slightly on the radius chosen, but the variation in the theoretical ratio with radii from 0.7 to 0.9 F is less than 0.5%. The answers shown were obtained with an rms radius of 0.8 F. The column labeled G_M/G_E scattering shows the ratio of magnetic to electric scattering predicted by the experimental fit to the Rosenbluth[21] cross section. Figure 6 shows the data plotted graphically as a function of momentum transfer. The dashed line is the best polynomial fit to the data passing through one at q^2=zero. The data show deviations from one (the first Born prediction) at higher momentum transfer and backward angles. It is probable that there are more two-photon corrections than predicted by Lewis's theory at the larger momentum transfers.

[21] M. Rosenbluth, Phys. Rev. **79**, 615 (1950).

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New Reduction of the Faddeev Equations and Its Application to the Pion as a Three-Particle Bound State*

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A new separation of the angular momentum in the Faddeev equations is given. This separation makes use of the relative angular momentum of two particles, which is combined with the angular momentum of the third particle in the over-all center-of-mass system. With the assumption that the two-body amplitudes factorize in the initial and final momenta, the Faddeev equations are reduced to a coupled set of integral equations in one variable. This set is furthermore simplified in the case of identical particles to only one integral equation. Thereby the statistics is correctly taken into account. The resulting equation is used to investigate possible bound states of three pions with total angular momentum zero, isospin one, and odd parity. The two-body amplitude which determines the kernel is approximated by the isospin-zero, *s*-wave effective-range formula of Chew and Mandelstam. Use is also made of relativistic kinematics. The pion is found as a bound state of three pions in this model. The outcome is, however, strongly dependent on a physical cutoff parameter in the two-body form factor. As a result a detailed investigation of the form factor is desirable.

1. INTRODUCTION

The Faddeev equations,[1–3] and their validity, for a system of nonrelativistic three particles interacting through two-body potentials between each pair of particles are now well known. These equations are clearly applicable to quantum-mechanical three-particle systems such as the problem of electron-hydrogen atom scattering. They can also be applied to three-body problems in low-energy nuclear physics in which the two-body interactions can be described by some sort of phenomenological potential. Thus in these problems the Faddeev equations are expected to play an important role. The accuracy of the results of such calculations merely depends on how accurately the computations can be carried out.

Our interest in the Faddeev equations is, however, based on their possible application to particle physics. Here, too, very little has been done with the three-particle problems. In nearly all the problems, the three-particle system has been regarded as being two particles, one of which is composed of two particles clumped together. The Faddeev equations, although nonrelativistic, are at least genuine three-particle equations. Furthermore, a remarkable property of the Faddeev equations is that they only require a knowledge of the two-body amplitude (off the energy shell). This is clearly an advantage because at least in the region of resonances and where the effective-range formulas are valid, the two-body amplitude is known fairly well, whereas very little is known about a corresponding po-

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[1] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].
[2] L. D. Faddeev, Dokl. Akad. Nauk SSSR **138**, 565 (1961) [English transl.: Soviet Phys.—Doklady **6**, 384 (1961)].
[3] L. D. Faddeev, Dokl. Akad. Nauk SSSR **145**, 301 (1962) [English transl.: Soviet Phys.—Doklady **7**, 600 (1963)].

tential (and even its possible existence). A prescription is needed, however, to find the off-shell two-body amplitude once the on-shell amplitude is known. As pointed out by Lovelace,[4,5] in the neighborhood of a resonance or a bound state the residue of the off-shell two-body amplitude factorizes in the initial and final momenta. The resulting functions of the momenta are called the form factors. In the case of a bound state these form factors are simply related to the bound-state wave function. Such considerations arise naturally if one considers the two-body bound-state solution for a separable potential.[6]

With this prescription of taking the two-body amplitude off the energy shell we apply the Faddeev equations to strongly interacting particles. We take as the simplest example the problem of possible bound states and resonances of three pions. The pions being spinless offer a relatively simpler problem. Furthermore, since we want to look for three-particle bound states and resonances, we only have to study the possible solutions of the homogeneous equations. In particular, in the present paper, we shall consider the homogeneous Faddeev equations for a system of three pions in a state of total angular momentum zero, isospin one, and odd parity. Thus, following the notion of composite particles often emphasized by Chew,[7] we want to know whether the homogeneous Faddeev equations have a solution at an energy corresponding to that of the pion and with the above quantum numbers. We would interpret a positive result to mean that, within the framework of the Faddeev equations and our approximations (to be discussed later), the pion can be understood as a bound state of three pions. Similarly, one can investigate whether three pions can form a resonance state corresponding to the ω particle.

Our first task is thus to write the Faddeev equations for a system of three pions in a given angular momentum, parity, and isospin state. A method of separation in angular momentum of the Faddeev equations has been discussed by Omnes[8] (this separation was utilized by one of us in a preliminary investigation of the ω problem[9]). Omnes uses the variables described as the energies of the three particles in the three-body center-of-mass system, the total angular momentum and its components in a body-fixed axis and on a space-fixed axis. We use instead the variables introduced by Dalitz[10] in connection with the three-pion decay of the τ meson. Using this separation of angular momentum together with the factorization of the off-shell two-body amplitude, we obtain an integral equation in one variable so that numerical computations can be carried out with reasonable confidence. Recently, a similar result has also been obtained by Basdevant[11] using the Omnes variables. The separation of angular momentum by Omnes and the one carried out here are merely two alternative ways of which one or the other may be more practical in a particular problem.

In the pion problem the two-body amplitude is considered to consist only of the isospin zero, s-wave state described by the ABC phenomenon, and the effective-range formula associated with it. The third pion is considered to combine with the ABC to form an over-all isospin one, angular-momentum zero state. This model (for every combination of the pions) is imbedded into the Faddeev equations, and conditions for a homogeneous solution are sought.

