Static Strong-Coupling Theory with π and η Fields*†

R. RAMACHANDRAN
The Enrico Fermi Institute for Nuclear Studies and the Department of Physics, The University of Chicago, Chicago, Illinois

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In the framework of the static strong-coupling approximation, a model with a nucleon source and π and η meson fields is considered. Characteristically different isobar spectra result, depending on whether the ratio of coupling constants of the nucleon to the η and π mesons is less than or greater than a certain critical value. In the first case the spectrum of excited states is entirely uninfluenced by the η-meson fields, with the result that the lowest lying excited state is the well-known (3,3) resonance. But if the coupling constants are such that case II is realized, then the isobar spectrum is drastiaclly changed; in particular a $T=1/2$, $J=3/2$ resonance would be expected as the first excited state. The leading term in the strong-coupling approximation of the nuclear force (potential) resulting from the nucleons exchanging π and η mesons is also derived.

I. INTRODUCTION

THE nucleon isobar states were first theoretically predicted[1] in the static strong-coupling approximation for scalar Yukawa interactions quite a long time ago. Subsequently, in the same scheme the pseudoscalar π meson fields were known to give an excited nucleon (isobar) spectrum,[2] with the energy levels

$$E_T=T(T+1)\,;\quad T=J=\tfrac{1}{2},\tfrac{3}{2},\cdots, \tag{1.1}$$

where T stands for isospin and J for spin-quantum numbers of the isobar. The first excited member of this family was identified with the now well-known (3,3) resonance which has $J=T=\frac{3}{2}$.

Recently, $SU(3)$ symmetry has had some success in classifying the various mesons and baryons according to the irreducible representations of the group. With this in view, the question we would like to ask is what baryon multiplets might be expected under the assumption of an extreme (Yukawa-type) coupling between the baryon octet and the meson octet. This has already been studied for an octet of scalar mesons obeying the exact internal symmetry.[3] However, such a treatment for the physically important pseudoscalar mesons seems quite complicated, so we shall attempt here a simpler problem, possibly indicative of what may happen in a more complete theory. Accordingly, we shall restrict ourselves to the zero-strangeness part of the realistic problem, our model consisting of two baryons (neutron and proton) and four mesons (π triplet and η singlet). We would like to know, in particular, if any isobar states may be expected besides those given by (1.1).

In Sec. II we give a brief description of the strong-coupling method, which is applied to the specified problem and worked out in some detail in Secs. III, IV, and V. Significantly, we find that, depending upon the ratio of the coupling constants $g_{\bar{N}N\pi}$ and $f_{\bar{N}N\eta}$, there exist two distinct sets of solutions which must be treated separately. In case I, which occurs when $(f/g)^2$ is less than a certain critical value, the isobar spectrum has the same structure as given in (1.1). We shall refer to this case as the dominant π coupling. In case II (dominant η coupling), $(f/g)^2$ is greater than the critical value and additional excited states, not included in (1.1), are found to occur. In particular, the lowest lying excited state in this case has quantum numbers $J=\frac{3}{2}$, $T=\frac{1}{2}$. In Sec. VI, we give a brief account of the strong-coupling approximation for the static nuclear forces between nucleons, exchanging both π and η mesons. These turn out to be closely related to the results in the conventional "weak"-coupling perturbation theory. In conclusion, we note the similarities and contrasts between our model and the Chew-Low static theory, or equivalently (ND^{-1}) dispersion-theoretic calculations.

II. METHOD

Omitting the bare nucleon mass as a constant, we may write the Hamiltonian for the problem as

$$H=H_0+H',$$

where H_0 corresponds to the noninteracting meson field energy:

$$H_0=\tfrac{1}{2}\sum_\rho\int d^3x\big[\pi_\rho^2(\mathbf{x})+\psi_\rho(\mathbf{x})(\mu_\pi^2-\Delta)\psi_\rho(\mathbf{x}) +\pi_\eta^2(\mathbf{x})+\psi_\eta(\mathbf{x})(\mu_\eta^2-\Delta)\psi_\eta(\mathbf{x})\big]. \tag{2.1}$$

$\psi(\mathbf{x})$ and $\pi(\mathbf{x})$ are the usual canonically conjugate field variables. The subscript $\rho(=1, 2, 3)$ denotes the isospin index of the π meson variables and the index η refers to the variables of the η meson fields. No particular simplification results if the rest masses μ_π and μ_η are chosen to be equal, so we shall impose no restriction on

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1 G. Wentzel, Helv. Phys. Acta **13**, 269 (1940); **14**, 633 (1941).
2 W. Pauli and S. M. Dancoff, Phys. Rev. **62**, 85 (1942); S. Tomonaga, Progr. Theoret. Phys. (Kyoto) **1**, 109 (1946); A. Houriet, Helv. Phys. Acta **18**, 473 (1945); further references may be found in G. Wentzel, Rev. Mod. Phys. **19**, 1 (1947).
3 G. Wentzel, EFINS Report 64-33 (unpublished); C. Dullemond, Ann. Phys. (N. Y.) (to be published).

them. The coupling of the meson fields to the (static) nucleon source function is assumed to be of the Yukawa-type; thus

$$H'=g\sum_{i\rho}\int d^3x\,\sigma_i\tau_\rho\psi_\rho(\mathbf{x})[\nabla_i\delta_a(\mathbf{x})]+f\sum_i\int d^3x\sigma_i\psi_\eta(\mathbf{x})[\nabla_i\delta_a(\mathbf{x})],\quad(2.2)$$

where the spherically symmetric source function $\delta_a(\mathbf{x})$ [which reduces to the Dirac delta function $\delta(\mathbf{x})$ when $a\rightarrow 0$] represents a spatial extension of the bare nucleon and is normalized to unity:

$$\int d^3x\,\delta_a(\mathbf{x})=1\,.$$

This "size" of the nucleon (or effectively the cutoff parameter) is conventionally defined in the strong-coupling theories by

$$a=\left[\int d^3xd^3\mathbf{x}'\delta_a(\mathbf{x})\frac{1}{|\mathbf{x}-\mathbf{x}'|}\delta_a(\mathbf{x}')\right]^{-1}.$$

We shall be primarily interested in the limit $a\ll\mu_\pi^{-1}$, μ_η^{-1}. In Eq. (2.2), σ_i and τ_ρ are the usual 2×2 Pauli matrices that operate on the spin and isospin spaces of the nucleon. The coupling constants g, f have dimensions of length and $g>0$, $f>0$ may be taken without any loss of generality.

After Wentzel,[4] we introduce a complete set of real orthogonal functions $K_r(\mathbf{x})$ in terms of which we may expand the field variables:

$$\int d^3x\,K_r(\mathbf{x})K_s(\mathbf{x})=\delta_{rs},\quad \sum_r K_r(\mathbf{x})K_r(\mathbf{x}')=\delta(\mathbf{x}-\mathbf{x}')\,,\quad(2.3)$$

$$\psi_\rho(\mathbf{x})=\sum_s K_s(\mathbf{x})q_{s\rho},\qquad \pi_\rho(\mathbf{x})=\sum_s K_s(\mathbf{x})p_{s\rho},\quad(2.4)$$

$$\psi_\eta(\mathbf{x})=\sum_s K_s(\mathbf{x})q_s{}^\eta,\qquad \pi_\eta(\mathbf{x})=\sum_s K_s(\mathbf{x})p_s{}^\eta.$$

The p and q obey canonical commutation rules:

$$i[p_{r\rho},q_{s\sigma}]=\delta_{rs}\delta_{\rho\sigma},\quad i[p_r{}^\eta,q_s{}^\eta]=\delta_{rs}.\quad(2.5)$$

All other pairs commute. This development serves the same purpose as a partial-wave expansion. In the static limit, the pseudoscalar-meson–nucleon interaction would pick out the three p waves only, and correspondingly in our expansion we shall need the explicit form for only the first three of the orthogonal functions $K_r(\mathbf{x})$. We choose them as[5]

$$K_i(x)=-\frac{1}{C}\frac{\partial}{\partial x_i}\delta_a(\mathbf{x}),\quad i=1,2,3\,,\quad(2.6)$$

where C is determined by the normalization condition (2.3). $C\sim a^{-5/2}$. Substituting (2.4) and (2.6) in (2.2), we obtain

$$H'=gC\sum_{i\rho}\sigma_i\tau_\rho q_{i\rho}+fC\sum_i\sigma_iq_i{}^\eta.\quad(2.7)$$

Here H' is a 4×4 matrix in spin-isospin space. In the strong-coupling theories, one first diagonalizes this "large" interaction Hamiltonian, and then, with the help of the unitary matrix employed to achieve this, obtains in successive perturbative approximations the contributions from H_0.

