Spin $\frac{5}{2}$:

$$d_{\frac{5}{2}\frac{5}{2}}(\beta)=\tfrac{1}{4}(1+\cos\beta)^2\cos\tfrac{1}{2}\beta\,,\qquad d_{\frac{5}{2}\frac{3}{2}}(\beta)=-(5)^{1/2}\tfrac{1}{4}(1+\cos\beta)^2\sin\tfrac{1}{2}\beta\,,$$

$$d_{\frac{5}{2}\frac{1}{2}}(\beta)=(10^{1/2}/4)\sin^2\beta\cos\tfrac{1}{2}\beta\,,\qquad d_{\frac{5}{2}-\frac{1}{2}}(\beta)=-(10^{1/2}/4)\sin^2\beta\sin\tfrac{1}{2}\beta\,,$$

$$d_{\frac{5}{2}-\frac{3}{2}}(\beta)=5^{1/2}\tfrac{1}{4}(1-\cos\beta)^2\cos\tfrac{1}{2}\beta\,,\qquad d_{\frac{5}{2}-\frac{5}{2}}(\beta)=-\tfrac{1}{4}(1+\cos\beta)^2\sin\tfrac{1}{2}\beta\,,$$

$$d_{\frac{3}{2}\frac{3}{2}}(\beta)=\tfrac{1}{2}(5\cos\beta-3)\cos^3\tfrac{1}{2}\beta\,,\qquad d_{\frac{3}{2}\frac{1}{2}}(\beta)=(1/\sqrt{2})(-(5\cos\beta-1))\cos^2\tfrac{1}{2}\beta\sin\tfrac{1}{2}\beta\,,$$

$$d_{\frac{3}{2}-\frac{1}{2}}(\beta)=(1/\sqrt{2})(1+5\cos\beta)\sin^2\tfrac{1}{2}\beta\cos\tfrac{1}{2}\beta\,,\qquad d_{\frac{3}{2}-\frac{3}{2}}(\beta)=-\tfrac{1}{2}(5\cos\beta+3)\sin^3\tfrac{1}{2}\beta\,,$$

$$d_{\frac{1}{2}\frac{1}{2}}(\beta)=\tfrac{1}{2}(5\cos^2\beta-2\cos\beta-1)\cos\tfrac{1}{2}\beta\,,\qquad d_{\frac{1}{2}-\frac{1}{2}}(\beta)=-\tfrac{1}{2}(5\cos^2\beta+2\cos\beta-1)\sin\tfrac{1}{2}\beta\,.$$

Spin 3:

$$d_{33}(\beta)=\tfrac{1}{8}(1+\cos\beta)^3\,,\qquad d_{32}(\beta)=-(6^{1/2}/8)\sin\beta(1+\cos\beta)^2\,,$$

$$d_{31}(\beta)=(15^{1/2}/8)\sin^2\beta(1+\cos\beta)\,,\qquad d_{30}(\beta)=-(5^{1/2}/4)\sin^3\beta\,,$$

$$d_{3-1}(\beta)=(15^{1/2}/8)\sin^2\beta(1-\cos\beta)\,,\qquad d_{3-2}(\beta)=-(6^{1/2}/8)\sin\beta(1-\cos\beta)^2\,,$$

$$d_{3-3}(\beta)=\tfrac{1}{8}(1-\cos\beta)^3\,,\qquad d_{22}(\beta)=\tfrac{1}{4}(1+\cos\beta)^2(3\cos\beta-2)\,,$$

$$d_{21}(\beta)=-(5^{1/2}/4\sqrt{2})\sin\beta(3\cos^2\beta+2\cos\beta-1)\,,\qquad d_{20}(\beta)=(15^{1/2}/2\sqrt{2})\cos\beta\sin^2\beta\,,$$

$$d_{2-1}(\beta)=(5^{1/2}/4\sqrt{2})\sin\beta(3\cos^2\beta-2\cos\beta-1)\,,\qquad d_{2-2}(\beta)=\tfrac{1}{4}(1-\cos\beta)^2(3\cos\beta+2)\,,$$

$$d_{11}(\beta)=\tfrac{1}{8}(1+\cos\beta)(15\cos^2\beta-10\cos\beta-1)\,,\qquad d_{10}(\beta)=-(\sqrt{3}/4)\sin\beta(5\cos^2\beta-1)\,,$$

$$d_{1-1}(\beta)=\tfrac{1}{8}(1-\cos\beta)(15\cos^2\beta+10\cos\beta-1)\,,\qquad d_{00}(\beta)=(5\cos^3\beta-3\cos\beta)/2\,.$$

