ (F)	σ_{el}' (F^2)	σ_r (F^2)
Present work	5.28	49.6°	4.07	38.9	7.7±2
GT	5.48	53.2°	4.39	40.2	10.3
SY	5.46	56°	4.52	···	11.6

There are no direct measurements of these reaction cross sections which could be used for comparison. In the p-α scattering case, there are complex phase shift analyses of the experimental data by Suwa and Yokosawa (SY)[8] and Giamati and Thaler (GT).[9] The reaction cross sections determined from these complex phase shifts are compared with the present work in Table II. The phase shifts of SY and GT fit the elastic cross section data equally well except at small angles (8 to 12°), where the GT fit is slightly better. This small angle region is of great importance however in determining the reaction cross section.

To investigate the sensitivity of the method to experimental error we used an optical-model code to generate "experimental data." The code also gave us the quantities mentioned in the headings of the columns of Table II for comparison. Varying the differential cross section from the optical-model calculations gave us the effect of "experimental error." The reaction cross section calculated is very sensitive to experimental error at small angles as one might expect, so that these small-angle measurements of $\sigma(\theta)$ must be made with considerable care.

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[8] S. Suwa and A. Yokosawa, Phys. Letters **5**, 351 (1963).
[9] C. C. Giamati and R. M. Thaler, Nucl. Phys. **59**, 159 (1964).