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Lorentz Invariance and Internal Symmetry*

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One of the outstanding problems of elementary-particle physics is the problem of combining Lorentz invariance and internal symmetry in a nontrivial way. In this paper we attack this problem by making a general investigation of the possibilities of finding a Lie algebra of *finite* order E such that the Lie algebras of the Lorentz and internal symmetry groups appear in it as subalgebras. We carry out this investigation by using some powerful standard results from the general theory of Lie algebras. The most relevant of these is Levi's result that any Lie algebra E is the semidirect product of a semisimple algebra G and an invariant solvable subalgebra S (called the radical). Using this result we show that if L, the algebra of the inhomogeneous Lorentz group, is a subalgebra of E, and M and P are the homogeneous and translation parts of L respectively, then either (a) M lies completely in G and P lies completely in S, or (b) L has no intersection with S. The relevance of this result is that it enables us to classify the ways in which L can be a subalgebra of E in a very simple way. The classification is carried out by subdividing case (a) into the three cases: (i) $S=P$, (ii) S Abelian but larger than, and containing P, and (iii) S solvable but not Abelian, and containing P; and by regarding case (b) as case (iv) $S\cap P=0$. Each of these four cases is considered in detail. It turns out that case (i) is essentially a direct sum of L and a semisimple Lie algebra; case (ii) is possible, but has the disadvantage of introducing a translation group of more than four dimensions; and case (iii) seems to be rather unphysical. Case (iv) is possible but is equivalent to imbedding L in a *simple* Lie algebra. The over-all picture which emerges is that while there are a number of ways in which L can be imbedded in an E, none of these (except the direct sum) seems to be particularly attractive from the physical point of view. In particular, it seems that, while it may be possible to make $SU(6)$ theory fully relativistic, it is probably not possible to do so within the context of a Lie algebra of finite order. [This does not contradict the $\tilde{U}(12)$ theory.] The question of explaining mass splitting within the context of a Lie algebra of finite order is considered, and it is shown that this cannot be done. The various negative-type theorems obtained by previous authors for special cases of E are rederived here within the general framework, most of them being derived from much weaker assumptions.

1. INTRODUCTION

RECENTLY, a large number of papers have appeared in which the problem of combining internal symmetry and Lorentz invariance has been discussed. Although this problem has always been of interest, attention has been focused on it in recent months for two reasons. The first concerns the mass splittings which are found experimentally to occur within the multiplets of particles and which are not explainable (except as symmetry-breaking phenomena) within the context of the internal symmetry groups. The hope is that they might be explainable within the context of a higher symmetry group E which would contain both the Lorentz and the internal symmetry group as subgroups. For example, if one lets T^+ denote the step-up operator for the isospin group, and P_μ the 4-momentum operator, then the commutator

$$[T^+,P_\mu] \tag{1.1}$$

which is zero (and hence precludes mass splitting) for the ordinary internal symmetry algebra, would not necessarily be zero in E.

The second reason for the recent interest in this problem is the success which has been achieved with the group $SU(6)$.[1] In this group, an internal symmetry group $SU(3)$ and a space-time group, the nonrelativistic spin group $SU(2)$, have been combined in a nontrivial way. It is natural then to imagine that this theory could be extended and the whole inhomogeneous Lorentz group and an internal symmetry group be combined in a large group E in a nontrivial way. Attempts to carry out the relativistic extension of $SU(6)$, have, however, run into serious difficulties, though at present it seems as if many of these difficulties can be overcome.[2] In addition to the attempts to make $SU(6)$ relativistic, some specific Lie groups containing the Lorentz group, L and internal symmetry have been proposed[3–6] [many of these proposals were made, in fact, before the advent of $SU(6)$]. However, none of the groups proposed has been entirely satisfactory. Furthermore, a number of negative theorems have appeared, in which it is proved that, under certain conditions, the combination of

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internal symmetry and Lorentz invariance is necessarily trivial[7–10] (i.e., a direct product combination). The prototype of such theorems is that due to McGlinn[7]:

McGlinn's Theorem. Let L be the Lie algebra of the inhomogeneous Lorentz group, M and P the homogeneous and translation parts of L, respectively, and T any semisimple internal symmetry algebra. If (a) E is a Lie algebra whose basis consists of the basis of L and the basis of T, and if (b)

$$[T,M]=0\,, \tag{1.2}$$

i.e., the internal symmetry is Lorentz-invariant, then

$$[T,P]=0\,, \tag{1.3}$$

i.e., the internal symmetry is translational-invariant. Hence

$$E=L\oplus T\,, \tag{1.4}$$

where $\oplus$ denotes direct sum.

Most of the later theorems have been concerned with weakening condition (b) of McGlinn's theorem, and perhaps the most refined result of this kind is that obtained independently by Michel and Sudarshan.[9] These authors showed that if only one element of T (say the charge operator) is Lorentz-invariant [i.e., satisfies (1.2)], then McGlinn's result follows (up to a redefinition). However, it is clear that it is assumption (a) of McGlinn's theorem which is the really restrictive one (just how restrictive will be seen in Sec. 7 of this article) and hence McGlinn's result is by no means general.

It is clear from the above discussion that most of the results which have been obtained so far, in connection with combining the Lorentz algebra L and an internal symmetry algebra T into a larger symmetry Lie algebra E, have been obtained for specific models or under spcific conditions. For this reason, we think it worthwhile to investigate in this paper the problem of determining the most general way in which L can be imbedded as a subalgebra in a larger Lie algebra E, assuming only that E is of finite order. To carry out this investigation, it is necessary to use some of the more powerful standard results concerning the structure of Lie algebras. These results are summarized briefly in the next section, the most important of them, for our purpose, being Levi's radical-splitting theorem, which states that every Lie algebra E of finite order is the semidirect product of a semisimple Lie algebra G and an invariant solvable subalgebra S. We use this result to show that if L is a subalgebra of E, there are only two possibilities: (a) P, the translation part of L is completely contained in S, the radical of E, or (b) L has no intersection whatsoever with S. The relevance of this result is that it affords us a convenient method of classifying the ways in which L can be imbedded in an enveloping Lie algebra E. To carry out the classification, it is convenient to subdivide case (a) into three classes; i.e., (i) $S=P$; (ii) S Abelian but larger than, and containing, P; (iii) S solvable but not Abelian, and containing P, and to write class (b) as class (iv), where (iv) $S\cap P=0$, where $\cap$ denotes intersection. In all cases, $M\cap S$ is zero, where M is the homogeneous part of L.