In the following section we describe the reduction in the angular momentum. Subsequently, the pion problem is considered in Sec. 3. Finally, in the last section we give a discussion of the results.

2. REDUCTION OF THE FADDEEV EQUATIONS

Let us consider the case of three nonidentical spinless particles with masses m_1, m_2, m_3. At the end of this section we shall examine how the reduced equations can be further simplified for identical particles.

In the nonrelativistic case the equations for the three-particle scattering matrix T has been given by Faddeev.[1] They can be written in a formal way as

$$\begin{aligned} T^1(s)&=T_1(s)-T_1(s)G_0(s)\{T^2(s)+T^3(s)\}\,,\\ T^2(s)&=T_2(s)-T_2(s)G_0(s)\{T^1(s)+T^3(s)\}\,,\\ T^3(s)&=T_3(s)-T_3(s)G_0(s)\{T^1(s)+T^2(s)\}\,, \end{aligned} \tag{2.1}$$

with

$$\begin{aligned} T&=T^1+T^2+T^3\,,\\ G_0(s)&=1/(H_0-s)\,,\\ T_i(s)&=V_i-V_iG_0(s)T_i(s)\,. \end{aligned} \tag{2.2}$$

Here H_0 denotes the total kinetic energy of the three particles and V_i is the potential between particles j and $k(\neq i)$. From (2.2) we see that T_i is the two particle scattering matrix in the Hilbert space of the three particle states. This should in principle be known off the energy shell for solving the Faddeev equations.

The $T^i(s)$ are defined by the equation

$$T^i(s)=V_i-V_iG_0(s)T(s)\,. \tag{2.3}$$

[4] C. Lovelace, in *Strong Interactions and High Energy Physics*, edited by R. G. Moorhouse (Oliver and Boyd, London, 1964).

[5] C. Lovelace, Phys. Rev. **135**, B1225 (1964).

[6] See, for example, Yoshio Yamagouchi, Phys. Rev. **95**, 1628 (1954).

[7] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961).

[8] R. Omnes, Phys. Rev. **134**, B1358 (1964).

[9] Akbar Ahmadzadeh, Lawrence Radiation Laboratory Report UCRL-11749, 1964 (unpublished). Note that Eq. (11) of this paper should read: $f_{23}=-a_{23}/2\pi^2$. Although this error makes the eigenvalues considerably smaller, a preliminary study of the integral equation given in the present paper showed that it is possible to produce eigenvalues of the order of 1 by taking the physical cutoff parameter p_m to be of the order of 5. We intend to re-examine this problem within the framework of the present paper.

[10] R. H. Dalitz, Phys. Rev. **94**, 1046 (1954).

[11] Jean Louis Basdevant, Phys. Rev. **138**, B892 (1965); Lawrence Radiation Laboratory Report UCRL-11838, 1964 (unpublished). We are grateful to Dr. Basdevant for communicating his results to us.

Owing to the conservation of the total momentum we may assume without any loss of generality that we are in the center-of-mass system of the three particles. The Eqs. (2.1) have been further reduced by Omnes[8] by using as variables the Eulerian angles and the absolute values of the momenta $\mathbf{k}_i(i=1,2,3)$. We shall here carry out the reduction with a different set of variables.

Following Lovelace,[4,5] instead of characterizing a three particle state by the momenta $\mathbf{k}_i(i=1,2,3)$, we can use certain combinations of them, namely

$$\mathbf{p}_1=\frac{1}{(2m_2m_3(m_2+m_3))^{1/2}}[m_3\mathbf{k}_2-m_2\mathbf{k}_3],$$
$$\mathbf{q}_1=\frac{1}{(2m_1(m_2+m_3)(m_1+m_2+m_3))^{1/2}}\times[m_1(\mathbf{k}_2+\mathbf{k}_3)-(m_2+m_3)\mathbf{k}_1]. \tag{2.4}$$

The corresponding normalized state we shall denote by $|\mathbf{p}_1;\mathbf{q}_1\rangle_1$. The extra subscript 1 is needed to stress the fact that we use the combination (2.4). Furthermore, we shall also need the corresponding partial-wave states, which are denoted by $|p_1lm_l;q_1Lm_L\rangle_1$. These wave functions are normalized as

$$_1\langle plm_l;qLm_L|p'l'm_{l'};q'L'm_{L'}\rangle_1 =(p^2q^2)^{-1}\delta(p-p')\delta(q-q')\delta_{ll'}\delta_{LL'}\delta_{m_lm_{l'}}\delta_{m_Lm_{L'}}.$$

In addition to the set $\mathbf{p}_1$, $\mathbf{q}_1$, we shall also use the other sets $\mathbf{p}_2$, $\mathbf{q}_2$ and $\mathbf{p}_3$, $\mathbf{q}_3$, which are defined by cyclic permutation of the subscripts in Eq. (2.4). The corresponding normalized states are denoted, respectively, by $|\mathbf{p}_2;\mathbf{q}_2\rangle_2$ and $|\mathbf{p}_3;\mathbf{q}_3\rangle_3$. Needless to say, these representations describe the same state of the three particles. Hence

$$|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle=|\mathbf{p}_1;\mathbf{q}_1\rangle_1=|\mathbf{p}_2;\mathbf{q}_2\rangle_2=|\mathbf{p}_3;\mathbf{q}_3\rangle_3. \tag{2.5}$$