PHYSICAL REVIEW VOLUME 139, NUMBER 4B 23 AUGUST 1965

Introduction to the N-Quantum Approximation in Quantum Field Theory*

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(Received 1 April 1965)

The N-quantum approximation is designed to find approximate operator solutions of theories characterized by a specific Hamiltonian. The Heisenberg field operators of the theory are approximated by finite-degree normal-ordered expansions in an irreducible set of in-fields. The c-number functions which are the coefficients of these expansions are the unknown quantities in the approximation. The approximation assumes that the dominant contributions to the vertex function, scattering function, and other low-order functions come from functions of similar low order. The c-number functions correspond to the connected graphs with a given number of external lines. Thus in graphical language the approximation assumes that the connected graphs with few external lines dominate. The N-quantum approximation is manifestly covariant, treats positive and negative frequencies in a symmetric way, allows a calculation of several different physical processes simultaneously, allows incorporation of bound states, and requires extrapolation off the mass shell in fewer variables than the usual Green's function approaches. After describing the N-quantum approximation, it is shown to be compatible with renormalization theory in first order of the approximation in the model with $\mathcal{L}_I=gA^3$. It should be emphasized that all powers of the coupling constant occur in first order of the N-quantum approximation in this model. A quadratic integral equation is obtained for the vertex function, and it is shown that the vertex function satisfies the renormalization criteria that the particles in the theory have a given observed mass, and that the vertex function has a given coupling constant as the residue of a pole $(m^2-k^2)^{-1}$ in the unphysical region. It is also shown that the power-series-expansion solution is finite term by term in all orders of the coupling constant.

1. INTRODUCTION

QUANTUM electrodynamics is the only quantum field theory which provides a quantitative description of relativistic particle interactions. Even this theory has rather restricted scope, since it applies only to purely electromagnetic interactions and is not valid when effects due to strong and weak interactions enter, for example, at high energies. After a number of unsuccessful attempts to treat specific theories of strong interactions without using perturbation theory, these attempts largely have been abandoned, and interest has shifted to approaches in which various general requirements, such as relativistic invariance, spectrum, locality,

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unitarity, analyticity, and crossing symmetry, are substituted for the characterization of a theory by a specific Hamiltonian. It is not clear that any of these schemes determines dynamics.

We propose to return to the analysis of theories characterized by specific Hamiltonians using the general information that has been gained in the last several years. Our goal is to put strong interaction physics in the same position as quantum electrodynamics; that is, to learn how to extract experimental predictions from theories of strong interactions, leaving aside deep and difficult questions such as the existence of solutions to field theories and the domains of definition of field operators. The main idea of the N-quantum approximation[1] is to approximate the Heisenberg field operator of a specific theory by a finite-degree normal-ordered expansion in an irreducible set of in-fields.[2] The rationale for this approximation is that the dominant contributions to the vertex function, and other low-order functions (which are the c-number coefficients of low-degree normal-ordered products of in-fields in the expansion of the Heisenberg fields) come from functions of similar low order. In graphical language this rationale is that the connected graphs with few external lines dominate. This approach is reminiscent of the Tamm-Dancoff method, but because the central object in the calculation is the Heisenberg field rather than some state vector, the procedure differs qualitatively from the Tamm-Dancoff method. In particular, the N-quantum approach is manifestly covariant, seems to be compatible with renormalization,[3] and treats positive and negative frequencies in a symmetric way. In contrast to the linear equations which result in the usual Tamm-Dancoff method,[4] the N-quantum approach will generally lead to nonlinear integral equations. Some further desirable properties of this method are that the unknown functions introduced are closely related to S-matrix elements—in fact are as little extrapolated off the mass shell as is possible in a field theory; the method automatically deals only with the connected (truncated) functions; several different physical processes can be treated simultaneously; and bound states can be easily incorporated.

Up to now the N-quantum approach has been applied to the nonrelativistic description of bound states and rearrangement collisions,[5] the separation of two-body and three-body effects in potential scattering, a relativistic model of the deuteron,[5] higher sectors of the Lee model,[6] and a simplified model of pion-pion scattering.[7] These applications will be reported separately.