We then discuss each of the four classes listed in turn. The first class seems to be the most attractive from the physical point of view, but it is shown that (up to a redefinition) this case reduces to a direct sum of L and a semisimple algebra T. Case (ii) cannot be reduced to a direct sum in this way, but has the disadvantage of introducing a translation algebra of more than four dimensions. Case (iii) appears to be rather unphysical, and no algebra of this kind has been proposed so far. For case (iv), we find that (again up to a redefinition) this case is equivalent to imbedding the Lorentzian algebra as a subalgebra in a *simple* Lie algebra. This is not impossible, as is shown by an example, but the fact that the simple algebras are classified, means that we can examine the possibilities for this case systematically. The over-all picture which emerges is that while it is not impossible to imbed L in a larger algebra E, the ways in which this may be done are restricted and none of them (apart from the direct sum) seems to be particularly attractive from the physical point of view. It might be worth mentioning at this point that the outlook for Lie algebras of infinite order is not so bright either.[11]

Our results are, of course, obtained only modulo some redefinitions. We take the view here that such redefinitions are trivial. However we discuss the alternative point of view in Sec 8. It is assumed throughout, of course, that E is of finite order, and for Lie algebras of infinite order the situation may be quite different.

We come now to one of the most important questions which occurs in connection with combining Lorentz and internal symmetry, namely, the question of mass splitting. In this connection, we have already obtained the result,[12] that for a Lie algebra of finite order, no mass splitting is possible. This result is discussed in some detail in Sec. 6. Furthermore, as the result appears to be in contradiction to the results of some other papers in which mass splittings have been obtained or proposed, two of these papers[3,5] are examined in detail, and it is shown that there is, in fact, no contradiction.

The relation between the present work and the work in which the various negative-type theorems have been obtained is also discussed. It is shown that if one makes McGlinn's first assumption, namely, that the

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enveloping Lie algebra E consists only of the elements of L and of an internal semisimple algebra T, then (modulo same redefinitions, and assuming the linear independence of L and T) McGlinn's direct-sum result follows, without making *any* assumptions concerning the commutativity of M and T. In other words, as has sometimes been suspected, McGlinn's assumption (a) is already so restrictive as to preclude anything but a direct sum (or some redefinitions thereof). A theorem due to Michel and Sakita[13] is also rederived as a special case of our general results. Finally a link between the present work and the nonrelativistic $SU(6)$ and $SU(6)\otimes O_3$ theories[14] is found. However, no connection with "relativistic" $SU(6)$ is found. On the contrary, our results would seem to indicate, that, while it may be possible to make $SU(6)$ theory fully relativistic, it is probably not possible to do this within the context of a Lie algebra of finite order, which contains the *translational*, as well as the homogeneous, part of L. This does not contradict the results of Ref. 2.

Throughout the paper we confine ourselves to the study of Lie algebras, rather than Lie groups (or other topological groups). The algebra L of the inhomogeneous Lorentz group (Poincaré group) we define to be the algebra consisting of the homogeneous part M with commutation relations

$$[M_{\mu\nu},M_{\sigma\lambda}]=g_{\nu\sigma}M_{\mu\lambda}-g_{\mu\lambda}M_{\nu\sigma}-g_{\mu\sigma}M_{\nu\lambda}+g_{\nu\lambda}M_{\mu\sigma}\,, \tag{1.5}$$

μ, ν, σ, $\lambda=1\cdots 4$, and the translation part P with generators P_μ satisfying the relations

$$[M_{\mu\nu},P_\sigma]=g_{\nu\sigma}P_\mu-g_{\mu\sigma}P_\nu\,, \tag{1.6}$$

and

$$[P_\mu,P_\nu]=0\,.$$

2. SOME STANDARD MATHEMATICAL RESULTS

In this section we should like to introduce the standard result on Lie algebras which is the point of departure for the considerations of the present paper: This is the theorem of Levi. In order to introduce it, however, it might be worthwhile to define first the concept of a *solvable* Lie algebra. This concept is introduced as follows: Let E denote a Lie algebra and also any element of it, and consider the totality of elements of E of the form

$$E^{(1)}=[E,E]\,. \tag{2.1}$$

It is easy to see that the set of elements $E^{(1)}$ form not only a subalgebra of E, but an *invariant* subalgebra. This invariant subalgebra, which we shall denote by $E^{(1)}$, is called the *first-derived algebra* of E. In general, $E^{(1)}$ is smaller than E. The extreme cases are $E^{(1)}=0$ and $E^{(1)}=E$. The first occurs if, and only if, E is Abelian. The second occurs for semisimple algebras (as can easily be checked by inspection of the Cartan canonical form), but it also occurs for a wider class of algebras, e.g., it occurs for L, the Lie algebra of the inhomogeneous Lorentz group. The algebra

$$E^{(2)}=[E^{(1)},E^{(1)}]\,, \tag{2.2}$$

which is the first-derived algebra of $E^{(1)}$, is called the *second-derived algebra* of E. By using the Jacobi identity one can show that it is not only an invariant subalgebra of $E^{(1)}$, but also an invariant subalgebra of E. Continuing in this way, we can define the kth derivative algebra of E to be

$$E^{(k)}=[E^{(k-1)},E^{(k-1)}]\,, \tag{2.3}$$

and this is an invariant subalgebra of $E^{(r)}$, $r=0,\cdots,(k-1)$. A Lie algebra E is said to be *solvable* if, for some integer k,

$$E^{(k)}=0\,. \tag{2.4}$$

We are now in a position to introduce the standard result mentioned above (see Jacobson,[15] p. 91).

Levi's theorem. Every Lie algebra E can be written in the form

$$E=G \mathbin{+\!\!\!\!\supset} S\,, \tag{2.5}$$

where G is a semisimple subalgebra of E, S is an invariant solvable subalgebra of E, and $\mathbin{+\!\!\!\!\supset}$ denotes semidirect sum.

The semisimple subalgebra G is called the *Levi factor* of E, and the invariant solvable subalgebra S is called the *radical*. From the semisimplicity of G (Ref. 15, p. 24) it can easily be shown that S contains every invariant solvable subalgebra of E.

An important question for our later consideration is that of the uniqueness of G in (2.5). (The uniqueness of S is guaranteed by its invariance.) Clearly G is not completely unique, since the inner automorphisms

$$E\to E'=\exp(\tilde{E})E\exp(-\tilde{E}) \tag{2.6}$$

induced by elements $\tilde{E}$ which involve S, do not, in general, leave G invariant. Thus the question really is: Is G unique up to such inner automorphisms? The answer to this question is "yes." This is shown by the Malcev-Harish-Chandra theorem (Ref. 15, p. 92) which proves the following more general result: Let G_1 be any semisimple subalgebra of E. Then there exists an inner automorphism $\tilde{E}$ such that

$$G_1'=\exp(\tilde{E})G_1\exp(-\tilde{E}) \tag{2.7}$$

is a subalgebra of G in (2.5).