The relations between two sets of variables is simply a linear one. For example,

$$\mathbf{p}_2=-\left(\frac{m_1m_2}{(m_1+m_3)(m_2+m_3)}\right)^{1/2}\mathbf{p}_1+\left(\frac{m_3(m_1+m_2+m_3)}{(m_1+m_3)(m_2+m_3)}\right)^{1/2}\mathbf{q}_1,$$
$$\mathbf{q}_2=-\left(\frac{m_3(m_1+m_2+m_3)}{(m_1+m_3)(m_2+m_3)}\right)^{1/2}\mathbf{p}_1-\left(\frac{m_1m_2}{(m_1+m_3)(m_2+m_3)}\right)^{1/2}\mathbf{q}_1. \tag{2.6}$$

Since the total angular momentum J is a conserved quantity, in writing out the Faddeev equations explicitly it is more appropriate to use the states in which J is diagonal. These states are simply given by

$$|pqJMlL\rangle_i=\sum_{m_l,m_L}C\begin{pmatrix}J & l & L\\ M & m_l & m_L\end{pmatrix}|plm_l;qLm_L\rangle_i, \tag{2.7}$$

in which $C\begin{pmatrix}J & l & L\\ M & m_l & m_L\end{pmatrix}$ are the well-known Clebsch-Gordan coefficients. For convenience we shall denote the discrete quantum numbers $JMlL$ simply by α.

We are now in a position to write out the Faddeev equations in this representation. Let us consider the first equation in (2.1). The other two can be treated in a similar way. With the notation

$$\Psi_n{}^i(pq\alpha)={}_i\langle pq\alpha|T^i|n\rangle$$

(where $n=\mathbf{k}_1'',\mathbf{k}_2'',\mathbf{k}_3''$), it can be written as

$$\Psi_n{}^1(pq\alpha)=\Phi_n{}^1(pq\alpha)-\sum_{i=2}^{3}\sum_{\alpha_i}\int dp_idq_iK_i(pq\alpha|p_iq_i\alpha_i)\times\frac{p_i{}^2q_i{}^2}{p_i{}^2+q_i{}^2-s}\Psi_n{}^i(p_iq_i\alpha_i), \tag{2.8}$$

where

$$K_i(pq\alpha|p_iq_i\alpha_i)={}_1\langle pq\alpha|T_1|p_iq_i\alpha_i\rangle_i,$$
$$\Phi_n{}^1(pq\alpha)={}_1\langle pq\alpha|T_1|n\rangle.$$

We first proceed to compute K_2. With the aid of (2.5) and (2.7) we obtain

$$K_2=\sum_{\substack{m_l,m_L\\ m_{l'},m_{L'}}}\int d\Omega\, C\begin{pmatrix}J & l & L\\ M & m_l & m_L\end{pmatrix}\times C\begin{pmatrix}J' & l' & L'\\ M' & m_{l'} & m_{L'}\end{pmatrix}{}_1\langle\mathbf{pq}|T_1|\mathbf{p}_1\mathbf{q}_1\rangle_1Y_{m_l}{}^{l*}(\theta_p,\varphi_p)\times Y_{m_L}{}^{L*}(\theta_q,\varphi_q)Y_{m_{l'}}{}^{l'}(\theta_{p_2},\varphi_{p_2})Y_{m_{L'}}{}^{L'}(\theta_{q_2},\varphi_{q_2}),$$

where $d\Omega=d\cos\theta_pd\varphi_pd\cos\theta_qd\varphi_pd\cos\theta_{p_2}d\varphi_{p_2}d\cos\theta_{q_2}d\varphi_{q_2}$. Moreover, $\mathbf{p}_1$ and $\mathbf{q}_1$ are defined through the relation (2.6) as functions of $\mathbf{p}_2$ and $\mathbf{q}_2$. According to (2.2) the matrix element ${}_1\langle\mathbf{pq}|T_1|\mathbf{p}_1\mathbf{q}_1\rangle_1$ is given by

$${}_1\langle\mathbf{pq}|T_1|\mathbf{p}_1\mathbf{q}_1\rangle_1=\delta(\mathbf{q}-\mathbf{q}_1)\langle\mathbf{p}|\hat{T}_1(s-q^2)|\mathbf{p}_1\rangle, \tag{2.9}$$

where $\hat{T}_1$ is the two-particle scattering matrix in the Hilbert space of the two-particle states. It is now useful to make the decomposition

$$\langle\mathbf{p}|\hat{T}_1(s)|\mathbf{p}_1\rangle=\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta_{pp_1})\hat{t}_l{}^1(p,p_1;s). \tag{2.10}$$

Here θ_{pp_1} denotes the angle between $\mathbf{p}$ and $\mathbf{p}_1$. (A similar decomposition can be made for $\hat{T}_2$ and $\hat{T}_3$.) With the aid of (2.9) together with (2.10), we obtain for K_2, after integration over the angles θ_p, φ_p, θ_q, and φ_q the expression

$$K_2=\frac{8\pi}{q}\sum_{\substack{m_l,m_L\\ m_{l'},m_{L'}}}\int d\cos\theta_{p_2}d\varphi_{p_2}d\cos\theta_{q_2}d\varphi_{q_2}C\begin{pmatrix}J & l & L\\ M & m_l & m_L\end{pmatrix}C\begin{pmatrix}J' & l' & L'\\ M' & m_{l'} & m_{L'}\end{pmatrix}\delta(q^2-q_1^2)\cdot\hat{t}_l^1(p,p_1;s-q^2)$$
$$\times Y_{m_l}^{l*}(\theta_{p_1},\varphi_{p_1})Y_{m_L}^{L*}(\theta_{q_1},\varphi_{q_1})Y_{m_{l'}}^{l'}(\theta_{p_2},\varphi_{p_2})Y_{m_{L'}}^{L'}(\theta_{q_2},\varphi_{q_2}).\quad(2.11)$$