Towards this, we find it convenient to define a new set of variables in place of $q_{i\rho}$, following Wentzel[6]:

$$q_{i\rho}=\sum_{n=1}^{3}r_ns_{in}s_{\rho n}{}'\,,\quad(2.8)$$

where r_n are the positive square roots of the eigenvalues of the tensor $T_{ij}=\sum_\rho q_{i\rho}q_{j\rho}$, and s_{in} are the corresponding eigenvectors. $s_{\rho n}'$ are similar eigenvectors of the tensor $T_{\rho\sigma}=\sum_i q_{i\rho}q_{i\sigma}$, which has the same set of eigenvalues r_n^2. Orthogonality and completeness of these eigenvectors give

$$\sum_i s_{in}s_{im}=\delta_{nm}\,,\quad \sum_n s_{in}s_{jn}=\delta_{ij}.$$

Similarly

$$\sum_\rho s_{\rho n}'s_{\rho m}'=\delta_{nm}\,,\quad \sum_n s_{\rho n}'s_{\sigma n}'=\delta_{\rho\sigma}.$$

Thus, clearly the matrices s_{in} and $s_{\rho n}'$ denote rotation in ordinary and isospin 3-dimensional spaces, respectively; hence it is easy to construct unitary matrices, Y, Y' such that

$$Y^*(\sum_i\sigma_is_{in})Y=\sigma_n\quad\text{and}\quad Y'^*(\sum_\rho\tau_\rho s_{\rho n}')Y'=\tau_n\,,$$

Y acting only on the σ space and Y' on the τ space. We further introduce in place of the variables $q_i{}^\eta$

$$q_i{}^\eta=g/f\sum_n u_ns_{in};\quad u_n=f/g\sum_i q_i{}^\eta s_{in}.\quad(2.9)$$

Now, with Eqs. (2.8) and (2.9) in (2.7), we obtain

$$Y^*Y'^*H'Y'Y=gC\sum_n(r_n\sigma_n\tau_n+u_n\sigma_n).\quad(2.10)$$

[4] G. Wentzel, Helv. Phys. Acta **16**, 551 (1943), Sec. 3.

[5] Notation: We shall use the subscripts i, j, k when the indices assume values from 1 to 3, and r, s, t when they take all values from 1 to ∞. The restriction of the summation over r to $r>3$ will be denoted by Σ_r'. Greek indices will be used to denote the isospin subscripts only.

[6] See Sec. 4 of Ref. 4. See also Sec. 4 and the Appendix of W. Pauli and S. M. Dancoff (Ref. 2) where a somewhat similar transformation is employed.

This traceless 4×4 matrix can be diagonalized in the form

$$Z^* \sum_n (r_n\sigma_n\tau_n + u_n\sigma_n)Z = -(y_1\sigma_3 + y_2\tau_3 + y_3\sigma_3\tau_3); \quad y_n \geq 0, \quad (2.11)$$

where y_1^2, y_2^2, and y_3^2 are the roots of a certain cubic equation (the "resolvent" associated with the quartic equation resulting from the traceless 4×4 matrix). The algebraic steps involved are given in Sec. 11 of Ref. 4, where a similar problem is solved.[7] We shall here need only the coefficients of the cubic equation, expressed in terms of r_n and u_n.

$$\sum_n y_n^2 = \sum_n (r_n^2 + u_n^2)$$

$$\sum_{n<m} y_n^2 y_m^2 = \sum_{n<m} r_n^2 r_m^2 + \sum_n r_n^2 u_n^2 \quad (2.12)$$

$$(y_1y_2y_3)^2 = (r_1r_2r_3)^2; \quad (y_n \geq 0).$$

Writing $S = YY'Z$, we have the required unitary matrix that diagonalizes H'.

$$S^*H'S = -gC(y_1\sigma_3 + y_2\tau_3 + y_3\sigma_3\tau_3). \quad (2.13)$$

The four eigenvalues of H' are, in general, widely separated because of the "large" factor g.[8] (Actually $g \gg a$ is a necessary condition for the strong-coupling approximation to be valid.) In the Schrödinger equation

$$(-E + H)F = 0,$$

substituting for the 4-component Schrödinger function

$$F = SF', \quad (2.14)$$

we obtain, since S commutes with $\sum_n y_n$,

$$S^*(-E + H_0 + H')SF' = (-E + S^*KS + \Lambda)F' = 0, \quad (2.15)$$

where

$$K = H_0 - gC \sum_n y_n$$

$$\Lambda = gC[(1-\sigma_3)y_1 + (1-\tau_3)y_2 + (1-\sigma_3\tau_3)y_3].$$

In the limit of infinitely strong coupling, Λ vanishes for the lowest eigenvalue of H' (when $\sigma_3 = \tau_3 = 1$) and is $+\infty$ for the other three, so that the lowest eigenstate is completely decoupled from the higher ones. The system of low-lying excited states is then described by a one-component Schrödinger equation:

$$[-E + (S^*KS)_{00}]F' = 0 \quad (2.16)$$

and the relevant isobar terms are contained in $(S^*KS)_{00}$. The corrections caused by mixing between these states and the higher eigenstates of H' (when g is finite, but large) are very small and will be neglected.

III. POTENTIAL VALLEY AND THE TWO CASES

In terms of the field variables p and q defined in Eq. (2.4), the free Hamiltonian H_0 takes the form

$$H_0 = \tfrac{1}{2}\Big[\sum_\rho \Big\{\sum_s p_{s\rho}^2 + \sum_{rs} B_{rs}q_{r\rho}q_{s\rho}\Big\} + \Big\{\sum_s (p_s^\eta)^2 + \sum_{rs} B_{rs}'q_r^\eta q_s^\eta\Big\}\Big], \quad (3.1)$$

where

$$B_{rs} = \int d^3x\, K_r(\mathbf{x})(\mu_\pi^2 - \Delta)K_s(\mathbf{x}),$$
$$B_{rs}' = \int d^3x\, K_r(\mathbf{x})(\mu_\eta^2 - \Delta)K_s(\mathbf{x}). \quad (3.2)$$

Later we will need their inverse matrices

$$\bar{B}_{rs} = \int d^3x\, K_r(\mathbf{x})(\mu_\pi^2 - \Delta)^{-1}K_s(\mathbf{x}),$$
$$\bar{B}_{rs}' = \int d^3x\, K_r(\mathbf{x})(\mu_\eta^2 - \Delta)^{-1}K_s(\mathbf{x}), \quad (3.3)$$
$$\sum_s \bar{B}_{rs}B_{st} = \delta_{rt}, \quad \sum_s \bar{B}_{rs}'B_{st}' = \delta_{rt}.$$

We may note that the submatrices $\bar{B}_{ij}$ and $\bar{B}_{ij}'$ (when $i, j = 1, 2, 3$) are diagonal.

$$\bar{B}_{ij} = \delta_{ij}Y_\pi, \quad Y_\pi = \frac{1}{3C^2}\int d^3x\, \delta_a(\mathbf{x})\frac{-\Delta}{(\mu_\pi^2 - \Delta)}\delta_a(\mathbf{x}), \quad (3.4)$$

and

$$\bar{B}_{ij}' = \delta_{ij}Y_\eta, \quad Y_\eta = \frac{1}{3C^2}\int d^3x\, \delta_a(\mathbf{x})\frac{-\Delta}{(\mu_\eta^2 - \Delta)}\delta_a(\mathbf{x}). \quad (3.5)$$

In the limit $a \ll \mu_\pi^{-1}, \mu_\eta^{-1}$, we have $Y_\pi \simeq Y_\eta \sim a^2$ (and when $a \gg \mu_\pi^{-1}, \mu_\eta^{-1}$, we have $Y_\pi \sim \mu_\pi^{-2}$ and $Y_\eta \sim \mu_\eta^{-2}$).

From Eq. (2.16), we see that the q-dependent terms of the free-meson energy H_0, together with the lowest eigenvalue of H', $-gC\sum_n y_n$, plays the role of a potential energy for the Schrödinger problem. This potential energy exhibits a valley in the q space, the rotational (and vibrational) motions in which furnish the low-lying excited states. The remainder of this section will be devoted to locating this valley or, equivalently, finding the minimum of K as a function of the q variables. In the course of this we shall see that there exist two domains for the ratio f/g, which have characteristically different solutions.

[7] In the case of the general vector meson interaction, the coupling involving both the transverse and longitudinal mesons, G. Wentzel (Ref. 4) has shown that a similar structure for H' results (with σ and τ interchanged), and hence the diagonalization procedure is quite similar.

[8] We are, however, concerned only that the lowest eigenstate should be nondegenerate, and infinitely separated from the other three in the strong-coupling limit. This is satisfied if we require at least two of the three y_n to be nonzero.