3. CLASSIFICATION THEOREM

In this section we shall use the standard results just mentioned to establish the theorem stated below. This theorem will enable us to make a classification of the

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possibilities for combining Lorentz invariance and internal symmetry, and to discuss these possibilities in a systematic way.[16]

Theorem. Let L be the Lie algebra of the inhomogeneous Lorentz group, consisting of the homogeneous part M and translation part P. Let E be any Lie algebra, with radical S and Levi factor G. If L is a subalgebra of E, then either

$$\text{(a)}\quad M\subset G;\qquad P\subset S,$$

or

$$\text{(b)}\quad L\cap S=0.$$

Here $\subset$ denotes "is a subalgebra of," and $\cap$ denotes intersection.

Proof. Since M is semisimple, it follows from the Malcev–Harish-Chandra theorem that G can be defined so that $M\subset G$. From the invariance of S and the Lorentz relation

$$[M,P]=P, \tag{3.1}$$

we have

$$[M,S\cap P]=S\cap P. \tag{3.2}$$

Thus, with respect to M, $P\cap S$ is an invariant subspace of the space P. But P is irreducible with respect to M. Hence

$$P\cap S=0 \quad\text{or}\quad P\cap S=P. \tag{3.3}$$

From this the theorem follows.

The relevance of this theorem is that, as mentioned above, it enables us to classify the ways in which L may be a subalgebra of a larger algebra E. We make this classification by subdividing the case (a) above into the following three cases: (i) $S=P$; (ii) S Abelian but larger than, and containing, P; (iii) S solvable but not Abelian, and containing P. If we add to these case (b), which may be written (iv) $P\cap S=0$, we see that we have, with this classification, four possible cases. These will be discussed in Sec. 5. To facilitate that discussion, some preliminary theorems are proved in the next section.

4. SOME PRELIMINARY THEOREMS

In order to facilitate the discussion of the next section, it is convenient to establish here some theorems which are of a rather technical nature. They arise in the following way: Let E be any algebra with Levi factor G and radical S, and U any Abelian subalgebra of S (including S itself) which is invariant with respect to G, i.e.,

$$[G,U]=U. \tag{4.1}$$

In this way, U forms a representation space for G. The question arises as to the reducibility of U with respect to G, and further, as to the reducibility of U with respect to the *simple* algebras $G^{(a)}$ in the direct sum expansion

$$G=\sum_a \oplus G^{(a)} \tag{4.2}$$

of G. It is to handle these questions that the following theorems are proved.

Theorem A. If U_1 is a 1-dimensional invariant subspace of U with respect to any $G^{(a)}$ of (4.2), i.e., if

$$[G^{(a)},U_1]=U_1, \tag{4.3}$$

then

$$[G^{(a)},U_1]=0. \tag{4.4}$$

Proof. From (4.3) we have

$$[[G^{(a)},G^{(a)}],U_1]=0. \tag{4.5}$$

That is to say, any commutator in $G^{(a)}$ commutes with U_1. But $G^{(a)}$ is simple. Hence (Sec. 2) its first derived algebra is equal to itself. Hence every element of $G^{(a)}$ can be expressed as a commutator. Hence (4.5) implies (4.4). Q.E.D.

Theorem B. If $D^{(b)}$ is the set of matrices representing the linear transformations induced on U by the elements of $G^{(b)}$ in (4.2), then

$$[D^{(b)},D^{(c)}]=0, \tag{4.6}$$

for *all* elements in $G^{(c)}$ and $G^{(b)}$.

Proof. Since G is a direct sum, we have

$$[G^{(b)},G^{(c)}]=0. \tag{4.7}$$

Hence from the Jacobi identity, we have

$$\begin{aligned}0&=[[G^{(b)}G^{(c)}]U]\\&=[G^{(c)}[G^{(b)}U]]-[G^{(b)}[G^{(c)},U]]\\&=[G^{(c)},D^{(b)}u]-[G^{(b)},D^{(c)}U]\\&=D^{(b)}D^{(c)}U-D^{(c)}D^{(b)}U.\quad\text{Q.E.D.}\end{aligned} \tag{4.8}$$

Theorem C. For any $G^{(c)}$, let $D^{(c)}$ which is completely reducible (Ref. 15, p. 79) be written as a direct sum of irreducible representations $D_q^{(c)}$, $q=1\cdots m$. If any $D_q^{(c)}$ occurs only once in the reduction, then the subspace U_q on which $D_q^{(c)}$ operates, is an invariant subspace of U with respect to G.

Proof. U_q is obviously invariant with respect to $G^{(c)}$. But from theorem B, and Schur's Lemma (in the form

$$SM=M'S\rightarrow S=0, \tag{4.9}$$

for M and M' inequivalent) U_q is also invariant with respect to $D^{(b)}$, and hence $G^{(b)}$, $b\neq c$. Hence, U_q is invariant with respect to all $G^{(a)}$. Q.E.D.

We now specialize to the case where P is an invariant subalgebra of the algebra E of Sec. 3. This includes in the first place, the case (i) of the classification made above (i.e., $S=P$), but includes also cases (ii) and (iii) with the restriction that P be invariant. We now prove:

Theorem D. If P is an invariant subalgebra of E, then the Levi factor G of E can be written in the form

$$G=G_0\oplus G_r, \tag{4.10}$$

where $\oplus$ denotes direct sum, the complex extension $\tilde{G}_0$ of G_0 is

$$\tilde{G}_0=A_3 \quad\text{or}\quad B_2 \quad\text{or}\quad A_1\oplus A_1, \tag{4.11}$$

[16] L. O'Raifeartaigh, Phys. Rev. Letters **14**, 332 (1965).

(in the Cartan notation) (Ref. 15, p. 146) and G_r (r=remainder) is a semisimple algebra, satisfying the relation

$$[G_r,P]=0. \tag{4.12}$$

Proof. From the invariance of P we have

$$[G,P]=P. \tag{4.13}$$

Hence, for the $G^{(a)}$ of (4.2), we have

$$[G^{(a)},P]=P. \tag{4.14}$$

If, with respect to every $G^{(a)}$, P reduces to a set of four 1-dimensional spaces, then by theorem A, we have

$$[G^{(a)},P]=0, \tag{4.15}$$

for all a, whence

$$[G,P]=0. \tag{4.16}$$

This is clearly incompatible with the fact that G contains as a subalgebra the homogeneous Lorentz algebra M, with respect to which P transforms irreducibly. Hence there exists at least one $G^{(a)}$, $G^{(1)}$ say, with respect to which P contains an irreducible subspace P_d of more than one dimension. There are now four possibilities (i) P_d is 4 dimensional ($P_d=P$). (ii) P_d is 3 dimensional, and (iii) P_d is 2 dimensional, and $G^{(1)}$ (which is simple and therefore fully reducible on P), induces an inequivalent representation on the complementary 2-space $P-P_d$. (iv) P_d is 2 dimensional, and $G^{(1)}$ induces an equivalent representation on $P-P_d$. Cases (ii) and (iii), however, are ruled out by theorem C, since in these cases P_d would be an invariant subspace of P with respect to G, but since G contains M, there are no invariant subspaces of P with respect to G, other than P and O.