We have, furthermore, made use of the relations

$$\delta(\mathbf{q}-\mathbf{q}_1)=(2/q)\delta(q^2-q_1^2)\delta(\cos\theta_q-\cos\theta_{q_1})\delta(\varphi_q-\varphi_{q_1}),$$
$$P_l(\cos\theta_{pp_1})=\frac{4\pi}{2l+1}\sum_m Y_m^l(\theta_p,\varphi_p)Y_m^{l*}(\theta_{p_1},\varphi_{p_1}).$$

After some calculation (see Appendix) the expression (2.11) can be simplified to

$$K_2=\frac{32\pi^{5/2}(2l+1)^{1/2}}{q}(-1)^{l+l'-L-L'}\delta_{JJ'}\delta_{MM'}\sum_{\substack{m_l,m_L\\ m_{L'}}}\begin{pmatrix}l & L & J\\ m_l & m_L & -m_{L'}\end{pmatrix}\begin{pmatrix}l' & L' & J\\ 0 & m_{L'} & -m_{L'}\end{pmatrix}$$
$$\times\int d\cos\theta_{q_2p_2}\delta(q^2-q_1^2)\hat{t}_l^1(p,p_1;s-q^2)Y_{m_l}^{l*}(\theta_{p_1p_2},0)Y_{m_L}^{L*}(\theta_{q_1p_2},0)Y_{m_{L'}}^{L'}(\theta_{q_2p_2},0),\quad(2.12)$$

in which the angles $\theta_{\ldots}$ should all be expressed with the aid of (2.6) as functions of p_2, q_2 and $\theta_{q_2p_2}$. Here the symbols in large parentheses are the Wigner 3-j symbols (see, for example, Ref. 12). It should be noted from (2.12) that J and M are conserved, which was to be expected.

In the same way one finds for K_3:

$$K_3=\frac{32\pi^{5/2}(2l+1)^{1/2}}{q}(-1)^{l+l'-L-L'}\delta_{JJ'}\delta_{MM'}\sum_{\substack{m_l,m_L\\ m_{L'}}}\begin{pmatrix}l & L & J\\ m_l & m_L & -m_{L'}\end{pmatrix}\begin{pmatrix}l' & L' & J\\ 0 & m_{L'} & -m_{L'}\end{pmatrix}$$
$$\times\int d\cos\theta_{q_3p_3}\delta(q^2-q_1^2)\hat{t}_l^1(p,p_1;s-q^2)Y_{m_l}^{l*}(\theta_{p_1p_3},0)Y_{m_L}^{L*}(\theta_{q_1p_3},0)Y_{m_{L'}}^{L'}(\theta_{q_3p_3},0).\quad(2.13)$$

Here p_1, q_1, and $\theta_{\ldots}$ are related to p_3, q_3 and $\theta_{q_3p_3}$ through a relation between $\mathbf{p}_1$, $\mathbf{q}_1$ and $\mathbf{p}_3$, $\mathbf{q}_3$ analogous to Eq. (2.6).

The inhomogeneous part of (2.8) can be calculated in a straightforward way. We shall not write this out explicitly, since in the applications considered in this paper, we shall only be concerned with the solutions of the homogeneous part of (2.1).

The Faddeev equations, which according to (2.8) have been reduced to a coupled set of integral equations with only two continuous variables, can be further simplified to a set with only one variable, if we make an approximation for the two-body partial-wave amplitudes. It was pointed out by Lovelace that when the partial-wave amplitudes are dominated by a bound state or resonance the two-body scattering matrix off the energy shell factorizes out in a good approximation in the following way[5]

$$\hat{t}_l^i(p,p';s)=g_l^i(p)g_l^i(p')t_l^i(s).\quad(2.14)$$

Assuming the validity of (2.14), the p dependence in the Faddeev equations can then easily be separated out. From (2.8) one sees, namely, in view of (2.12) and (2.13), that the solution is simply given in the form

$$\Psi_n^i(pq\alpha)=g_l^i(p)\tilde{\Psi}_n^i(q\alpha).$$

With this, Eq. (2.8) reduces to

$$\tilde{\Psi}_n^1(q\alpha)=\tilde{\Phi}_n^1(q\alpha)-\sum_{i=2}^{3}\sum_{\alpha_i}\int_0^\infty dq_i\tilde{K}_i(q\alpha|q_i\alpha_i)\tilde{\Psi}_n^i(q_i\alpha_i),\quad(2.15)$$

where

$$\tilde{\Phi}_n^1=\Phi_n^1(pq\alpha)/g_l^1(p),$$

$$\tilde{K}_i=\int_0^\infty dp_iK_i(pq\alpha|p_iq_i\alpha_i)\frac{p_i^2q_i^2}{p_i^2+q_i^2-s}\frac{g_{l_i}^i(p_i)}{g_l^1(p)}.\quad(2.16)$$

The generalization to the case that a certain partial wave contains more than one resonance is quite obvious.