The pertinent terms to be minimized are

$$K_q=\tfrac{1}{2}\left[\sum_{rs}\sum_{\rho} B_{rs}q_{r\rho}q_{s\rho}+\sum_{rs} B_{rs}'q_r^{\eta}q_s^{\eta}\right]-gC\sum_n y_n \quad (3.6)$$

with constraints implied by Eqs. (2.8) and (2.9) together with (2.12). Using Lagrange multipliers $\alpha_{i\rho}$ and β_i, we write

$$dK_q-\sum_{i\rho}\alpha_{i\rho}d\left(q_{i\rho}-\sum_n r_n s_{in}s_{\rho n}'\right)-\sum_i \beta_i d\left(q_i^{\eta}-\sum_n (g/f)u_n s_{in}\right)=0. \quad (3.7)$$

Variation with respect to $q_{s\rho}$ gives

$$\sum_r B_{rs}q_{r\rho}-\sum_i \alpha_{i\rho}\delta_{si}=0, \quad (3.8)$$

and by varying with respect to r_m, we get

$$gC\frac{\partial}{\partial r_m}\sum_n y_n-\sum_{i\rho}\alpha_{i\rho}s_{im}s_{\rho m}'=0. \quad (3.9)$$

From Eq. (3.8), with the help of the inverse matrix $\bar{B}$, we obtain

$$q_{r\rho}=\sum_{i\rho}\alpha_{i\rho}\bar{B}_{ir};\quad q_{i\rho}=\alpha_{i\rho}Y_{\pi}. \quad (3.10)$$

Thus

$$\alpha_{i\rho}=Y_{\pi}^{-1}\sum_n r_n s_{in}s_{\rho n}'. \quad (3.11)$$

Substituting for $\alpha_{i\rho}$ in (3.9), we get

$$\Gamma\frac{\partial}{\partial r_m}\sum_n y_n=r_m;\quad \Gamma=gCY_{\pi}. \quad (3.12)$$

Similar variations with respect to q_r^{η} and u_n yield

$$q_r^{\eta}=\sum_i \beta_i\bar{B}_{ir}', \quad (3.13)$$

$$\beta_i=(g/f)Y_{\eta}^{-1}\sum_n u_n s_{in}, \quad (3.14)$$

$$\Gamma'\frac{\partial}{\partial u_m}\sum_n y_n=u_m;\quad \Gamma'=(f/g)^2 gCY_{\eta}. \quad (3.15)$$

On combining (3.10), (3.12), (3.13), and (3.15), the equilibrium value of K_q is readily expressed as

$$K_q^{(0)}=\tfrac{1}{2}\sum_n\left[Y_{\pi}^{-1}r_n^2+Y_{\eta}^{-1}(g/f)^2u_n^2-2gCy_n\right]. \quad (3.16)$$

It may be noted that (2.12) does not provide a simple expression for $\sum_n y_n$; yet its derivatives with respect to r_m and u_m can be computed after a few simple but lengthy operations, so that (3.12) and (3.15) may be replaced by the six algebraic equations:

$$r_m\left\{1-\Gamma\left[(y_1+y_2)(y_2+y_3)(y_3+y_1)\right]^{-1}\left[\sum_{n<n'}y_ny_{n'}+\sum_n y_n^2-r_m^2+u_m^2+\frac{r_1r_2r_3}{r_m^2}\sum_n y_n\right]\right\}=0 \quad (3.17)$$

$$u_m\left\{1-\Gamma'\left[\frac{\sum_{n<n'}y_ny_{n'}+r_m^2}{(y_1+y_2)(y_2+y_3)(y_3+y_1)}\right]\right\}=0. \quad (3.18)$$

Equations (3.17) and (3.18), or equivalently (3.12) and (3.15), have several solutions whose $K_q^{(0)}$ we must now compare to determine the over-all minimum.

Case I. All u_n are zero. This corresponds to the situation where no η but only π mesons are bound. With the help of (2.12) and (3.12), we obtain

$$y_n=r_n=\Gamma \text{ for all } n.$$

Thus, for case I,

$$u_n^{(0)}=0,\quad r_n^{(0)}=\Gamma;\quad K_q^{(0)\mathrm{I}}=-\tfrac{3}{2}gC\Gamma. \quad (3.19)$$

Case II. $u_1=u_2=0$, $u_3\neq 0$. Now (2.12) gives

$$y_{1,2}=\tfrac{1}{2}\{[(r_1+r_2)^2+u_3^2]^{1/2}\pm[(r_1-r_2)^2+u_3^2]^{1/2}\},\quad y_3=r_3.$$

With (3.15) for $m=3$, we get

$$[(r_1+r_2)^2+u_3^2]^{1/2}=\Gamma'.$$

Similarly, (3.12) for $m=3$ gives $r_3=\Gamma$, and for $m=1, 2$ it gives two linear homogeneous equation in r_1 and r_2, the determinant of which vanishes for $\Gamma'=2\Gamma$. Thus, when $\Gamma'\neq 2\Gamma$,

$$r_1^{(0)}=r_2^{(0)}=0,\quad r_3^{(0)}=\Gamma,\quad u_3^{(0)}=\Gamma';\quad K_q^{(0)\mathrm{II}}=-\tfrac{1}{2}gC(\Gamma+\Gamma'). \quad (3.20)$$

Notice that $K_q^{(0)\mathrm{I}}\gtrless K_q^{(0)\mathrm{II}}$, according as $\Gamma'\gtrless 2\Gamma$.

Case III. $u_1\neq 0$, $u_2\neq 0$, $u_3=0$. Then Eq. (3.18) for $m=1, 2$ requires $r_1=r_2=r$, and if $r\neq 0$, then Eq. (3.17) for $m=1, 2$ requires $u_1=\pm u_2=u$. With this, (2.12) gives

$$y_{1,2}=\tfrac{1}{2}\{[(r+r_3)^2+2u^2]^{1/2}\pm[(r-r_3)^2+2u^2]^{1/2}\},\quad y_3=r.$$

From (3.18), follows

$$[(r+r_3)^2+2u^2]^{1/2}=\Gamma',$$

and from (3.17), we get

$$r^{(0)}=\frac{\Gamma(\Gamma'-\Gamma)}{2\Gamma'-3\Gamma},\quad r_3^{(0)}=\frac{\Gamma^2}{2\Gamma'-3\Gamma}.$$

Further, in order that u be real, it is necessary that

$$r^{(0)}+r_3^{(0)}<\Gamma' \quad\text{or}\quad \Gamma'>2\Gamma.$$

Thus, the extremum III (with $r\neq 0$) has the value

$$K_q^{(0)\mathrm{III}} = -\tfrac{1}{2}gC\left[\Gamma' + \frac{\Gamma(\Gamma'-\Gamma)}{(2\Gamma'-3\Gamma)}\right],$$

$$\text{only when } \Gamma' > 2\Gamma. \quad (3.21)$$

In the event that $r=0$, then

$$y_1 = y_2 = 0, \quad y_3 = (r_3^2 + u_1^2 + u_2^2)^{1/2}.$$

Then, either $\Gamma' = \Gamma = y_3$ and $K_q^{(0)\mathrm{III}a} = -\frac{1}{2}gC\Gamma$; or $\Gamma' \neq \Gamma$, whence $r_3 = 0$ and $y_3 = (u_1^2 + u_2^2) = \Gamma'$. Thus all r_n are zero. This situation will be taken up later as Case V.

Case IV. All $u_m \neq 0$. Consequently from (3.18) $r_1 = r_2 = r_3 = r$; and if $r \neq 0$, with (3.17), we get $u_1^2 = u_2^2 = u_3^2 = u^2$. Then it follows from (2.12) that

$$y_{1,2} = \tfrac{1}{2}\{[4r^2 + 3u^2]^{1/2} \pm \sqrt{3}u\}, \quad y_3 = r.$$

Further, from (3.18) and (3.17), we get

$$[4r^{(0)\,2} + 3u^{(0)\,2}]^{1/2} = \Gamma', \quad r^{(0)} = \Gamma\Gamma'/(3\Gamma' - 4\Gamma).$$

This extremum exists only (since u must be real) when $\Gamma' \geq 2\Gamma$. Then, using (3.16), we get

$$K_q^{(0)\mathrm{IV}} = -\tfrac{1}{2}gC\Gamma'\left(1 + \frac{\Gamma}{3\Gamma' - 4\Gamma}\right). \quad (3.22)$$

Case V. Finally, suppose all $r_m = 0$. This corresponds to the situation where only η mesons are bound. One has then $y_1 = y_2 = 0$, $y_3 = [\sum_m u_m^2]^{1/2} = \Gamma'$. Thus

$$K_q^{(0)\mathrm{V}} = -\tfrac{1}{2}gC\Gamma'. \quad (3.23)$$

Comparing the five solutions above, we notice that when $\Gamma' < 2\Gamma$, we have $K_q^{(0)\mathrm{I}} < K_q^{(0)\mathrm{II}}$ and no other solutions are valid; and when $\Gamma' > 2\Gamma$, we have $K_q^{(0)\mathrm{II}} < K_q^{(0)\mathrm{III}} < K_q^{(0)\mathrm{IV}}$ and $K_q^{(0)\mathrm{II}} < K_q^{(0)\mathrm{V}}$, and Case I is not valid. Thus, we get two general solutions, depending on whether

$$\Gamma'/2\Gamma = \tfrac{1}{2}(f/g)^2 Y_\eta Y_\pi^{-1} < 1 \quad \text{(Case I)} \quad (3.24\text{a})$$

or

$$\Gamma'/2\Gamma = \tfrac{1}{2}(f/g)^2 Y_\eta Y_\pi^{-1} > 1 \quad \text{(Case II)}, \quad (3.24\text{b})$$

each providing the corresponding lowest potential valley, the rotations and oscillations in which, as already mentioned, give rise to the spectrum of low-lying excited states. It is also seen that the transition from Case I to Case II takes place within a very short interval, as the ratio of coupling constants $(f/g)^2$ changes through the critical value $2Y_\pi Y_\eta^{-1}$. Indeed, the complications through mixing of states corresponding to Cases I and II occur only if $|K_q^{(0)\mathrm{I}} - K_q^{(0)\mathrm{II}}|$ is less than the vibrational zero-point energy ($\sim 1/a$ if $\mu a \ll 1$), implying

$$\left|\frac{\Gamma'}{2\Gamma} - 1\right| < (a/g)^2. \quad (3.25)$$

Excluding this narrow transition region (note $a \ll g$), we shall assume either Case I or Case II (unmixed).