In case (i), $G^{(1)}$ is a simple algebra with a 4 dimensional irreducible representation. Hence, from the general classification of simple Lie algebras and their finite representations, we have for the complex extension $\tilde{G}^{(1)}$ of $G^{(1)}$

$$\tilde{G}^{(1)}=A_3 \quad \text{or} \quad B_2(=C_2) \quad \text{or} \quad A_1. \tag{4.17}$$

On the other hand, since $D^{(1)}$ is irreducible, we have from Schur's lemma (in the form

$$SM=MS\rightarrow S=kI, \tag{4.18}$$

where k is a number and I the unit matrix),

$$D^{(a)}=kI, \quad a\neq 1, \tag{4.19}$$

whence from theorem A,

$$[G^{(a)},P]=0, \quad a\neq 1. \tag{4.20}$$

Hence in case (i) we may take

$$G_0=G^{(1)}, \quad G_r=\sum_{a\neq 1}\oplus G^{(a)}{}_1, \tag{4.21}$$

where $G^{(1)}$ is A_3 or B_2, since A_1 is ruled out by the fact that G must contain M.

In case (iv), the situation is a little more complicated. By a change of basis in P we can arrange that the representations of $G^{(1)}$ in the 2-space P_d and $P-P_d$ are not merely equivalent but equal, and then using theorem B and Schur's lemma [in the form (4.18)] we have

$$D^{(a)}=\begin{bmatrix}\alpha I & \beta I\\ \gamma I & \delta I\end{bmatrix}, \quad a\neq 1, \tag{4.22}$$

where I is the unit 2×2 matrix. The matrices (4.22) cannot be reducible for every $a\neq 1$, since any one which is reducible is diagonal, and hence zero, by theorem A. Hence if they are all reducible, P_d and $P-P_d$ are invariant with respect to all $G^{(a)}$, hence with respect to G, and hence with respect to M, which is impossible. Hence there exists at least one a, $a=2$, say such that $D^{(2)}$ is irreducible. But then, from theorem B and Schur's lemma [in the form (4.18)] we have

$$D^{(a)}=kI, \quad a\neq 1,2, \tag{4.23}$$

whence from theorem A,

$$[G^{(a)},P]=0, \quad a\neq 1,2. \tag{4.24}$$

On the other hand, $G^{(1)}$ and $G^{(2)}$ are simple Lie algebras with irreducible 2 dimensional representations. Hence, from the general classification mentioned above, we have for their complex extensions $\tilde{G}^{(1)}$ and $\tilde{G}^{(2)}$

$$\tilde{G}^{(1)}=\tilde{G}^{(2)}=A_1. \tag{4.25}$$

Hence, in case (iv), we may take

$$G_0=G^{(1)}\oplus G^{(2)}, \quad G_r=\sum_{a\neq 1,2}\oplus G^{(a)}. \tag{4.26}$$

Combining the two cases (i) and (iv) we obtain the required result.

5. GENERAL DISCUSSION

In Sec. 3, we made a classification of the ways in in which the Lie algebra L of the inhomogeneous Lorentz group could be imbedded as a subalgebra of a larger Lie algebra E. In this section we should like to discuss each of the four ways which we classified in turn.

Case (i): $S=P$

In this case, the translation algebra P is an invariant subalgebra of G (and is, in fact, the only Abelian invariant subalgebra of G). Hence this case would seem to be the most attractive from the physical point of view. However, we shall now show that this case can amount to no more than a direct sum.

To show this, we note that this is a special case of the case considered in theorem D of Sec. 4. Since G can therefore be expanded as in (4.2), we can expand M in the form

$$M=M_{G0}+M_{Gr}. \tag{5.1}$$

Clearly the algebras M_{G0} and M_{Gr} are homorphic to M. But since, from (4.12)

$$[M_{Gr},P]=0, \tag{5.2}$$

we see that the Lorentz transformations

$$[M,P]=P\,, \tag{5.3}$$

induced by M, are, in fact, induced by M_{G0} alone. Thus M_{G0} must be isomorphic to M. Since (M_{G0},P) has therefore all the properties of $L=(M,P)$, and M_{Gr} is trivial on account of (5.2), it is only a matter of redefinition[17] to set

$$M_{Gr}=0\,. \tag{5.4}$$

Thus (modulo such a redefinition) we can say that M is contained in G_0 of (4.10). If we how have an internal symmetry algebra T, then since the vector spaces $P=S$ and M of G_0 are "already occupied" by the Lorentz algebra L,T can consist only of two parts i.e., T_{G0}= remainder of G_0 when M is removed, and $T_{Gr}=G_r$. But since

$$\tilde{G}_0=A_3 \quad \text{or} \quad B_2 \quad \text{or} \quad A_1\oplus A_1\,, \tag{5.5}$$

it is clear that T_{G0} is, from the physical point of view, an embarrassment rather than a help, unless we choose

$$T_{G0}=0\colon\;\; \tilde{G}_0=A_1\oplus A_1\colon\;\; G_0=M\,, \tag{5.6}$$

and in this case we have just

$$E=L\oplus T \tag{5.7}$$

where $\oplus$ denotes direct sum. Thus case (i) amounts to no more than a direct sum.

Case (ii): *S* Abelian, but Larger Than and Containing *P*

In this case S is again a representation space for G. However, the choice of simple algebras in the expansion of G which do not commute completely with S will be restricted only by the dimension of S, and this is variable. S itself may, of course, consist of several irreducible invariant substances S_q, one of which will contain P. The 1-dimensional subspaces commute with everything (from theorem A of Sec 4) and are therefore the generators of the gauge-transformations, such as that corresponding to baryon number conservation. The S_q of more than 1 dimension, and not containing P, seem to have no particular physical significance. The important question is whether the *irreducible* S_q containing P is larger than P. If not, then we can simply omit the other S_q and we are back at case (i). Thus, by case (ii) we mean essentially the case where S is Abelian, larger than P, and *irreducible* with respect to G. This case has been proposed in Refs. 3 and 6. Its chief disadvantage is that it introduces an invariant translational algebra of more than four dimensions. This is not easy to interpret physically. Furthermore, we note that since all the elements of S commute with those of P, they represent internal quantum numbers which can be measured simultaneously with momentum and energy, and the question is: What are these numbers physically? Moreover, unless the spectra of these operators are limited in some artificial way (i.e., in some way extraneous to the Lie algebra, as is done by Barut[18] for example) they will be continuous. This is hardly what we expect for an internal variable. Thus, case (ii) cannot be ruled out, but it is also not particularly attractive.