Finally, we shall examine how the equations are simplified when the three particles are identical. In this

[12] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, New Jersey, 1957).

case, the matrix elements of two different T^i can be related to each other. For example, we have according to (2.3)

$$\langle \mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|T^1|\mathbf{k}_1'\mathbf{k}_2'\mathbf{k}_3'\rangle = \langle \mathbf{k}_2\mathbf{k}_1\mathbf{k}_3|T^2|\mathbf{k}_2'\mathbf{k}_1'\mathbf{k}_3'\rangle. \quad (2.17)$$

With the aid of these relations the matrix elements of T^2 and T^3 in (2.1) can be replaced by the corresponding ones of T^1. Furthermore, we also have to take the proper statistics into account. It is readily verified using (2.17) that the requirement that the particles should satisfy the Bose or Fermi statistics is equivalent to the condition

$${}_i\langle \mathbf{pq}|T^i|n_\pm\rangle = \pm {}_i\langle -\mathbf{pq}|T^i|n_\pm\rangle. \quad (2.18)$$

Here $|n_\pm\rangle$ stands respectively for a totally symmetric or antisymmetric wave function with respect to interchange of any two particles. The condition (2.18) simply amounts to the requirement that the matrix elements of T^i in (2.18) have only to be symmetric or antisymmetric with respect to interchange in the initial state of the two particles j and $k(\neq i)$. From (2.15) we see that (2.18) can be satisfied by imposing on the kernel of the integral equation the condition that l should be even or odd, respectively. Using this, one finds that the Faddeev equations are reduced to only one integral equation in one variable which is given by

$$\tilde{\Psi}_{n\pm}{}^1(q\alpha) = \tilde{\Phi}_{n\pm}{}^1(q\alpha) - \sum_{\alpha'} \int_0^\infty dq' \tilde{K}(q\alpha|q'\alpha')\tilde{\Psi}_{n\pm}{}^1(q'\alpha'), \quad (2.19)$$

where

$$\tilde{K}(q\alpha|q'\alpha') = \sum_{i=2}^{3} \tilde{K}_i(q\alpha|q'\alpha'). \quad (2.20)$$

Up to now we confined ourselves to the case that the particles did not have any internal degrees of freedom. Since we are concerned in this paper with pions, a word should be said about the influence of the isotopic spin. The generalization to this case is obvious. The three-particle states are now represented by $\langle \mathbf{k}_1\mathbf{k}_2\mathbf{k}_3 I_1 I_{1z} I_2 I_{2z} I_3 I_{3z}|$, where I_j, I_{jz} designate the isotopic spin of the jth particle and its third component. In writing out the Faddeev equations it is now useful to take for the representation in which for example particle one plays a special role, ${}_1\langle \mathbf{pq} I_1(I_2I_3)I_{23}II_z|$. The notation is clear. In these wave functions I_2 and I_3 are coupled together to form I_{23} which is in turn coupled to I_1 to form the total isotopic spin I. Since we are dealing with strong interactions, the isotopic spin dependence can be separated out easily. One finds

$$\tilde{\Psi}_{n\pm}{}^1(q\alpha I_{23}) = \tilde{\Phi}_{n\pm}{}^1(q\alpha I_{23}) - \sum_{\alpha', I_{23}'} \int_0^\infty dq' \times \tilde{K}(q\alpha I_{23}|q'\alpha' I_{23}')\tilde{\Psi}_{n\pm}{}^1(q'\alpha' I_{23}'), \quad (2.21)$$

where

$$\tilde{K}(q\alpha I_{23}|q'\alpha' I_{23}') = \langle I_1(I_2I_3)I_{23}II_z|(I_3I_1)I_{31}I_2II_z\rangle \times \tilde{K}^{I_{23}}(q\alpha|q'\alpha') \quad (\text{with } I_{31}\equiv I_{23}'). \quad (2.22)$$

Here $\tilde{K}$ is defined by (2.20). We have used an additional superscript I_{23} to emphasize that the two-body scattering matrix also depends on I_{23}. Furthermore, we have for convenience only written explicitly the dependence on I_{23}. The matrix elements $\langle\cdots|\cdots\rangle$ are directly related to the 6-j symbols (printed in curly brackets) according to

$$\langle I_1(I_2I_3)I_{23}II_z|(I_3I_1)I_{31}I_2II_z\rangle = (-1)^{I_1+I_2+I_3+I}[(2I_{31}+1)(2I_{23}+1)]^{1/2} \times \begin{Bmatrix} I_2 & I_3 & I_{23} \\ I_1 & I & I_{31} \end{Bmatrix}.$$

The condition for satisfying the statistics should also be modified slightly. Instead of (2.16) we now have

$${}_1\langle \mathbf{pq} I_1I_{z1}I_2I_{2z}I_3I_{3z}|T^1|n_\pm\rangle = \pm {}_1\langle -\mathbf{pq} I_1I_{z1}I_3I_{3z}I_2I_{2z}|T^2|n_\pm\rangle. \quad (2.23)$$

In some practical problems the Eq. (2.21) assumes a simpler form. For example, one could try to find a solution of the homogeneous Faddeev equations for a system of three pions in a state with the characteristics of the ω particle. In this problem the two-body scattering matrix can be taken to be dominated by the ρ resonance, i.e., $l=1$, $I_{23}=1$. Moreover, since $J=1$, we have $L=0$, 1, 2 of which only $L=1$ is allowed due to the odd parity of the three-particle state. With these the homogeneous part of Eq. (2.21) becomes

$$\Psi(q) = \int_0^\infty dq_2 K(q;q_2;s)\Psi(q_2), \quad (2.24)$$

where

$$K(q;q_2;s) = -\frac{12\pi}{q}\int_0^\infty dp_2 \int_{-1}^{+1} d\cos\theta_{q_2p_2}\delta(q^2-q_1^2) \times \hat{t}_1(p_2;p_1;s-q^2)\frac{p_2{}^2q_2{}^2}{p_2{}^2+q_2{}^2-s} \times \cos\theta_{p_1p_2}|\sin\theta_{q_1p_2}||\sin\theta_{q_2p_2}|. \quad (2.25)$$

It should be noted that the condition (2.23) is automatically satisfied since $l=1$ and $I_{23}=1$.