In the present article we introduce the N-quantum approximation, and, to show that it is compatible with renormalization theory, use it to find a finite, renormalized equation for the vertex function in the model with $\mathcal{L}_I=gA^3$. The quadratic integral equation which we obtain for the vertex function satisfies the renormalization criteria that the particles in the theory have a given observed mass, and the vertex function has a given coupling constant as the residue of a pole $(m^2-k^2)^{-1}$ in the unphysical region. We show that the iterative solution to this integral equation is finite for all finite iterations. This is equivalent to showing that the power-series expansion of the solution is finite term by term. Convergence of the iterative procedure (which would be equivalent to demonstration of existence of a solution) is not discussed.

We want to emphasize that the present article is an introduction, and that our main interest is to apply the N-quantum approach to specific problems in particle physics, and to make use of the guidance furnished by comparison with experiment to see whether a given theory is valid, and, if so, to get more accurate approximate solutions to the theory.

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[1] A preliminary version of this work was reported at the Washington meeting of the American Physical Society in April 1964, Bull. Am. Phys. Soc. **9**, 448 (1964).

[2] After this work was completed, Professor K. Symanzik and Professor J. G. Taylor drew our attention to related work of theirs. Symanzik's pioneering work, *Lectures on High Energy Physics*, edited by B. Jaksic (Federal Nuclear Energy Commission of Yugoslavia, Hercegnovi, 1961), Vol. II, p. 485, and earlier references cited there, gives an analysis of the structure of the complete set of Green's function equations (with all variables off the mass shell) and, among other things, applies this analysis to the $\mathcal{L}_I=gA^3$ model in six-dimensional space-time. He shows that this model can be renormalized. He does not consider nonperturbative approximations to this model. The main interest of the present article is to find nonperturbative approximations. We emphasize the computation of the Heisenberg field operators as the central problem in the solution of a field theory, and consider that the expansion functions $f^{(n)}$, defined below, of the Heisenberg fields in normal-ordered products of an irreducible set of in-fields have a privileged position among the various Green's functions. These $f^{(n)}$ have all variables but one on the mass shell. Taylor's extensive work [Nuovo Cimento, Suppl. **1**, 862 (1963)] deals, among other things, with the complete set of r-function equations. He concludes that although the unrenormalized equations can be written with only one leg off the mass shell, the renormalized equations require up to four legs off the mass shell. Taylor considers approximations in which connected graphs with more than N external lines are dropped (his N-particle approximations) which are similar to the N-quantum approximation studied in the present article. Most of Taylor's equations are written in diagrammatic, rather than analytic, form. Taylor does not give a set of equations for the r functions of his N-particle approximation which have finite iterative solutions. In the present article we find an explicit renormalized equation for a nonperturbative approximation to our $f^{(2)}$ (the retarded vertex function with two legs on the mass shell) which has a finite iterative solution. Other related work was done by M. Wellner, Nuovo Cimento **24**, 913 (1962); and H. M. Fried, J. Math. Phys. **3**, 1107; **4**, 451 (1963). Wellner considers a nonperturbative iterative approximation to the connected part of the time-ordered functions of the A^3 model in which the lowest approximation uses a solution of the classical equation of motion. Fried studies axiomatic perturbation theory without a specific iteration.

[3] We will exhibit a renormalized first N-quantum approximation to the theory with $\mathcal{L}_I=gA^3$ in Sec. 3.

[4] Professor K. Symanzik has pointed out that if the usual Tamm-Dancoff method were reformulated in terms of truncated functions then it would also lead to nonlinear equations.

[5] O. W. Greenberg, Bull. Am. Phys. Soc. **10**, 484 (1965).

[6] A. Pagnamenta, Bull. Am. Phys. Soc. **9**, 449 (1964).

[7] D. G. Currie and O. W. Greenberg, Bull. Am. Phys. Soc. **9**, 449 (1964).

2. PROPERTIES OF THE EXPANSION OF THE HEISENBERG FIELD IN IN-FIELDS

(a) The Expansion of the Heisenberg Fields

The expansion of the Heisenberg field in in-fields[8] is the central mathematical tool of the N-quantum approach. In this section we want to review the main properties of this expansion. The exposition will refer to relativistic theory; however it should be clear what changes should be made for nonrelativistic theories. For simplicity we will assume that there is only a single neutral scalar Heisenberg field $\tilde{A}(x)$ and that a single neutral scalar in-field $\tilde{A}_{\rm in}(x)$ is irreducible. Realistic theories require several Heisenberg fields and, what is particularly interesting, more in-fields than Heisenberg fields when bound states occur. Such cases can be treated in a straightforward way. The in-field expansion, which in general is an infinite series, is

$$\tilde{A}(x)=\tilde{A}_{\rm in}(x)+\sum_{n=2}^{\infty}\frac{1}{n!}\int \tilde{f}^{(n)}(x-x_1,\, x-x_2,\, \cdots,\, x-x_n)\times:\prod_1^n \tilde{A}_{\rm in}(x_j):\prod_1^n dx_j.$$