The unitary S matrix appropriate for diagonalizing H' in either case is now easily approximated. In Case I, since in equilibrium η mesons are not bound, we are content with diagonalizing the dominant pion part of the H' alone and leave the remaining small nondiagonal terms to be treated as perturbations. Thus with $S_\mathrm{I} = YY'Z_\mathrm{I}$; $Z_\mathrm{I} = (\sigma_1 + i\tau_2)/\sqrt{2}$,

$$S_\mathrm{I}^* H' S_\mathrm{I} = gCZ_\mathrm{I}^*[\sum_n (r_n\sigma_n\tau_n + u_n\sigma_n)]Z_\mathrm{I}$$

$$= -gC(r_1\sigma_3 + r_2\tau_3 + r_3\sigma_3\tau_3) + gC\{Z_\mathrm{I}^*(\sum_n u_n\sigma_n)Z_\mathrm{I}\} \quad \text{(Case I)}. \quad (3.26)$$

Similarly in Case II, we require only terms with r_3 and u_3 to be diagonal; thus with $S_\mathrm{II} = YY'$,

$$S_\mathrm{II}^* H' S_\mathrm{II} = gC(r_3\sigma_3\tau_3 + u_3\sigma_3) + gC\{\sum_{n=1}^{2}(r_n\sigma_n\tau_n + u_n\sigma_n)\} \quad \text{(Case II)}. \quad (3.27)$$

The perturbation terms (within curly brackets) in (3.26) and (3.27) give rise to second-order corrections in $[-gC\sum_n y_n]$:

$$-gC\sum_n y_n = -gC\left[\sum_n r_n + \frac{1}{4\Gamma}\sum_n u_n^2\right] \quad \text{(Case I)} \quad (3.28)$$

$$-gC\sum_n y_n = -gC\left[r_3 + u_3 + \frac{1}{2}\left\{\frac{(r_1+r_2)^2}{\Gamma'} + \frac{u_1^2+u_2^2}{\Gamma+\Gamma'}\right\}\right] \quad \text{(Case II)}. \quad (3.28')$$

These additional terms give a contribution to the "potential energy" at small deviations from the equilibrium positions and give rise to a weak scattering of the "free" mesons by the bound system. We shall not, however, consider these effects in this paper.

IV. DOMINANT π COUPLING ($\Gamma' < 2\Gamma$)

With the help of (2.1), (2.9), (3.1), and (3.28), the Hamiltonian describing the strong-coupling effects in Case I is given by

$$K = \tfrac{1}{2}[\sum_\rho\{\sum_s p_{s\rho}^2 + \sum_{rs} B_{rs}q_{r\rho}q_{s\rho}\} - 2gC\sum r_n]$$

$$+ \frac{1}{2}\left[\sum_s (p_s^\eta)^2 + \sum_{rs} B_{rs}' q_r^\eta q_s^\eta - \frac{1}{2Y_\eta}\frac{\Gamma'}{2\Gamma}\sum_i (q_i^\eta)^2\right]. \quad (4.1)$$

The new Hamiltonian K appears *separated* into π and η parts, and the *strong* interaction ($\sim g$) survives only in the π part. Then, the binding of the π mesons occurs as though no η mesons were present, and the results of

Pauli and Dancoff or Houriet[2] can be taken over without any change. The main feature of this case, accordingly, is a nucleon isobar spectrum, given by

$$E_T=\epsilon T(T+1);\quad T=J,\quad T\geqslant|T_3|,\quad \geqslant|J_z|,$$

where

$$\epsilon=3\pi a/g^2 \quad\text{for}\quad a\mu\ll 1 \quad (g\gg a),$$
$$\simeq a^5\mu^4/g^2 \quad\text{for}\quad a\mu\gg 1 \quad (g^2\gg a^5\mu^3).$$

T, T_3, and J_z are all half-odd integers. Only isobar states with equal spin (J) and isospin (T) occur.

We feel, at this point, it is necessary to emphasize that the absence of strong binding of η mesons is *not* to be construed as due to weak ηN coupling. This dominance of pion binding persists so long as the ratio of the coupling constants $(f/g)^2$ is below the critical value (3.24), however large the magnitude of f^2 itself may be. As the ηN coupling is increased so that $(f/g)^2$ exceeds the critical value, the transition from Case I to Case II takes place almost abruptly, and immediately the η mesons begin to be strongly bound. This remarkable distinction, peculiar to the strong-coupling theory, is associated with the wide separation between the two potential valleys (because of the "large" value of g).

Finally we observe that as this transition takes place, it is simultaneously accompanied by an instability developing in the solution corresponding to Case I. The term $-(1/4Y_\eta)(\Gamma'/2\Gamma)\sum_i(q_i{}^\eta)^2$ in Eq. (4.1) (which can be regarded as a "pair interaction"), in conjunction with the free η-meson Hamiltonian, causes a shift in the eigenfrequencies $\omega_k=(\mu_\eta{}^2+k^2)^{1/2}$ of the ψ_η field. The continuous spectrum is unchanged, except for the lowest eigenvalue $\omega_0{}^2$, which, as $\Gamma'/2\Gamma$ is increased from 0 to 1, detaches itself from the continuum, decreases from $\mu_\eta{}^2$ to zero, and becomes *negative* as $\Gamma'/2\Gamma$ is further increased, confirming the fact that "equilibrium I" becomes unstable in Case II.

V. DOMINANT η COUPLING ($\Gamma'>2\Gamma$)

From Eq. (3.20), we know that the potential valley for this case is situated at

$$r_1{}^{(0)}=r_2{}^{(0)}=u_1{}^{(0)}=u_2{}^{(0)}=0,\quad r_3{}^{(0)}=\Gamma,\quad u_3{}^{(0)}=\Gamma'.$$

Convenient polar coordinates, in terms of which the field variables $q_{i\rho}$ and $q_i{}^\eta$ may be expressed, are defined with the help of two unit vectors

$$\begin{aligned}&e_i=s_{i3}\,,\quad e_\rho'=s_{\rho3}'\\ &e_1=\sin\theta\cos\phi\,,\quad e_2=\sin\theta\sin\phi\,,\quad e_3=\cos\theta\,,\\ &e_1'=\sin\theta'\cos\phi'\,,\quad e_2'=\sin\theta'\sin\phi'\,,\quad e_3'=\cos\theta'.\end{aligned}\tag{5.1a}$$

The remaining components of s_{in} and $s_{\rho n}'$ are given by

$$\begin{aligned}s_{i1}\pm is_{i2}&=\exp(\pm i\psi)\frac{\partial e_i}{\partial\theta}\pm i\frac{\exp(\pm i\psi)}{\sin\theta}\frac{\partial e_i}{\partial\phi}\,,\\ s_{\rho1}'\pm is_{\rho2}'&=\exp(\pm i\psi')\frac{\partial e_\rho'}{\partial\theta'}\pm i\frac{\exp(\pm i\psi')}{\sin\theta'}\frac{\partial e_\rho'}{\partial\phi'}.\end{aligned}\tag{5.1b}$$

These definitions are consistent with the orthogonality conditions of s_{in} and $s_{\rho n}'$, since

$$\begin{aligned}&\sum_i e_i\frac{\partial e_i}{\partial\theta}=\sum_i e_i\frac{\partial e_i}{\partial\phi}=\sum_i\frac{\partial e_i}{\partial\theta}\frac{\partial e_i}{\partial\phi}=0\,,\\ &\sum_i e_i{}^2=\sum_i\left(\frac{\partial e_i}{\partial\theta}\right)^2=\sum_i\frac{1}{\sin^2\theta}\left(\frac{\partial e_i}{\partial\phi}\right)^2=1\,,\\ &e_ie_j+\frac{\partial e_i}{\partial\theta}\frac{\partial e_j}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial e_i}{\partial\phi}\frac{\partial e_j}{\partial\phi}=\delta_{ij}.\end{aligned}\tag{5.2}$$

Similar relations hold for e_ρ'.

The nine field variables $q_{i\rho}$ are then readily replaced:

$$\begin{aligned}q_{i\rho}=re_ie_\rho'&+\xi_1\frac{\partial e_i}{\partial\theta}\frac{\partial e_\rho'}{\partial\theta'}+\xi_2\frac{\partial e_i}{\partial\theta}\frac{1}{\sin\theta'}\frac{\partial e_\rho'}{\partial\phi'}\\ &+\xi_3\frac{1}{\sin\theta}\frac{\partial e_i}{\partial\phi}\frac{\partial e_\rho'}{\partial\theta'}+\xi_4\frac{1}{\sin\theta}\frac{\partial e_i}{\partial\phi}\frac{1}{\sin\theta'}\frac{\partial e_\rho'}{\partial\phi'}\,,\end{aligned}\tag{5.3}$$

where we have denoted r_3 by r, and ξ_1, ξ_2, ξ_3, and ξ_4 are linear functions of r_1 and r_2 ($\ll r$), dependent on ψ and ψ'; we are not interested in their explicit form. New variables for the remaining $q_{r\rho}(r>3)$ and $q_s{}^\eta$ are introduced by shifting their origin to their equilibrium values, as given by (3.10), (3.13), and (3.20). Thus

$$\begin{aligned}q_{r\rho}&=q_{r\rho}'+gC\sum_i\bar{B}_{ir}e_ie_\rho'\,,\quad r>3\,,\\ q_s{}^\eta&=q_s'{}^\eta+fC\sum_i\bar{B}_{is}'e_i.\end{aligned}\tag{5.4}$$