3. THE PION PROBLEM

In this section we look for a possible solution of the homogeneous Faddeev equations for a system of three pions in a state of zero total angular momentum, odd parity, and isospin one. The two-body amplitude which determines the kernel of the Faddeev equation is assumed to be dominated by the s-wave amplitude with even parity and isospin zero. As an approximation for this we use the Chew-Mandelstam effective-range formula.[13]

According to (2.21) and (2.22) the homogeneous Faddeev equation in this case can be written in the form

$$\Psi(q)=\int_0^\infty dq_2 K(q;q_2;s)\Psi(q_2)\,, \tag{3.1}$$

where the kernel K is given by

$$K(q;q_2;s)=-\frac{8\pi}{3q}\int_0^\infty dp_2\int_{-1}^{+1} d\cos\theta_{p_2q_2}\delta(q^2-q_1^2) \times\hat{t}_0(p_2;p_1;s-q^2)\frac{p_2^2q_2^2}{p_2^2+q_2^2-s}\,. \tag{3.2}$$

This is readily found by inserting (2.16) together with (2.12) and (2.13) in (2.21) and taking $J=M=l=L=l'=L'=0$. Furthermore, we note that the condition (2.23) is also satisfied for this case in view of $l=0$, $I_{23}=0$.

Although the Faddeev equations are by nature nonrelativistic, we can at least make use of relativistic kinematics. In Eq. (3.2) the parameter s represents the total kinetic energy of three particles in the over-all center-of-mass system. Instead of s we shall use the parameter z which is to include the rest energy of the three pions. Furthermore, in the nonrelativistic case $\nu=s-q^2$ represents the square of the relative momentum of the two particles. Instead of this relation, for its relativistic analog, we proceed as follows. The invariant energy in the two-body center-of-mass system (taking $m_\pi=1$) is given by $E_{23}=(4p^2+4)^{1/2}$. The momentum of the third particle in the over-all center-of-mass system is given according to Eq. (2.4) by $\mathbf{k}_1=-2\mathbf{q}/\sqrt{3}$. Thus the total energy in the three-particle center-of-mass system is given by

$$z=(k_1^2+1)^{1/2}+(k_1^2+4p^2+4)^{1/2} =(\tfrac{4}{3}q^2+1)^{1/2}+(\tfrac{4}{3}q^2+4p^2+4)^{1/2}\,. \tag{3.3}$$

From (3.3) in identifying $\nu=p^2$ one finds

$$\nu=\tfrac{1}{4}[(z-(\tfrac{4}{3}q^2+1)^{1/2})^2-\tfrac{4}{3}q^2-4]\,. \tag{3.4}$$

This is now taken to be the relativistic analog of $s-q^2$ in the argument of the two-body scattering matrix in Eq. (3.2). On the energy shell, the two-body scattering matrix is related to the invariant amplitude by

$$\hat{t}_0(p_2;p_1;\nu)=-(1/2\pi^2)A_0(\nu)\quad\text{with}\quad p_1^2=p_2^2=\nu\,. \tag{3.5}$$

The off-shell scattering matrix is then taken to be given by

$$\hat{t}_0(p_2;p_1;\nu)=-[g(p_1)g(p_2)/2\pi^2]A_0(\nu)\,, \tag{3.6}$$

with $g(0)=1$. We expect the form factor to behave as a constant near $p=0$ (threshold behavior for s wave) and go to zero for large p. Finally, we modify the free resolvent [the factor $1/(p_2^2+q_2^2-s)$ in Eq. (3.2)] to

$$\frac{1}{(k_1^2+1)^{1/2}+(k_2^2+1)^{1/2}+(k_3^2+1)^{1/2}-z}\,, \tag{3.7}$$

where $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ are given in terms of $\mathbf{p}_2$ and $\mathbf{q}_2$ in the previous section.

We have already mentioned that the two-body amplitude is approximated by the s-wave, isospin zero π-π effective-range formula. This formula was first written down by Chew and Mandelstam[13] from the N/D formalism and was later used by Booth and Abashian to fit the ABC phenomenon.[14] It has also been utilized by Scotti and Wong in the nucleon-nucleon problem.[15] Following Chew and Mandelstam we write

$$A_0(\nu)=N(\nu)/D(\nu)\,, \tag{3.8}$$

where we take $N(\nu)=a_I$ and

$$D(\nu)=1-\frac{a_I(\nu+1)}{\pi}\int_0^\infty\left(\frac{\nu'}{\nu'+1}\right)^{1/2} \times\frac{1}{(\nu'-\nu)(\nu'+1)}d\nu'\,. \tag{3.9}$$

Integrating this equation $D(\nu)$ is given by

$$\begin{aligned} D(\nu)&=1-\frac{2a_I}{\pi}+\frac{a_I}{\pi}\left(\frac{\nu}{\nu+1}\right)^{1/2} \times\{2\ln[\nu^{1/2}+(\nu+1)^{1/2}]-i\pi\}\,;\quad \nu>0\\ &=1-\frac{2a_I}{\pi}+\frac{2a_I}{\pi}\left(\frac{-\nu}{\nu+1}\right)^{1/2} \times\tan^{-1}\left(\frac{\nu+1}{-\nu}\right)^{1/2};\quad -1<\nu<0\\ &=1-\frac{2a_I}{\pi}+\frac{2a_I}{\pi}\left(\frac{\nu}{\nu+1}\right)^{1/2} \times\ln[(-\nu)^{1/2}+(-\nu-1)^{1/2}];\quad \nu<-1 \end{aligned} \tag{3.10}$$

and $D(-1)=1$. Equations (3.4) through (3.10) are utilized in Eq. (3.1) which now determines the kernel. We have thus defined our use of the relativistic kinematics as well as a prescription of using the effective-range formula off the energy shell. As for the form factors $g(p_1)$ and $g(p_2)$, we shall use them as

$$\begin{aligned} g(p)&=1\quad\text{if}\quad p\leq p_m\\ &=0\quad\text{if}\quad p>p_m\,, \end{aligned} \tag{3.11}$$

[13] G. F. Chew and Stanley Mandelstam, Phys. Rev. **119**, 476 (1960).