Since the fields are neutral, $\tilde{A}(x)=\tilde{A}(x)^*$, $\tilde{A}_{\rm in}(x)=\tilde{A}_{\rm in}(x)^*$. For the exact $\tilde{f}^{(n)}$'s in a local quantum field theory there are formulas[9] which give the $\tilde{f}^{(n)}$'s in terms of the multiple retarded commutator functions, often called r functions, restricted to the mass shell in all variables but one. These r functions have analyticity properties which come from locality, spectrum, and Lorentz invariance. Because the approximate $\tilde{f}^{(n)}$'s are not local,[10] we will not make any use of the analyticity properties of r functions. The properties which we do use, and which are valid for both exact and approximate $\tilde{f}^{(n)}$'s, are (1) Lorentz invariance, $\tilde{f}^{(n)}(x_1,x_2,\cdots,x_n)=\tilde{f}^{(n)}(\Lambda x_1,\Lambda x_2,\cdots,\Lambda x_n)$, (2) *TCP*, Hermiticity, and other symmetries (for example, for a neutral field, Hermiticity requires the $\tilde{f}^{(n)}$ to be real), (3) the truncation property, $\tilde{f}^{(n)}(x_1,\, \cdots,\, x_j;\, x_{j+1}-a,\, \cdots,\, x_n-a)\rightarrow 0$ in $\mathcal{S}'$, the space of temperate distributions,[11] $a\rightarrow\infty$ space-like, which in graphical language means that the $\tilde{f}^{(n)}$ consist only of connected graphs. In addition the $\tilde{f}^{(n)}$'s are symmetric under permutation of their arguments. This last property is an elementary consequence of Bose statistics.

To discuss the physical significance of the $\tilde{f}^{(n)}$'s it is convenient to work in momentum space. Then there is a corresponding expansion of the Heisenberg field

$$A(k)=A_{\rm in}(k)\delta_m(k)+\sum_{n=2}^{\infty}\frac{1}{n!}\int f^{(n)}(k_1,k_2,\cdots,k_n)\times:\prod_1^n A_{\rm in}(k_j)\delta_m(k_j):\delta(k-\sum_1^n k_j)\prod_1^n dk_j. \quad (1)$$

Here, neutrality implies $A(k)=A(-k)^*$, $A_{\rm in}(k)\delta_m(k)=A_{\rm in}(-k)^*\delta_m(k)$. In these equations, the x-space and momentum-space objects are related by

$$\tilde{A}(x)=\int A(k)e^{-ikx}dk,\quad \tilde{A}_{\rm in}(x)=\int A_{\rm in}(k)\delta_m(k)e^{-ikx}dk,$$

$$\delta_m(k)\equiv\delta(k^2-m^2),$$

$$\tilde{f}^{(n)}(x_1,\cdots,x_n)=(2\pi)^{-4n}\int f^{(n)}(k_1,\cdots,k_n)\exp(-i\sum_1^n k_jx_j)\prod_1^n dk_j.$$

In addition

$$[\tilde{A}_{\rm in}(x),\tilde{A}_{\rm in}(y)]_-=i\Delta_{m^2}(x-y)=\frac{1}{(2\pi)^3}\int\epsilon(k)\delta_m(k)e^{-ik(x-y)}dk,$$

$$[A_{\rm in}(k_1)\delta_m(k_1),A_{\rm in}(k_2)\delta_m(k_2)]_-=\frac{1}{(2\pi)^3}\epsilon(k_1)\delta_m(k_1)\delta(k_1+k_2).$$

The properties (1), (2), and (3) above now translate into

(1) $f^{(n)}(\Lambda k_1,\cdots,\Lambda k_n)=f^{(n)}(k_1,\cdots,k_n)$,

(2) Hermiticity $\bar{f}^{(n)}(k_1,\cdots,k_n)=f^{(n)}(-k_1,\, \cdots,\, -k_n)$,

(3) $[\exp i\sum_{j+1}^n k_s\cdot a]\times f^{(n)}(k_1,\cdots,k_j;\, k_{j+1},\cdots,k_n)\rightarrow 0$, in $\mathcal{S}'$, $a\rightarrow\infty$ space-like.