The corresponding canonically conjugate p' variables are then defined by[5]

$$\begin{aligned}p_{i\rho}&=p_{i\rho}'+\sum_{r\sigma}{}'\lambda_{i\rho,r\sigma}p_{r\sigma}'+\sum_s\mu_{i\rho,s}p_s'{}^\eta\,,\\ p_{r\rho}&=p_{r\rho}'\,,\quad r>3\,,\\ p_s{}^\eta&=p_s'{}^\eta\,,\end{aligned}\tag{5.5}$$

where, in order to satisfy the canonical commutation relations, we have

$$\begin{aligned}\lambda_{i\rho,r\sigma}&=-igC\sum_j\bar{B}_{jr}[p_{i\rho}',e_je_\sigma']\,,\quad r>3\,,\\ \mu_{i\rho,s}&=-igC\sum_j\bar{B}_{js}'[p_{i\rho}',e_j].\end{aligned}\tag{5.6}$$

$p_{i\rho}'$ are expressible as linear functions of p_θ, p_ϕ, $p_{\theta'}$, $p_{\phi'}$, p_r, π_1, π_2, π_3, and π_4, which are the canonical conjugates of the field variables θ, ϕ, θ', ϕ', r, ξ_1, ξ_2, ξ_3, and ξ_4 introduced in (5.1,5.2,5.3):

$$p_{i\rho}'=p_{i\rho}{}^{(1)}+p_{i\rho}{}^{(2)}+\cdots,$$

where

$$p_{i\rho}^{(1)}=\frac{1}{r}\left\{e_i\left[\frac{\partial e_\rho'}{\partial\theta'}p_{\theta'}+\frac{1}{\sin^2\theta'}\frac{\partial e_\rho'}{\partial\phi'}p_{\phi'}\right]+e_\rho'\left[\frac{\partial e_i}{\partial\theta}p_\theta+\frac{1}{\sin^2\theta}\frac{\partial e_i}{\partial\phi}p_\phi\right]\right\}+\cdots,$$

$$p_{i\rho}^{(2)}=e_ie_\rho'p_r+\frac{\partial e_i}{\partial\theta}\frac{\partial e_\rho'}{\partial\theta'}\pi_1+\frac{\partial e_i}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial e_\rho'}{\partial\phi'}\pi_2+\frac{1}{\sin\theta}\frac{\partial e_i}{\partial\phi}\frac{\partial e_\rho'}{\partial\theta'}\pi_3+\frac{1}{\sin\theta}\frac{\partial e_i}{\partial\phi}\frac{1}{\sin\theta'}\frac{\partial e_\rho'}{\partial\phi'}\pi_4, \quad (5.7)$$

and the dots stand for terms containing ξ variables that are of no interest.

With the help of (5.2) and (5.7) the commutators in (5.6) can be determined:

$$\lambda_{i\rho,r\sigma}=-\sum_j\frac{\bar{B}_{jr}}{Y_\pi}\{e_ie_j(\delta_{\rho\sigma}-e_\rho'e_\sigma')+e_\rho'e_\sigma'(\delta_{ij}-e_ie_j)\},\quad r>3; \quad (5.8)$$

$$\mu_{i\rho,s}=-(f/g)\sum_j\frac{\bar{B}_{js}'}{Y_\pi}e_\rho'(\delta_{ij}-e_ie_j).$$

Thus substituting the new variables (5.3,5.4,5.5) in K, [Eq. (2.15)] we obtain

$$K=\tfrac{1}{2}\{\sum_{r\rho}{}' p_{r\rho}'^2+\sum_s(p_s'^\eta)^2+\sum_{rs}B_{rs}\sum_\rho q_{r\rho}'q_{s\rho}'+\sum_{rs}B_{rs}'q_r'^\eta q_s'^\eta-gC(\Gamma+\Gamma')+\sum_{i\rho}[p_{i\rho}'+\sum_{r\sigma}{}'\lambda_{i\rho,r\sigma}p_{r\sigma}'+\sum_s\mu_{i\rho,s}p_s'^\eta]^2\}+K_s', \quad (5.9)$$

where K_s' is the perturbation term (3.28):

$$K_s'=-\tfrac{1}{2}gC\left[\frac{(r_1+r_2)^2}{\Gamma'}+\frac{u_1^2+u_2^2}{(\Gamma+\Gamma')}\right].$$

We still have to perform another (but weaker) transformation on the p variables ($p'=p'^{(0)}+p''$). This would separate the p-dependent terms of K into those containing only $p_{i\rho}'$, involving the "bound" meson variables, and the rest that denote the "free" mesons, only weakly interacting with the bound system. As usual, this is achieved by a variation of K with respect to $p_{r\sigma}'(r>3)$ and $p_s'^\eta$, the resulting minima $K_p^{(0)}$ containing the required compound nucleon terms.

From $(\partial K_p/\partial p_{r\sigma}')=0$, we get

$$p_{r\sigma}'^{(0)}=-\sum_{i\rho}\lambda_{i\rho,r\sigma}p_{i\rho}^{(0)},\quad r>3, \quad (5.10a)$$

and from $(\partial K_p/\partial p_s'^\eta)=0$, we get

$$p_s'^{\eta(0)}=-\sum_{i\rho}\mu_{i\rho,s}p_{i\rho}^{(0)}, \quad (5.10b)$$

where we have defined

$$p_{i\rho}^{(0)}=p_{i\rho}'+\sum_{r\sigma}{}'\lambda_{i\rho,r\sigma}p_{r\sigma}'^{(0)}+\sum_s\mu_{i\rho,s}p_s'^{\eta(0)}. \quad (5.11)$$

Substituting (5.10) in (5.11), we derive

$$p_{i\rho}^{(0)}+\sum_{j\tau}\Lambda_{i\rho,j\tau}p_{j\tau}^{(0)}=p_{i\rho}' \quad (5.12)$$

with

$$\Lambda_{i\rho,j\tau}=\sum_{r\sigma}{}'\lambda_{i\rho,r\sigma}\lambda_{j\tau,r\sigma}+\sum_s\mu_{i\rho,s}\mu_{j\tau,s}.$$

Substituting for λ and μ from Eq. (5.8), we obtain

$$\Lambda_{i\rho,j\tau}=\alpha e_\rho'e_\tau'(\delta_{ij}-e_ie_j)+\beta e_ie_j(\delta_{\rho\tau}-e_\rho'e_\tau'), \quad (5.13)$$

where

$$\alpha=\{[z+(f/g)^2z']/Y_\pi^2\}-1,\quad \beta=(z/Y_\pi^2)-1$$

with

$$z=\sum_r(\bar{B}_{ir})^2=\frac{1}{3C^2}\int d^3x\delta_a(x)\frac{-\Delta}{(\mu_\pi^2-\Delta)^2}\delta_a(x),$$

$$z'=\sum_r(\bar{B}_{ir}')^2,$$

$$C^2z=1/12\pi a \quad \text{when} \quad \mu_\pi a\ll1,$$
$$\sim a^{-5}\mu_\pi^{-4} \quad \text{when} \quad \mu_\pi a\gg1;$$
$$C^2z'=1/12\pi a \quad \text{when} \quad \mu_\eta a\ll1,$$
$$\sim a^{-5}\mu_\eta^{-4} \quad \text{when} \quad \mu_\eta a\gg1.$$

Substituting (5.13) for Λ in (5.12) we can solve this equation for $p_{i\rho}^{(0)}$, with the result

$$p_{i\rho}^{(0)}=p_{i\rho}'-\frac{\alpha}{1+\alpha}e_\rho'\sum_\sigma e_\sigma'p_{i\sigma}'-\frac{\beta}{1+\beta}e_i\sum_j e_jp_{j\rho}'+\left(\frac{\alpha}{1+\alpha}+\frac{\beta}{1+\beta}\right)e_ie_\rho'\sum_{j\sigma}e_je_\sigma'p_{j\sigma}'. \quad (5.14)$$

This, with (5.10), determines the location and the value of K_p minimum.

$$K_p^{(0)}=\tfrac{1}{2}\sum_{i\rho}p_{i\rho}'p_{i\rho}^{(0)}=\tfrac{1}{2}\sum_{i\rho}\left\{p_{i\rho}'^2-\frac{\alpha}{1+\alpha}p_{i\rho}'e_\rho'\sum_\sigma e_\sigma'p_{i\sigma}'-\frac{\beta}{1+\beta}p_{i\rho}'e_i\sum_j e_jp_{j\rho}'+\left(\frac{\alpha}{1+\alpha}+\frac{\beta}{1+\beta}\right)p_{i\rho}'e_ie_\rho'\sum_{j\sigma}e_je_\sigma'p_{j\sigma}'\right\}. \quad (5.15)$$

Now, with (5.7) for $p_{i\rho}'$, and using the orthogonality conditions (5.2), Eq. (5.15) becomes

$$K_p^{(0)}=\frac{1}{2\Gamma^2}\left(\frac{L^2}{1+\alpha}+\frac{L'^2}{1+\beta}\right)+\tfrac{1}{2}p_r^2+\tfrac{1}{2}\sum_{\nu=1}^{4}\pi_\nu^2, \quad (5.16)$$

where

$$L^2=(1/\sin\theta)p_\theta\sin\theta p_\theta+p_\phi^2/\sin^2\theta,$$

$$L'^2=(1/\sin\theta')p_{\theta'}\sin\theta' p_{\theta'}+p_{\phi'}^2/\sin^2\theta'.$$

The rest of the terms of K are of no interest to us since we do not wish to study here the "free"-meson scattering effects.