[14] Norman E. Booth and Alexander Abashian, Phys. Rev. **132**, 2314 (1963).

[15] A. Scotti and D. Y. Wong, Phys. Rev. Letters **10**, 142 (1963); also Phys. Rev. **138**, B145 (1965).

which means that we use a cutoff in the integrations over p_2. This cutoff obviously has a physical significance. We shall call p_m the cutoff parameter.

We are now in a position to discuss the solution of Eq. (3.1). The kernel in this equation is now a function of the parameters a_I and p_m as well as z. We now write Eq. (3.1) as

$$\Psi(q)=\int_0^\infty dq' K(q,q',z,p_m,a_0)\Psi(q'), \qquad (3.12)$$

where we have defined $a_0=a_I/(1-2a_I/\pi)$. The quantity a_0 is the conventional scattering length. The kernel is approximated by a finite $N\times N$ matrix by choosing finite mesh sizes in the integration. The number of steps N in the integration was taken to be 30. Varying this number to $N=60$ did not change the result. The problem amounts now to solve the resulting eigenvalue equation by appropriate variations of the parameters z, p_m, a_0, such that there is an eigenvalue one obtained. It turns out that this eigenvalue was the largest one in the region of the parameters we were considering.

Since we are interested in the question whether it is possible to find a three-particle bound state with the same mass of the pion, we have taken $z=1$ varying only p_m and a_0. The result is shown in Fig. 1. It was also found that for a given p_m, a_0 this was the only bound state. The position of the bound state, however, turned out to be very sensitive to these parameters.

The scattering length a_0 is determined by Booth and Abashian from the ABC experiment to be given by (2 ± 1). (Scotti and Wong[15] recently found from the nucleon-nucleon problem the value $a_0=2.7$.) Taking the value $a_0=2$, we find according to Fig. 1 that the cutoff parameter of the form factor should be $p_m=5.3$.

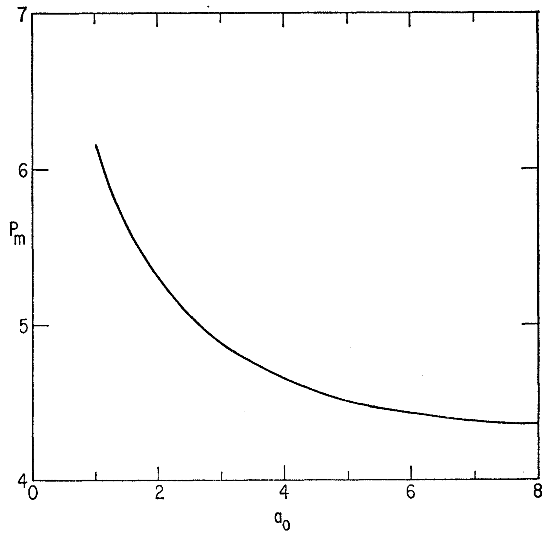

FIG. 1. The dependence of the cutoff parameter p_m on the scattering length a_0 for $z=1$.

4. DISCUSSION

In Sec. 2 we have made a new separation in angular momentum of the Faddeev equations. With this separation and the assumption that the two-body amplitude off the energy shell factorizes into terms containing the initial and final momenta we have been able to reduce the Faddeev equations to an integral equation in only one variable. Moreover, in the separation the statistics for identical particles can easily be taken into account. The method of separation adopted here differs from that of Omnes[8] in the choice of variables, and, depending on the particular problem under consideration, one or the other separation may be more convenient. The reduction to an integral equation in one variable is not, however, an exclusive feature of the separation used here. As shown by Basdevant,[11] once the two-body amplitude is assumed to factorize in terms consisting of the initial and final momenta, the Omnes separation of angular momentum also gives rise to integral equations in one variable. This simplification is of considerable practical importance and under this condition the numerical solution can be carried out with reliable accuracy.

In the previous section we have considered a model in which the pion is a bound state of a system of three pions in the Faddeev equations. We have thus utilized a notion often discussed by Chew, namely, that the strongly interacting particles are composite of one another. We have, however, neglected all other channels to which the pion is coupled. This approximation is quite analogous to that in the current bootstrap problems in which only the contribution of the nearby singularities are taken into account. Here it is the free resolvent as well as the two-body scattering matrix which suppress the effect of the more massive particles. The pion is, for example, also strongly coupled to the $N\bar{N}$ system. It would in fact be interesting to consider as a model for this the $\pi N\bar{N}$ system in which the two body amplitudes πN, $\pi\bar{N}$, and $N\bar{N}$ are approximated by appropriate bound states and resonances. A situation in which the eigenvalues of the Faddeev kernel corresponding to such a $\pi N\bar{N}$ model are considerably smaller than unity would be in support of our three-pion model. One would, of course, like to have a way of combining all these effects. Such a problem is, however, not a simple one in practice. Furthermore, in our three-pion model we have, for example, neglected the ρ contribution to the two-body amplitude since the π-ρ system is considerably more massive than the π-ABC system.