Case (iii): *S* Solvable but not Abelian, and Containing *P*

This case is very similar to cases (i) and (ii). It does not seem to have any particular advantage over those cases. Moreover, solvable non-Abelian algebras are not usually considered in physics, and, in fact, no such algebra has been proposed in connection with higher symmetry so far. One of the reasons for this is the result due to Lie (Ref. 14, p. 50) that every finite dimensional representation of such an algebra is triangular, i.e., has a basis such that every matrix in the representation has zero elements above the diagonal. This means that, except for the trivial Abelian representations, for finite-dimensional representations Hermitian conjugation cannot be defined.

Case (iv): $P\cap S=0$

In this case we see that no element of the translation algebra P lies completely in S. This does not mean that P lies completely in the Levi- factor G, but only that if we expand P in the form

$$P=P_{\mathrm{G}}+P_S \tag{5.8}$$

then for each P, P_{G} is nonzero. On the other hand, it does mean that P_{G} is isomorphic to P and transforms under M (which is contained in G) in the same way as P. Hence it is only a matter of redefinition[17] to take P_{G} rather than P as the translation algebra, in which case we have,

$$L\subset G\,. \tag{5.9}$$

Let us now expand G in a direct sum of semisimple algebras

$$G=\sum_a\oplus G^{(a)}\,. \tag{5.10}$$

In an obvious notation, we then have

$$L=\sum_a\oplus L^{(a)}\,. \tag{5.11}$$

It is clear that each $L^{(a)}$ is homomorphic to L and, since L cannot be expressed as a direct sum of two algebras, that at least one $L^{(a)}$, $L^{(1)}$ say, is isomorphic to L. By a further redefinition, we can take $L^{(1)}$ and not L to be a Lorentz algebra in which case we have the result: Case (iv) is equivalent to imbedding the Lorentz algebra L as a subalgebra in a simple Lie algebra. (If we do not allow the redefinitions made

[17] The redefinition problem is discussed separately in Sec. 8.

[18] A. O. Barut, University of Colorado Report (unpublished).

above, we can make the weaker statement: case (iv) is equivalent to imbedding an algebra isomorphic to the Lorentz algebra L in a simple Lie algebra.]

At first sight it would appear to be impossible to imbed L (or an algebra isomorphic to L) in this way, as L is not even semisimple. However, we can easily construct an example: Let M_{ab} be the generators of the real orthogonal group in six dimensions. As a basis for M choose M_{rs} and $M_{r0}=iM_{r4}$, $r, s, =1, 2, 3$, and as a basis for P, $P_\mu=M_{\mu5}+iM_{\mu6}$, $\mu=1, 2, 3, 4$. It is easy to verify that the M and P so chosen satisfy the relations (1.2) of the inhomogeneous Lorentz algebra. (Note that the relation

$$[P_\mu,P_\nu]=0\,, \tag{5.12}$$

depends critically on the i in $M_{\mu s}+iM_{\mu6}$). Less trivial examples, containing internal symmetry, can also be constructed.[19]

On the other hand, although case (iv) is therefore possible, the fact that the simple algebras are classified (Ref. 15, p. 146) provides us with a systematic way of determining the various simple algebras in which L can be imbedded.

There is one feature of case (iv), however, which should be mentioned, as it may have serious consequences for the physical interpretation. This is the fact that the generators P_μ are necessarily *nonreal* linear combinations of the generators of the compact form of the simple algebra in which L is imbedded. This we have already seen in the example given above, and that it holds, in general, is shown in the Appendix. This means that the parameters corresponding to the P_μ have a noncompact range. This may lead to serious difficulties in defining multiplets, even in the absence of mass splitting (see next section). No algebra belonging to case (iv) has been proposed in the literature cited.

This completes our general discussion of cases (i)–(iv). It appears that none of the four cases is particularly attractive, except for the direct product, though case (iv) deserves some further investigation. This negative conclusion is not, of course, in contradiction with the "relativistic $SU(6)$" results of Ref. 2, since we are considering here (a) exact symmetry and (b) the *inhomogenous* Lorentz group.

6. MULTIPLETS AND MASS SPLITTING

As mentioned in the Introduction, one of the reasons for attempting to combine Lorentz invariance and internal symmetry in a higher symmetry algebra is the hope that the observed mass differences of the elementary particles might be explained within the context of the higher symmetry. In this section we wish to discuss the possibility of explaining the mass differences in this way.

One of the difficulties confronting any attempt to explain the mass differences in this way is the difficulty of defining what is meant by a *multiplet* of particles. For a direct sum algebra, $L\oplus T$, where T is semisimple and compact, there is no difficulty. Each particle is represented by the direct product of a vector in a finite representation of T with a vector space which is a one-particle representation space of L. For the more complicated case when L and T are combined in a nontrivial way, the situation is not so simple. However, it seems reasonable to make, at any rate, the following three assumptions:

(a) Two particles belonging to the same physical multiplet should be represented by vectors belonging to the same irreducible representation of the combined algebra E.

(b) Each such vector should be an eigenvector of the mass operator $P_\mu P^\mu$ corresponding to a discrete eigenvalue.

(c) $P_\mu P^\mu$, which is an observable, should be self-adjoint.

The assumptions (a), (b), and (c) are not, of course, *sufficient* to define what is meant by a multiplet. However, what one can now show, is that these three assumptions are already enough to preclude mass-splitting among the particles belonging to the same multiplet. More specifically, one can establish the following theorem:

Theorem: Let L be the Lie algebra of the inhomogeneous Lorentz group, E any Lie algebra containing L, H a Hilbert space on which any irreducible representation of the group generated by E operates. If, on H, the mass operator

$$P^2=P_\mu P^\mu \tag{6.1}$$

has a discrete eigenvalue m^2, and P^2 is self-adjoint on H, then the eigenspace H_m belonging to the eigenvalue m^2 of P^2, is closed, and is invariant with respect to the elements representing E on H. Hence the elements representing E can produce no mass-splitting. The proof of this theorem has been given in Ref. 11. Here we shall merely discuss its implications.

If we now make the assumption that H, which is irreducible with respect to the group generated by E, is irreducible with respect to the operators representing E, or the local group of transformations generated by these operators, then we see that the theorem implies that

$$H=H_m. \tag{6.2}$$

We see that what this theorem proves essentially is that, on H, the mass operator has either a continuous spectrum or a spectrum consisting of one point. But a continuous spectrum cannot correspond to a multiplet. Hence what the theorem shows, is that, under assumptions (a), (b), and (c), the only multiplets which are possible are equal mass multiplets. This does not imply that for any given algebra E, equal mass multiplets exist. It may well be that even the equal mass

[19] D. L. Pursey (private communication).

multiplets are not possible (except in the direct sum case mentioned above). However, this is another question and will not be discussed here.