Finally, we have only to calculate $S^*K_p^{(0)}S$, the crucial term in the Schrödinger equation (2.15). We split the unitary matrix S_{II} (Eq. 3.27) so that

$$S_{\text{II}}=S^{(0)}S', \quad (5.17)$$

where $S^{(0)}$ makes $H'^{(0)}$ (the value of H' at equilibrium) diagonal, and S' provides perturbative corrections containing ξ_ν and r, which however, play no role in our problem. Accordingly, using (3.20), we have

$$H'^{(0)}=gC\sum_{i\rho}\sigma_i\tau_\rho q_{i\rho}^{(0)}+fC\sum_i\sigma_i q_i^{\eta(0)}$$

$$=gC[\Gamma\sum_{i\rho}\sigma_i\tau_\rho e_i e_\rho'+\Gamma'\sum_i\sigma_i e_i]. \quad (5.18)$$

Hence

$$S^{(0)*}H'^{(0)}S^{(0)}=gC(\Gamma\sigma_3\tau_3+\Gamma'\sigma_3).$$

This diagonalization is achieved by

$$S^{(0)}=YY';\quad Y^*(\sum_i\sigma_i e_i)Y=\sigma_3,\quad Y'^*(\sum_\rho\tau_\rho e_\rho')Y=\tau_3.$$

$$Y=(\cos\phi/2-i\sigma_3\sin\phi/2)(\cos\theta/2-i\sigma_2\sin\theta/2)$$
$$Y'=(\cos\phi'/2-i\tau_3\sin\phi'/2)(\cos\theta'/2-i\tau_2\sin\theta'/2). \quad (5.19)$$

On performing this $S^{(0)}$ transformation on the first term of (5.16), we obtain the isobar terms

$$H_T=\frac{1}{2\Gamma^2}\left[\frac{Y^*L^2Y}{1+\alpha}+\frac{Y'^*L'^2Y'}{1+\beta}\right],$$

$$Y^*L^2Y=\frac{1}{\sin\theta}p_\theta\sin\theta p_\theta+\frac{p_\phi^2-\cos\theta p_\phi+\frac{1}{4}}{\sin^2\theta}+\tfrac{1}{4}, \quad (5.20)$$

$$Y'^*L'^2Y'=\frac{1}{\sin\theta'}p_{\theta'}\sin\theta' p_{\theta'}+\frac{p_{\phi'}^2-\cos\theta' p_{\phi'}+\frac{1}{4}}{\sin^2\theta}+\tfrac{1}{4}.$$

Substituting for α and β from (5.13), we get

$$H_T=\frac{1}{2C^2}\left\{(g^2z+f^2z')^{-1}\left[\frac{1}{\sin\theta}p_\theta\sin\theta p_\theta+\frac{p_\phi^2-\cos\theta p_\phi+\frac{1}{4}}{\sin^2\theta}+\tfrac{1}{4}\right]+(g^2z)^{-1}\left[\frac{1}{\sin\theta'}p_{\theta'}\sin\theta' p_{\theta'}+\frac{p_{\phi'}^2-\cos\theta' p_{\phi'}+\frac{1}{4}}{\sin^2\theta'}+\tfrac{1}{4}\right]\right\} \quad (5.21)$$

as the Hamiltonian of the Schrödinger equation for the rotational states.

$$(-E_T+H_T)F'=0. \quad (5.22)$$

Since ϕ and ϕ' are cyclical variables, F' has the form

$$F'=e^{i(m\phi+n\phi')}u(\theta)u(\theta').$$

The requirement that $F=SF'$ be periodic in the variables ϕ and ϕ', with $S\sim e^{i(\pm\phi\pm\phi')/2}$, restricts m and n to half-odd integers. The eigenvalues of H_T are then

$$E_{T,J}=\frac{J(J+1)}{2C^2(g^2z+f^2z')}+\frac{T(T+1)}{2g^2C^2z}, \quad (5.23)$$

where $J\geqslant|m|$, $T\geqslant|n|$, and J, T, $m=J_z$, and $n=T_3$ are all half-odd integers. In the limit of small a ($\ll u_\pi^{-1}$, $\ll\mu_\eta^{-1}$),

$$E_{T,J}=6\pi a\left[\frac{J(J+1)}{g^2+f^2}+\frac{T(T+1)}{g^2}\right]. \quad (5.24)$$

Note that the condition (3.24b) implies in this limit $f^2>2g^2$.

Thus, the isobar spectrum in Case II is quite different from that of Case I. Whereas when $\Gamma'<2\Gamma$ only isobars with equal spin and isospin could be realized, now when $\Gamma'>2\Gamma$, isobars with $T\neq J$ are possible as well. In particular, the lowest excited state has quantum numbers $T=\frac{1}{2}$, $J=\frac{3}{2}$, and is followed by states with higher T and J values. In further contrast, when ηN coupling is dominant we find that both the π and η fields participate in forming the above bound states. Their strong cooperation is most manifest in the larger denominator (g^2+f^2) in (5.24), which demonstrates that both meson types contribute to the "moment of inertia" of rotational states with spin $J>\frac{1}{2}$. Higher T values can be achieved only by binding π mesons, and correspondingly the "moment of inertia" ($\sim g^2$) is smaller.

VI. NUCLEAR FORCE

The strong-coupling approximation for nucleon-nucleon interaction by exchange of scalar mesons was given by Wentzel,[1] and the result was found to be similar to that obtained in the conventional perturbation theory. We will seek here the equivalent form when π and η mesons are exchanged.

Consider N nucleons at rest, located $\mathbf{x}_\nu(\nu=0,1\cdots,N-1)$. The interaction Hamiltonian is then

$$H'=\sum_{\nu=0}^{N-1}H_\nu'=g\sum_\nu\int d^3x\,\sigma_i^{(\nu)}\tau_\rho^{(\nu)}\psi_\rho(\mathbf{x})(\nabla_i\delta_a(\mathbf{x}-\mathbf{x}_\nu))+f\sum_\nu\int d^3x\,\sigma_i^{(\nu)}\psi_\eta(\mathbf{x})(\nabla_i\delta_a(\mathbf{x}-\mathbf{x}_\nu)). \quad (6.1)$$

Among the complete set of orthogonal functions, the first $3N$ will now be defined by

$$K_{3\nu+i}=-\frac{1}{C}\frac{\partial}{\partial x_i}\delta_a(\mathbf{x}-\mathbf{x}_\nu);\quad \nu=0,\cdots,N-1, i=1,2,3,$$

the orthogonality between these functions being ensured by assuming

$$|\mathbf{x}_\mu-\mathbf{x}_\nu|\gg a \quad\text{or}\quad \int d^3x\,\delta_a(\mathbf{x}-\mathbf{x}_\mu)\delta_a(\mathbf{x}-\mathbf{x}_\nu)=\delta_{\mu\nu}.$$

Thus, by (2.4)

$$H'=\sum_\nu\{gC\sum_{i\rho}\sigma_i^{(\nu)}\tau_\rho^{(\nu)}q_{3\nu+i,\rho}+fC\sum_i\sigma_i^{(\nu)}q_{3\nu+i}^\eta\}. \quad (6.2)$$

H' is now a $4N$-dimensional matrix, diagonalized by $S=\prod_\nu S_\nu$, where S_ν diagonalizes the submatrix H_ν'. We obtain the lowest eigenvalue of H' as

$$-gC\sum_{\nu=0}^{N-1}\sum_{n=1}^{3}y_n^{(\nu)},$$

with the result

$$K=H_0-gC\sum_\nu\sum_n y_n^{(\nu)}. \quad (6.3)$$

We are again interested in the minimum of K_q, in particular its dependence on the nucleon distances $|\mathbf{x}_\mu-\mathbf{x}_\nu|$, because this gives the leading term in the strong-coupling approximation for the nucleon-nucleon interaction.

In order to find the equilibrium values of the q variables, we have to perform a variational calculation on K_q with constraints

$$\begin{aligned}q_{3\nu+i,\rho}&=\sum_n r_n^{(\nu)}s_{in}^{(\nu)}s_{\rho n}'^{(\nu)},\\ q_{3\nu+i}^\eta&=(g/f)\sum_n u_n^{(\nu)}s_{in}^{(\nu)}.\end{aligned} \quad (6.4)$$