The Faddeev equations, although basically nonrelativistic, are here assumed to be applicable to strong interactions. We believe that such an approach is justified because these equations have at least a correct nonrelativistic foundation and offer a good starting point. Furthermore, although these equations are derived in potential theory, once the two-body amplitudes are given no further knowledge of the potentials is required. In our model for the pion we have at least made use of relativistic kinematics in order to make the treatment more reasonable.

A drawback in this sort of calculation is our lack of detailed information about the form factors. For simplicity, we have taken here the form factor to be a constant up to a certain value p_m and zero after that. The fact that our results strongly depend on the cutoff parameter p_m makes a study of form factors highly desirable. It should be stressed that this cut-off parameter is not merely a mathematical artifice, but that it can in principle be determined from the two-body interactions. We would like to make the passing remark that instead of Eq. (3.11) one can introduce a form factor of the type[16]

$$g(p)=\frac{1}{(1+p^2/\beta^2)^{1/2}}.$$

In so doing, for $a_0=2$ we obtain $\beta\approx 3.5$.

With the approximations mentioned above we may conclude that the pion can be understood as a bound state of three pions in the Faddeev equations. We intend to make a detailed study of the ω particle along the same lines.

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APPENDIX

In this appendix we shall indicate how to reduce (2.11) to (2.12). All the equalities we thereby shall use can be found in Ref. 12. One relation we need is

$$Y_m{}^l(\theta_{p_1},\varphi_{p_1})=\sum_{m'} \mathfrak{D}_{m'm}{}^l(\alpha,\theta_{p_2},\varphi_{p_2})Y_{m'}{}^l(\theta_{p_2p_1},0). \quad \text{(I)}$$

Here $\mathfrak{D}_{m'm}{}^l(\alpha,\theta_{p_2},\varphi_{p_2})$ are the well-known rotation matrices and $\theta_{p_2p_1}$ is the angle between $\mathbf{p}_2$ and $\mathbf{p}_1$. The relation (I) corresponds to a rotation from a coordinate system with the z axis parallel to $\mathbf{p}_2$ and the x axis in the plane through $\mathbf{p}_1$ and $\mathbf{p}_2$ to the space-fixed coordinate system. The same rotation is now being used for the four $Y_m{}^l$'s in (2.11). Moreover, in (2.11) we have to integrate over the angles θ_{p_2}, φ_{p_2}, θ_{q_2}, φ_{q_2}. Instead of these variables we may as well use α, θ_{p_2}, φ_{p_2}, and $\theta_{q_2p_2}$. So we find

$$K_2=\frac{4\pi^{1/2}(2l+1)^{1/2}}{q}\sum_{\substack{n_l,n_L\\ n_{l'},n_{L'}}} A_{n_l,n_L,n_{L'}} \times\int d\cos\theta_{q_2p_2}\delta(q^2-q_1^2)\hat{t}_l{}^1(p;p_1;s-q^2) \times Y_{n_l}{}^{l*}(\theta_{p_1p_2},0)Y_{n_L}{}^{L*}(\theta_{q_1p_2},0)Y_{n_{L'}}{}^{L'}(\theta_{q_2p_2},0), \quad \text{(II)}$$

with

$$A_{n_l,n_L,n_{L'}}=\sum_{\{m\}} C\begin{pmatrix} J & l & L\\ M & m_l & m_L\end{pmatrix} C\begin{pmatrix} J' & l' & L'\\ M' & m_{l'} & m_{L'}\end{pmatrix} \times\int d\omega\, \mathfrak{D}_{n_lm_l}{}^{l*}(\omega) \times\mathfrak{D}_{n_Lm_L}{}^{L*}(\omega)\mathfrak{D}_{0m_{l'}}{}^{l'}(\omega)\mathfrak{D}_{n_{L'}m_{L'}}{}^{L'}(\omega),$$

where we have denoted the angles α, θ_{p_2}, φ_{p_2} by ω and $d\omega=d\cos\theta_{p_2}d\alpha d\varphi_{p_2}$. The computation of the expression A can now readily be carried out. With the aid of the relation

$$\mathfrak{D}_{m_1'm_1}{}^{j_1}(\omega)\mathfrak{D}_{m_2'm_2}{}^{j_2}(\omega)=\sum_{j,m,m'}(2j+1)\begin{pmatrix} j_1 & j_2 & j\\ m_1' & m_2' & m'\end{pmatrix} \times\begin{pmatrix} j_1 & j_2 & j\\ m_1 & m_2 & m\end{pmatrix}\mathfrak{D}_{m'm}{}^{j*}(\omega),$$

A can be reduced to an expression in which there are only two rotation matrices involved. Using subsequently the orthogonality relations for the rotation matrices

$$\int d\omega\, \mathfrak{D}_{mn}{}^j(\omega)\mathfrak{D}_{m'n'}{}^{j'*}(\omega)=\frac{8\pi^2}{2j+1}\delta_{mm'}\delta_{nn'}\delta_{jj'}$$

and for the 3-j symbols

$$\sum_{m_1,m_2}\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3'\\ m_1 & m_2 & m_3'\end{pmatrix} =\frac{1}{2j_3+1}\delta_{j_3j_3'}\delta_{m_3m_3'}\delta(j_1j_2j_3)$$

we find for A as a result

$$A_{n_l,n_L,n_{L'}}=8\pi^2\delta_{JJ'}\delta_{MM'}(-1)^{l+l'-L-L'} \times\begin{pmatrix} l & L & J\\ n_l & n_L & -n_{L'}\end{pmatrix}\begin{pmatrix} l' & L' & J\\ 0 & n_{L'} & -n_{L'}\end{pmatrix}.$$

Inserting this in formula (II) gives at once the expression (2.12).

[16] See, for example, M. Bander, Phys. Rev. **138**, B322 (1965).