Since, in fact, mass splittings do occur in nature, we must now discuss the question as to which of the assumptions we make, in order to obtain the result that there can be no mass splitting, is incorrect.

Of the assumptions (a), (b), and (c), (a) appears to be the strongest. (b) and (c) are, of course, only idealizations, since in practice, on account of the decays of the particles, m^2 is a bump, rather than a discrete point, in the mass spectrum, and P^2 contains a small non-self-adjoint part corresponding to the line breadth. Perhaps it is this idealization which is incorrect. It is much more likely, however, that it is the underlying hypothesis, namely, that one should imbed the inhomogeneous Lorentz algebra in a larger Lie algebra,[20] and in particular in a Lie algebra of finite order, which is incompatible with mass splitting. This is the conclusion which we prefer to draw.

7. CONNECTION WITH OTHER RESULTS

In this section we should like to consider the connection between the results obtained here, and those obtained by other authors.

We begin by considering the special case where the enveloping Lie algebra E (considered as a vector space) consists only of the elements of L and the elements of some semisimple internal symmetry algebra T, i.e.,

$$E=M+P+T. \qquad (7.1)$$

This is the case considered by McGlinn and in Refs. 8, 9, and 10. In this case, M lies in the Levi factor G of E, while T, because it is semisimple, satisfies the relation

$$T\cap S=0, \qquad (7.2)$$

where S is the radical. Hence we can redefine T so that T lies in G. From the general classification theorem of Sec. 3, we have two possibilities for P,

$$S\cap P=0 \quad \text{and} \quad P\subset S. \qquad (7.3)$$

In the first case, we can redefine L so that it is a subalgebra of a simple algebra E. But then the simple algebra E contains as subalgebras, L and the semisimple algebra T, which we assume to be linearly independent of L. This is obviously incompatible with (7.1) (the "remainder" of a simple algebra when L is removed is not semisimple). Hence the first case, $S\cap P=0$ is ruled out.

In the second case, M and T are both contained in G. Hence from (7.1) (again assuming that L and T are linearly independent), S can contain only P. Thus we are dealing with case (i) of our general classification. But we have already seen in Sec. 5 that this case reduces essentially to the direct sum of L and T. Hence we have the result: The condition (7.1) with T semisimple, is already enough to reduce E to

$$E=L\oplus T. \qquad (7.4)$$

Thus from McGlinn's first assumption alone we obtain the result obtained by him and by the authors of Refs. 8, 9, and 10. No assumptions concerning the commutativity of the elements of M and T are necessary. They are replaced by the much weaker assumptions that L and T be linearly independent and that certain redefinitions are legitimate.

We next consider the theorem of Michel and Sakita.[12] In this theorem it is assumed that (a) P is an invariant subalgebra[21] of an enveloping Lie algebra E, and (b) E "preserves the Minkowski metric," and from these assumptions alone, the direct-sum relation

$$G=M\oplus T, \qquad (7.5)$$

where G is the Levi factor of E and T is a semisimple algebra, is deduced.

It is easy to see that this result follows also from our general considerations. For if we make assumption (a) above, then the general discussion of Sec. 5 for $S=P$ carries through unchanged, and we have

$$G=G_0\oplus T \qquad (7.6)$$

where

$$\tilde{G}_0=A_3 \quad \text{or} \quad B_2 \quad \text{or} \quad A_1\oplus A_1. \qquad (7.7)$$

(In fact, theorem D of Sec. 4 was stated in just such a way that the discussion for $S=P$ could be generalized immediately to the case P invariant.) Assumption (b) then rules out A_3 and B_2, leaving (7.5).

Note that from theorem B of Sec. 4, we have, in addition, the McGlinn type result

$$[T,P]=0. \qquad (7.8)$$

This is not surprising because, in fact, theorems A and B of Sec. 4 are a generalization of McGlinn's theorem.

In view of the strong negative result concerning mass splitting obtained in the last section, it might be well to discuss some papers in which mass splitting has been obtained or proposed. One of these is discussed in detail in the next section, in connection with the redefinition problem. Here we shall discuss another, due to Barut.[3] In the latter article the enveloping algebra E is taken to be the algebra of the inhomogeneous six-dimensional orthogonal group, with the commutation relations

$$\begin{aligned}[][M_{ab},M_{cd}]&=\delta_{bc}M_{ad}-\delta_{ad}M_{bc}-\delta_{ac}M_{bd}+\delta_{bd}M_{ac},\\ [M_{ab},P_c]&=\delta_{bc}P_a-\delta_{ac}P_b,\\ [P_a,P_b]&=0.\end{aligned} \qquad (7.9)$$

[20] This includes the assumption that we are dealing with a Lie algebra in the first place, or the assumption made above that H is an irreducible representation of the algebra if it is an irreducible representation of the group. It is possible that this is not true, and that the situation is better described by the global properties of a Lie group (or some other kind of group). I am grateful to Professor G. F. Dell'Antonio for pointing out this possibility.

[21] It should be pointed out that our considerations are limited to Lie algebras, whereas Michel and Sakita discuss the more general case of connected topological groups.

The $M_{\mu\nu}$ and P_μ, μ, $\nu=1\cdots4$ (with appropriate i factors) are identified with the generators of L, and M_{45}, M_{56}, and M_{64} with those of the isospin algebra. (Clearly M_{56} must be the "charge" operator since it is the only one of the three which commutes with $M_{\mu\nu}$ i.e., it is Lorentz-invariant). The "momentum-like" invariant of this algebra is clearly not $P_\mu P^\mu$ but

$$P_a P_a = P_\mu P^\mu + P_5^2 + P_6^2\,, \tag{7.10}$$

which suggests that $P_\mu P^\mu$ might have different values within an irreducible representation of E. However, if we let $|p\rangle$, $|n\rangle$ represent a 1-proton and 1-neutron state, respectively, and assume that they are eigenstates belonging to discrete points in the spectrum of $p_\mu p^\mu$, then, in some one Lorentz frame at any rate, we should have

$$|p\rangle = T^+|n\rangle\,, \tag{7.11}$$

where T^+, the "step-up" operator for isospin, is the linear combination

$$T^+ = M_{45} + iM_{64}\,. \tag{7.12}$$

Otherwise the statement that M_{45}, M_{56}, and M_{65} are to be identified with the generators or isotopic spin has not much content. But then, from the invariance of P we have

$$\begin{aligned}(P_\mu P^\mu - m_n^2)^2|p\rangle &= (P_\mu P^\mu - m_n^2)T^+|n\rangle \\ &= [P_\mu P^\mu[P_\mu P^\mu, M_{45}+iM_{64}]]|n\rangle \\ &= [P_\mu P^\mu, 2P_4(-P_5+iP_6)]|n\rangle = 0\,,\end{aligned} \tag{7.13}$$

where m_n is the mass of the neutron, whence

$$m_n^2 = m_p^2\,, \tag{7.14}$$

where m_p is the mass of the proton. Hence for the nucleon system, there is, in spite of (7.10), no mass splitting. Similarly for any isotopic multiplet. We see, therefore, that in this example, which belongs to case (ii) of our general classification, there is no contradiction with our theorem.