This yields

$$\begin{aligned}&\sum_r B_{rs}q_{r\rho}-\sum_i\alpha_{i\rho}^{(\nu)}\delta_{3\nu+i,s}=0,\\ &gC\frac{\partial}{\partial r_m^{(\nu)}}\sum_n y_n^{(\nu)}-\sum_{i\rho}\alpha_{i\rho}^{(\nu)}s_{im}^{(\nu)}s_{\rho m}'^{(\nu)}=0,\\ &\sum_r B_{rs}'q_r^\eta-\sum_i\beta_i^{(\nu)}\delta_{3\nu+i,s}=0,\\ &gC\frac{\partial}{\partial u_m^{(\nu)}}\sum_n y_n^{(\nu)}-(g/f)\sum_i\beta_i^{(\nu)}s_{im}^{(\nu)}=0,\end{aligned} \quad (6.5)$$

from which we derive

$$K_q^{(0)}=\tfrac{1}{2}gC\sum_{\nu,m}\left[r_m^{(\nu)}\frac{\partial}{\partial r_m^{(\nu)}}\sum_n y_n^{(\nu)}+u_m\frac{\partial}{\partial u_m^{(\nu)}}\sum_n y_n^{(\nu)}-2y_m^{(\nu)}\right], \quad (6.6)$$

$$\begin{aligned}q_{3\mu+i,\rho}^{(0)}&=\sum_{\nu,j}\bar{B}_{3\nu+j,3\mu+i}\alpha_{j\rho}^{(\nu)}\\ &=Y_\pi\alpha_{i\rho}^{(\mu)}+\sum_{\nu\neq\mu}\sum_j Y_\pi^{-1}\bar{B}_{3\nu+j,3\mu+i}\\ &\quad\times\sum_n r_n^{(0)}s_{jn}^{(\nu)}s_{\rho n}'^{(\nu)},\end{aligned} \quad (6.7a)$$

$$\begin{aligned}q_{3\mu+i}^{\eta(0)}&=\sum_{\nu j}\bar{B}'_{3\nu+j,3\mu+i}\beta_j^{(\nu)}\\ &=Y_\eta\beta_i^{(u)}+(g/f)\sum_{\nu\neq\mu}\sum_j Y_\eta^{-1}\bar{B}'_{3\nu+j,3\mu+i}\\ &\quad\times\sum_n u_n^{(0)}s_{jn}^{(\nu)},\end{aligned} \quad (6.7b)$$

where Y_π and Y_η are defined by (3.4) and (3.5). In the $\nu\neq\mu$ terms above, we have approximated for $\alpha_{j\rho}^{(\nu)}$ and $\beta_j^{(\nu)}$, using r_n and u_n, as though $|\mathbf{x}_\mu-\mathbf{x}_\nu|$ were ∞. Substituting (6.4) in (6.7) and using (6.5), we get, analogously to (3.12) and (3.15):

$$\begin{aligned}r_m^{(\mu)}&=\Gamma\frac{\partial}{\partial r_m^{(\mu)}}\sum_n y_n^{(\mu)}+Y_\pi^{-1}\sum_{\nu\neq\mu}\sum_{ij\rho}\bar{B}_{3\nu+j,3\mu+i}\\ &\quad\times\sum_n r_n^{(0)}s_{jn}^{(\nu)}s_{im}^{(\mu)}s_{\rho n}'^{(\nu)}s_{\rho m}'^{(\mu)},\end{aligned} \quad (6.8a)$$

$$\begin{aligned}u_m^{(\mu)}&=\Gamma'\frac{\partial}{\partial u_m^{(\mu)}}\sum_n y_n^{(\mu)}+Y_\eta^{-1}\sum_{\nu\neq\mu}\sum_{ij}\bar{B}'_{3\nu+j,3\mu+i}\\ &\quad\times\sum_n u_n^{(0)}s_{jn}^{(\nu)}s_{im}^{(\mu)}.\end{aligned} \quad (6.8b)$$

Further we may write

$$r_n^{(\nu)}=r_n^{(0)}+\delta r_n^{(\nu)},\quad u_n^{(\nu)}=u_n^{(0)}+\delta u_n^{(\nu)}$$

and consider $\delta r_n^{(\nu)}$, $\delta u_n^{(\nu)}$ as small perturbations. As in Sec. III, Eq. (6.8) has several solutions, and depending on whether $\Gamma'/2\Gamma<1$ or >1, Case I or II will be realized.

Case I. ($\Gamma'<2\Gamma$): In this domain $r_n^{(0)}=\Gamma=gCY_\pi$, $u_n^{(0)}=0$ and neglecting terms $\sim(\delta u)^2=u^2$

$$\sum_n y_n^{(\nu)}=\sum_n r_n^{(\nu)},\quad \frac{\partial}{\partial r_m^{(\nu)}}\sum_n y_n^{(\nu)}=1,\ \text{for every }\nu.$$

Thus Eq. (6.6) becomes

$$\begin{aligned}K_q^{(0)}&=-\tfrac{3}{2}gC\Gamma N+\tfrac{1}{2}\sum_{\mu\neq\nu}V_{\mu\nu},\\ V_{\mu\nu}&=-g^2C^2\sum_{ij}\bar{B}_{3\nu+j,3\mu+i}\sum_\rho S_{j\rho}^{(\nu)}S_{i\rho}^{(\mu)},\end{aligned} \quad (6.9)$$

where $S_{i\rho}^{(\nu)}=\sum_n s_{in}^{(\nu)}s_{\rho n}'^{(\nu)}$, which are functions of Euler angles Θ_ν, Φ_ν, Ψ_ν. Here $V_{\mu\nu}$ represents the interaction between nucleon pairs ($\mu\neq\nu$) or rather its dominant term in the static strong-coupling limit. Since

$$\bar{B}_{3\nu+j,3\mu+i}=\int d^3x\, K_{3\nu+j}(\mathbf{x})(\mu_\pi{}^2-\Delta)^{-1}K_{3\nu+i}(\mathbf{x})$$
$$=\frac{1}{C^2}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{e^{-\mu_\pi r}}{4\pi r},$$

where $r=|\mathbf{x}_{(\mu)}-\mathbf{x}_{(\nu)}|$ and $x_i=(\mathbf{x}_{(\mu)}-\mathbf{x}_{(\nu)})_i$, Eq. (6.9) becomes

$$V_{\mu\nu}=g^2\sum_\rho\left(\sum_i S_{i\rho}{}^{(\mu)}\frac{\partial}{\partial x_i}\right)\left(\sum_j S_{j\rho}{}^{(\nu)}\frac{\partial}{\partial x_j}\right)\frac{e^{-\mu_\pi r}}{4\pi r} \tag{6.10}$$

which is to be regarded as a matrix between spin-isospin states of nucleons at $\mathbf{x}_\mu$ and $\mathbf{x}_\nu$. General expressions for these matrix elements have been derived by Fierz.[9] In particular, for the nucleon ground states $J^{(\nu)}=T^{(\nu)}=\frac{1}{2}$, the results can be expressed by merely writing $S_{i\rho}{}^{(\nu)}=\frac{1}{3}\sigma_i{}^{(\nu)}\tau_\rho{}^{(\nu)}$, thus

$$V_{\mu\nu}=\frac{g^2}{9}(\boldsymbol{\tau}^{(\mu)}\cdot\boldsymbol{\tau}^{(\nu)})(\boldsymbol{\sigma}^{(\mu)}\cdot\nabla)(\boldsymbol{\sigma}^{(\nu)}\cdot\nabla)\frac{e^{-\mu_\pi r}}{4\pi r}. \tag{6.11}$$

Except for the numerical factor $\frac{1}{9}$, this agrees with the one-pion exchange interaction, as computed in the conventional perturbation theory.

Case II. $(\Gamma'>2\Gamma)$: In this case $r_3{}^{(0)(\nu)}=\Gamma=gCY_\pi$, $u_3{}^{(0)(\nu)}=\Gamma'=(f/g)^2gCY_\eta$, the rest of the r and u vanishing. Neglecting the terms quadratic in δr and δu, we have

$$\sum_n y_n{}^{(\nu)}=r_3{}^{(\nu)}+u_3{}^{(\nu)},\quad \frac{\partial}{\partial r_3{}^{(\nu)}}\sum_n y_n{}^{(\nu)}=\frac{\partial}{\partial u_3{}^{(\nu)}}\sum_n y_n{}^{(\nu)}=1,$$
$$\frac{\partial}{\partial r_1{}^{(\nu)}}\sum_n y_n{}^{(\nu)}=\frac{\partial}{\partial r_2{}^{(\nu)}}\sum_n y_n{}^{(\nu)}=\frac{r_1{}^{(\nu)}+r_2{}^{(\nu)}}{\Gamma'}, \tag{6.12}$$
$$\frac{\partial}{\partial u_{1,2}{}^{(\nu)}}\sum_n y_n{}^{(\nu)}=\frac{u_{1,2}{}^{(\nu)}}{(\Gamma+\Gamma')}.$$

Equation (6.8) then yields

$$r_3{}^{(\mu)}=\Gamma+gC\sum_{\nu\neq\mu}\sum_{ij}\sum_\rho \bar{B}_{3\nu+j,3\mu+i}\times e_j{}^{(\nu)}e_i{}^{(\mu)}e_\rho{}'^{(\nu)}e_\rho{}'^{(\mu)}, \tag{6.13a}$$
$$u_3{}^{(\mu)}=\Gamma'+gC(f/g)^2\sum_{\nu\neq\mu}\sum_{ij}\bar{B}'_{3\nu+j,3\mu+i}e_j{}^{(\nu)}e_i{}^{(\mu)}, \tag{6.13b}$$

where we have set $s_{i3}{}^{(\nu)}=e_i{}^{(\nu)}$, $s_{\rho 3}{}'^{(\nu)}=e_\rho{}'^{(\nu)}$. Substituting (6.13) in (6.6), we obtain

$$K_q{}^{(0)}=-\tfrac{1}{2}gC(\Gamma+\Gamma')N+\tfrac{1}{2}\sum_{\mu\neq\nu}V_{\mu\nu},$$
$$V_{\mu\nu}=V_{\mu\nu}{}^{(g)}+V_{\mu\nu}{}^{(f)},$$
$$V_{\mu\nu}{}^{(g)}=-g^2C^2\sum_{ij}\sum_\rho \bar{B}_{3\nu+j,3\mu+i}e_j{}^{(\nu)}e_i{}^{(\mu)}e_\rho{}'^{(\nu)}e_\rho{}'^{(\mu)}, \tag{6.14}$$
$$V_{\mu\nu}{}^{(f)}=-f^2C^2\sum_{ij}\bar{B}'_{3\nu+j,3\mu+i}e_j{}^{(\nu)}e_i{}^{(\mu)}.$$

[9] M. Fierz, Helv. Phys. Acta 17, 181 (1944); 18, 158 (1945).