Finally, although it is somewhat irrelevant to the general purpose of this paper, we should like, for completeness, to mention an extension of a theorem due to Han.[10]

Extended Han theorem: Let L be a subalgebra of any Lie algebra E. If E_0 is any element of E, and

$$\omega^2 = (\epsilon_{\mu\nu\lambda\sigma} M_{\nu\lambda} P_\sigma)^2 / P^2\,, \tag{7.15}$$

where $\epsilon_{\mu\nu\lambda\sigma}$ is the Levi-Civita symbol, then

$$[E_0, \omega^2] = 0 \tag{7.16}$$

implies

$$[E_0, P^2] = \alpha P^2\,, \tag{7.17}$$

where α is a constant, and conversely.

We shall not present the proof of this theorem here, as it is rather similar to that given by Han, depending, in particular, on the use of the Poincaré-Birkhoff-Witt theorem (Ref. 15, p. 159). From a slight extension of theorem A of Sec. 4, we also have the corollary: If E_0 is a whole semisimple subalgebra of E, rather than one element, $\alpha=0$.

8. THE REDEFINITION PROBLEM

The results which we obtained in Secs. 5 and 7 depended, in certain cases, as we have seen, on making a redefinition of the Lorentz algebra L, or of the internal symmetry algebra T. (We must emphasize, however, that the mass-splitting theorem of Sec. 6 is independent of any such redefinition.) In connection with the redefinitions, there are two questions which arise:

(1) Are the results obtained "the best possible" or could one go further and obtain "redefinition-independent" results?

(2) Are the redefinitions, which are trivial mathematically, also trivial physically?

The answer to the first question is that the results obtained are, in fact, the best possible which can be obtained from general arguments concerning the structure of E, as used in this paper. To illustrate this, we consider the following example: In case (iv) of our classification, we showed that, modulo a redefinition, L could be imbedded as a subalgebra of a *simple* algebra E. Suppose, however, we *start* from the situation where L, with no redefinition, is a subalgebra of a simple algebra E, which, in turn, is a subalgebra of a direct-sum algebra $\tilde{E}$ of E and E',

$$\tilde{E} = E \oplus E' \tag{8.1}$$

where E' contains a subalgebra L' isomorphic to L. Then, if we redefine the Lorentz algebra to be

$$\tilde{L} = L \oplus L'\,, \tag{8.2}$$

it is no longer true that the Lorentz algebra is a subalgebra of a simple algebra. Similar results hold in cases (i), (ii), and (iii). We see, therefore, that even if we start with a clear cut result of the type we expect, the situation can be changed by a redefinition. Hence we cannot hope to obtain "redefinition-independent" results. Of course, if we put in some specific assumptions regarding the identification of L and the internal symmetry algebra T, then we may obtain clear-cut results. For example, McGlinn's assumption (a) is of a general nature and therefore leads to conclusions which are valid only up to a redefinition, but his assumption (b) is specific, in the sense that the relation

$$[T, M] = 0\,, \tag{8.3}$$

which he assumes, is not invariant under redefinitions, and hence leads to results which are independent of redefinitions.

The second question above is not so easy to answer, as it is to a certain extent a matter of opinion. We take the view in this paper that redefinitions *are* trivial, physically as well as mathematically. To see, however, just how bad things can become if we do not take this view, we consider the very instructive proposal of Ref. 5. In this paper the authors consider an algebra $\tilde{L}$ isomorphic to L, and a semisimple internal algebra T, which (at least for certain finite representations) contains a subalgebra L' isomorphic to L. They then let E, the containing Lie algebra, be the direct sum

$$E=T\oplus\tilde{L}\,, \tag{8.4}$$

and define L, the "actual" Lorentz algebra to be

$$L=L'\oplus\tilde{L}\,. \tag{8.5}$$

The criterion for distinguishing L, the "actual" Lorentz algebra, is the identification of the mass squared operator m^2 as $P_\mu P^\mu$ (and not $\tilde{P}_\mu\tilde{P}^\mu$, say). We then have,

$$m^2=P_\mu P^\mu=\tilde{P}_\mu\tilde{P}^\mu+2\tilde{P}_\mu P'^\mu+P'_\mu P'^\mu\,. \tag{8.6}$$

One now takes a multiplet, or finite-dimensional representation of T, and by regarding $\tilde{P}_\mu\tilde{P}^\mu$ as an "external" mass-squared, common to all the particles of the multiplet, one obtains mass splittings from the term $P_\mu'P'^\mu$ in (8.6) (and line breadths from the term $2\tilde{P}_\mu P'^\mu$) by taking expectation values of these operators with respect to the states of the multiplet.

The basic question, of course, is the validity of the formula (8.6) for the physical mass. However, let us accept this definition of m^2 as legitimate. Even then, one runs into a serious difficulty. This arises from the fact that, if we use a unitary representation of T, as is normally done in physics, then in order to obtain a finite representation R we must use the compact form of T. But since L' is not compact, R is a nonunitary representation of L'. Hence the operators L' and, in particular, the operator $P_\mu'P'^\mu$ will not be Hermitian,[22] and, in general, cannot be diagonalized. This difficulty is recognized by the authors of the above paper, and for this reason they take the *expectation* values rather than the *eigenvalues* of $P_\mu'P'^\mu$. If one takes the eigenvalues (calculated from the characteristic equation) one obtains no mass-splitting.[23] This leaves the validity of the procedure adopted here open to some doubt. This example seems to indicate that even if one does not allow redefinitions and uses the extra freedom allowed in this way, the possibilities for doing anything which is physically meaningful are still very much restricted. Some further difficulties connected with this type of model are discussed in Ref. 22.

It is perhaps interesting to note that if we wished to retain the idea of Ref. 5 without running into the difficulties which come from the noncompactness of L, we could set

$$E=T\oplus\tilde{L} \tag{8.7}$$

where $\tilde{L}$ is isomorphic to L, $\oplus$ denotes direct sum, and T is a compact semisimple algebra, which contains as a subalgebra an algebra T', not isomorphic to L, but isomorphic to the largest compact semisimple algebra contained in L. But this is just O_3 or SU_2. Then one could define the Lorentz algebra, in analogy to (7.5) as

$$\begin{aligned} M_{rs}&=\tilde{M}_{rs}+T_{rs}'\,,\\ M_{r0}&=\tilde{M}_{r0}\,,\\ P_\mu&=\tilde{P}_\mu\,, \end{aligned} \tag{8.8}$$

where r, $s=1\cdots3$, $\mu=0, 1, 2, 3$. This will not lead to any mass-splitting, of course. Further, the splitting of $M_{\mu\nu}$ into M_{rs} and M_{r0} is not Lorentz-invariant. Also if we allow redefinitions, T_{rs}' can be transformed away. However, if we choose not to allow redefinitions and are interested only in the nonrelativistic limit, we can regard the T_{rs}' part of M_{rs} as an $SU(2)$ subalgebra of T. If we require T to contain also $SU(3)$, then $SU(6)$ is obviously a suitable choice for T. In this way we obtain a link between the general analysis given here and $SU(6)$ theory. Equation (8.8) also suggests that one should not expect it to be possible to make $SU(6)$ theory relativistic in the strict sense, and suggests further that the $SU(2)$ part of M should commute with everything else in space-time, in particular with the "orbital angular momentum" M_{rs}. Finally, if we consider the little group of E in (8.7) with respect to the invariant subgroup generated by P, we see that it is precisely the group $SU(6)\otimes O_3$ considered by Mahanthappa and Sudarshan.[14]

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Note added in proof. The author wishes to thank Professor S. Coleman for a prepublication copy of his closely related paper.[24]

APPENDIX

We wish to show that if L is a subalgebra of a simple algebra G, then P is not a subalgebra of the compact form of G.

To show this we choose a Cartan basis (Ref. 15, p.121)

$$\begin{aligned} [H_i,H_j]&=0\,,\quad i,j=1\cdots l\,,\\ [H_i,E_\alpha]&=r_i(\alpha)E_\alpha\,,\quad \text{etc.} \end{aligned} \tag{A1}$$

[22] For this reason the fact that one obtains mass splittings in this model does not contradict the theorem of Sec. 6.

[23] R. Roskies, J. Math. Phys. (to be published).

[24] S. Coleman, Phys. Rev. **138**, B1262 (1965).

for G, such that the iH_i, $(E_\alpha+E_{-\alpha})$, and $i(E_\alpha-E_{-\alpha})$ belong to the compact form of G (Ref. 15, p. 149). Clearly there is no loss in generality in choosing $H_1=iM_{12}$ and $H_2=iM_{34}=M_{30}$. Since L is contained in G we can now write

$$P_1+iP_2=\sum_i a_iH_i+\sum_\alpha b_\alpha E_\alpha, \tag{A2}$$

where the a_i and b_α are numerical coefficients. But from (1.4) we have

$$[iM_{12},P_1+iP_2]=P_1+iP_2. \tag{A3}$$

Hence

$$\sum_i a_iH_i+\sum_\alpha b_\alpha E_\alpha$$
$$=[H_1,\sum_i a_iH_i+\sum_\alpha b_\alpha E_\alpha]=\sum_\alpha r_1(\alpha)b_\alpha E_\alpha. \tag{A4}$$

Thus

$$P_1+iP_2=\sum_{r_1(\alpha)=1} b_\alpha E_\alpha. \tag{A5}$$

Similiarly,

$$P_1-iP_2=\sum_{r_1(\alpha)=1} d_\alpha E_{-\alpha}, \tag{A5}$$

where the d_α are numerical coefficients. Hence

$$P_1=\tfrac{1}{4}\sum_{r_1(\alpha)=1}(b_\alpha+d_\alpha)(E_\alpha+E_{-\alpha})+\left[\frac{1}{i}(b_\alpha-d_\alpha)\right][i(E_\alpha-E_{-\alpha})]. \tag{A6}$$

But if P is a subalgebra of the compact form of G, $(b_\alpha+d_\alpha)$ and $1/i(b_\alpha-d_\alpha)$ are real, whence

$$d_\alpha=b_\alpha^*. \tag{A7}$$

In this case

$$0=[P_1+iP_2,P_1-iP_2]=\sum_{r_1(\alpha)=1;\,r_1(\beta)=1} b_\alpha b_\beta^*[E_\alpha,E_{-\beta}]$$
$$=\sum_{r_1(\alpha)=1}|b_\alpha|^2H_1+\sum_{i\neq 1}c_iH_i+\sum_\alpha c_\alpha E_\alpha, \tag{A8}$$

where the c_i and c_α are numerical coefficients, whence

$$b_\alpha=0,\quad r_i(\alpha)=1. \tag{A9}$$

Since this is impossible for

$$P_1\neq 0,\quad P_2\neq 0, \tag{A10}$$

we see that P cannot be a subalgebra of the compact form of G.

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π^+ Decay of $_\Lambda$Li^7

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An event unambiguously identified as the π^+ decay of a Li^7 hyperfragment is reported. The event was observed in a stack of L_4 hypersensitized Ilford emulsions exposed to a 1.5-GeV/c K^- beam at CERN. The charge of the hyperfragment was uniquely determined as three, by comparing its measured mean track width with the curves (mean track width versus dip angle) established for $Z=1$, 2, 3, 4, and 5 nuclides found in the same emulsions. The branching ratio R of the π^+/π^- decay modes for $_\Lambda$Li^7, on the basis of the present π^+ decay together with the available world data on π^- decays, is estimated as $R(_\Lambda\text{Li}^7)\sim 1\%$.

I. INTRODUCTION

ALTHOUGH the π^+ emission in free Λ decay is forbidden by the conservation laws, in the presence of a proton a Λ can generate π^+ by virtue of the "stimulation process"

$$\Lambda+p\to n+n+\pi^++35\ \text{MeV}. \tag{1}$$

There are several mechanisms which could conceivably contribute to this decay interaction; among them the following have been considered by various authors[1–5]:

(i) The Λ may undergo transition to a virtual Σ^+ state in the presence of a proton inside a hypernucleus and subsequently decay from this state with the emission of a π^+ meson; i.e.,

$$\Lambda+p\to(\Sigma^++n)\to\pi^++n+n. \tag{2}$$

(ii) The Λ may decay through the π^0-mesonic mode and the π^0 so produced may undergo charge exchange with a proton of the hypernucleus; i.e.,

$$\Lambda+p\to(n+\pi^0)+p\to n+n+\pi^+. \tag{3}$$

(iii) The Λ may generate the decay interaction $\Lambda\to n+(\pi^++\pi^-)$, by virtue of the four-fermion weak interaction $(\bar{\Lambda}p)(\bar{p}n)$, and the π^- produced may be subsequently absorbed on a proton inside the hypernucleus; i.e.,

$$\Lambda\to n+(p+\bar{p})\to n+(\pi^++\pi^-), \tag{4}$$
$$\Lambda+p\to n+n+\pi^+.$$

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