Substituting for $\bar{B}$ and $\bar{B}'$, we get

$$V_{\mu\nu}{}^{(g)}=g^2\sum_\rho(e_\rho{}'^{(\mu)}e_\rho{}'^{(\nu)})\times\left[\left(\sum_i e_i{}^{(\mu)}\frac{\partial}{\partial x_i}\right)\left(\sum_j e_j{}^{(\nu)}\frac{\partial}{\partial x_j}\right)\frac{e^{-\mu_\pi r}}{4\pi r}\right], \tag{6.15a}$$

$$V_{\mu\nu}{}^{(f)}=f^2\left(\sum_i e_i{}^{(\mu)}\frac{\partial}{\partial x_i}\right)\left(\sum_j e_j{}^{(\nu)}\frac{\partial}{\partial x_j}\right)\frac{e^{-\mu_\eta r}}{4\pi r}. \tag{6.15b}$$

For the nucleon ground states $J^{(\nu)}=T^{(\nu)}=\frac{1}{2}$ we can again write $e_i=\frac{1}{3}\sigma_i$, $e_\rho{}'=\frac{1}{3}\tau_\rho$. Thus,

$$V_{\mu\nu}{}^{(g)}=\frac{g^2}{81}(\boldsymbol{\tau}^{(\mu)}\cdot\boldsymbol{\tau}^{(\nu)})\left[(\boldsymbol{\sigma}^{(\mu)}\cdot\nabla)(\boldsymbol{\sigma}^{(\nu)}\cdot\nabla)\frac{e^{-\mu_\pi r}}{4\pi r}\right],$$
$$V_{\mu\nu}{}^{(f)}=\frac{f^2}{9}(\boldsymbol{\sigma}^{(\mu)}\cdot\nabla)(\boldsymbol{\sigma}^{(\nu)}\cdot\nabla)\frac{e^{-\mu_\eta r}}{4\pi r}.$$

This result is again in formal agreement with a weak-coupling theory involving single π and η exchange, with renormalized coupling constants suitably defined.

VII. CONCLUDING REMARKS

The Chew-Low-Wick[10] static theory, considering only zero- and one-pion intermediate states, successfully explained the genesis of the $T=\frac{3}{2}, J=\frac{3}{2}$ resonance. It further showed that no binding was possible in the other channels $(\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},\frac{3}{2})$, or $(\frac{3}{2},\frac{1}{2})$, essentially because the dominant Born-approximation phase shifts of these channels were negative. If isoscalar η mesons alone were considered, then only the $T=\frac{1}{2}, J=\frac{3}{2}$ channel would be found to resonate in this formalism, for much the same reasons.

Now, for a system with both π and η mesons interacting with nucleons, we have a multichannel problem, involving a 2×2 matrix for the scattering amplitude. Capps[11] has made an ND^{-1} calculation assuming in N a simple pole, the residue of which is given by the Born amplitudes. D is then readily computed by appealing to the unitarity requirement, and $\det|D|=0$ provides the resonance condition. For the $(\frac{3}{2},\frac{3}{2})$ channel, this reduces to the Chew-Low condition (neglecting the refinements due to crossing)

$$1-2\omega_{33}\gamma_\pi g^2=0, \tag{7.1}$$

and for the $(\frac{1}{2},\frac{3}{2})$ state it takes the form

$$\det D=1+\omega_{13}\gamma_\pi g^2-\omega_{13}\gamma_\eta f^2-4\omega_{13}{}^2\gamma_\pi\gamma_\eta g^2f^2=0, \tag{7.2}$$

where γ_π and γ_η are integals involving kinematical factors and are only weakly energy-dependent. The other channels $(\frac{1}{2},\frac{1}{2})$ and $(\frac{3}{2},\frac{1}{2})$ cannot resonate, again

[10] G. F. Chew and F. E. Low, Phys. Rev. 101, 1570 (1956); G. C. Wick, Rev. Mod. Phys. 27, 339 (1955).

[11] R. H. Capps, Nuovo Cimento 27, 1208 (1963); see also A. Martin and K. C. Wali, Phys. Rev. 130, 2455 (1963).

because of negative phase shifts in the Born approximation.

Clearly from Eq. (7.2), a large enough $f^2\gamma_\eta$ is necessary to offset the $g^2\gamma_\pi$ term if it is to cause a binding in the $(\frac{1}{2},\frac{3}{2})$ channel. Granting this, the relative position of the two resonances is dependent on the ratio $g^2\gamma_\pi/f^2\gamma_\eta$. In particular $\omega_{13}<\omega_{33}$, when

$$f^2/g^2>\gamma_\pi\gamma_\eta^{-1}. \quad (7.3)$$

This is, indeed, at least outwardly similar to the relation $\Gamma'>2\Gamma$ or $(f^2/g^2>2Y_\pi Y_\eta^{-1})$ obtained in Sec. III for the strong-coupling approximation.

Although in both theories, as intuitively expected, a $T=\frac{1}{2}$ resonance is enhanced only when the ηN interaction is dominant, there is considerable difference in their detailed mechanism. A characteristic feature of the strong-coupling theory is the abrupt emergence of the $T=\frac{1}{2}$ state as the coupling constant f is varied through the critical value, and further manifest in the structure of this theory is a curious interplay between the bound π and η meson fields. Furthermore, the strong-coupling theory predicts a $T=\frac{3}{2}$, $J=\frac{1}{2}$ state with its energy value lying between those of the states $(\frac{1}{2},\frac{3}{2})$ and $(\frac{3}{2},\frac{3}{2})$, when the ηN coupling is dominant (Case II). This effect is entirely peculiar to the strong-coupling model, and specifically a $J=\frac{1}{2}$ resonance cannot be obtained by the Chew-Low approach or in any theory that uses the Born amplitude as the boundary value, since these dominant Born terms have negative phase shifts. Finally, the higher excited states that arise in our scheme may be of some relevance when more detailed investigation of high-energy resonances is completed.

We may now attempt to correlate the conclusions of our model with the present experimental situation. We notice that the lowest lying nucleon isobar is $N_{3/2,3/2}^*(1238)$; thus we may associate the nucleon interactions with Case I in our model, which in turn implies $(f_{\bar{N}N\eta}/g_{\bar{N}N\pi})^2<2$ (in the limit $a\mu\ll1$). It is easy to see that similar strong-coupling calculations can be made with the Ξ doublet (instead of nucleons) to generate $S=-2$ excited states. Here, however, since $\Xi_{1/2,3/2}^*$ at 1529 Mev happens to be the lowest excited state, Case II is probably realized, which would indeed mean $(f_{\bar{\Xi}\Xi\eta}/g_{\bar{\Xi}\Xi\pi})^2>2$.

We could examine these inequalities in the light of the octet version of the unitary symmetry for strong interactions. The $SU(3)$-invariant interaction between a baryon octet and a meson octet is of two distinct types, D and F (related to d_{ijk} and f_{ijk} of Gell-Mann[12]), and the ratio of their mixing is given by a parameter α.[13] Precisely, $\alpha=0$ refers to pure D and $\alpha=1$ to pure F coupling. In terms of this mixing parameter, the ratios of our coupling constants are easily expressed:

$$\left(\frac{f_{\bar{N}N\eta}}{g_{\bar{N}N\pi}}\right)^2=\left[\frac{4\alpha-1}{\sqrt{3}}\right]^2,\quad \left(\frac{f_{\bar{\Xi}\Xi\eta}}{g_{\bar{\Xi}\Xi\pi}}\right)^2=\left[\frac{2\alpha+1}{(2\alpha-1)\sqrt{3}}\right]^2. \quad (7.4)$$

For α between 0.21 and 0.86 the above ratios are such as to fall in the Case I domain for nucleon interaction and in the region of Case II for Ξ interaction. This is indeed the region within which the recent experimental fit seems to fall[14] and is consistent with what we have observed in the previous paragraph.

Indeed, it should be interesting to incorporate hypercharge into the scheme of internal symmetry while diagonalizing the strong coupling interaction Hamiltonian, with K-meson effects properly included. *If* such a unified pseudoscalar strong-coupling calculation does not drastically change the results of our restricted model, then we have partially established that the lowest lying set of baryon resonances belongs to the $SU(3)$ multiplet $\{10\}$ and not to $\{\overline{10}\}$, $\{27\}$, or $\{8\}$, since the combination $N_{3/2}^*$ and $\Xi_{1/2}$ occurs only in the $\{10\}$ representation.[15]

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[12] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).

[13] Our parameter α corresponds to α_p of J. J. de Swart, Rev. Mod. Phys. **35**, 916 (1963), Sec. 17.

[14] For example, W. Willis *et al.*, Phys. Rev. Letters **13**, 291 (1964). There are two solutions obtained as best fits, giving $\alpha=0.37$ [agreeing with the solution of N. Cabibbo, Phys. Rev. Letters **10**, 531 (1963)] and $\alpha=0.63$. Both of these values of α satisfy our criterion.

[15] The $\{27\}$ representation requires $T=\frac{1}{2},\frac{3}{2}$ members for both N^* and Ξ^*.