The distribution $f^{(n)}$ is relevant on $k_j^2=m^2$. The Heisenberg field introduced is always the renormalized field.

For exact solutions of the operator field equations the in-fields make exact eigenstates of the total Hamiltonian and transform like Heisenberg fields under Poincaré transformations which do not reverse the sense of time. The Hamiltonian and other generators of the Poincaré group (when expressed in in-fields) are the free-field functionals of the in-field and are in "diagonal" form. For approximate solutions these properties no longer hold exactly; for example, the Hamiltonian expressed in

[8] R. Haag, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **29**, 12 (1955) introduced this expansion and gave its main properties. C. N. Yang and D. Feldman, Phys. Rev. **79**, 972 (1950); and G. Källen, Arkiv Physik **2**, 371 (1950) introduced in-fields in connection with the study of the equation of motion in the Heisenberg picture.

[9] V. Glaser, H. Lehmann, and W. Zimmermann, Nuovo Cimento **6**, 1122 (1957).

[10] It is an interesting open question whether or not the approximate $\tilde{f}^{(n)}$'s are approximately local, and, if so, in what sense.

[11] L. Schwartz, *Théorie des Distributions* (Hermann & Cie, Paris, 1959), Vol. 2.

the in-fields will contain terms with products of more than two in-fields in addition to the bilinear free-field functional of the in-fields. However, as part of the N-quantum approximation procedure, we choose the no-particle state of the in-fields to be the approximate vacuum state, and the other approximate eigenstates of the Hamiltonian to be the states created by the negative frequency parts of the in-fields acting on this approximate vacuum state. In addition, we take the in-fields to transform like Heisenberg fields (except under time reversals when in and out are interchanged).

What do these choices mean physically? Clearly an approximate solution can not have all the properties of the exact solution.[12] We have chosen to take the eigenstates to be simple (created by in-fields, which mathematically are free fields) and to put the complications of dynamics into the Heisenberg fields via the coefficients $f^{(n)}$ of the in-field expansion. To give a contrasting example, in the Tamm-Dancoff method the Hamiltonian is expressed in terms of Schrödinger fields which have simple commutation relations, but the eigenstates reflect the dynamical complications via the expansion of the eigenstates in terms of Schrödinger operators acting on the vacuum.

(b) Physical Interpretation of the $f^{(n)}$

The function $f^{(2)}$ contains all three pieces of the vertex function with two legs (those corresponding to the in-fields) on the mass shell and one leg (corresponding to the Heisenberg field) off the mass shell. To see this, compute

$$\langle 0|A(k)|k_1k_2\rangle_{\rm in}=(2\pi)^{-6}f^{(2)}(k_1,k_2)\delta(k-k_1-k_2)\times\theta(k_1)\delta_m(k_1)\theta(k_2)\delta_m(k_2),$$

$$_{\rm in}\langle k_1|A(k)|k_2\rangle_{\rm in}=(2\pi)^{-6}f^{(2)}(-k_1,k_2)\delta(k+k_1-k_2)\times\theta(k_1)\delta_m(k_1)\theta(k_2)\delta_m(k_2),$$

$$_{\rm in}\langle k_1,k_2|A(k)|0\rangle=(2\pi)^{-6}f^{(2)}(-k_1,-k_2)\delta(k+k_1+k_2)\times\theta(k_1)\delta_m(k_1)\theta(k_2)\delta_m(k_2),$$

where

$$|k_1,k_2\rangle_{\rm in}=A_{\rm in}(-k_1)\theta(k_1)\delta_m(k_1)A_{\rm in}(-k_2)\theta(k_2)\delta_m(k_2)|0\rangle.$$

These relations follow directly from Eq. (1). To interpret $f^{(3)}$ consider

$$S_{k_1k_2;k_3k_4}={}_{\rm out}\langle k_1,k_2|k_3,k_4\rangle_{\rm in}={}_{\rm in}\langle k_1|A_{\rm out}(k_2)\delta_m(k_2)|k_3,k_4\rangle_{\rm in}.$$

From the Lehmann-Symanzik Zimmermann (LSZ) limit applied to the Heisenberg field

$$A_{\rm out}(k)\delta_m(k)=A_{\rm in}(k)\delta_m(k)+2\pi i\epsilon(k)\delta_m(k)\times\sum_{n=3}^{\infty}\frac{1}{n!}\int[m^2-(\sum_1^n k_j)^2]f^{(n)}(k_1,\cdots,k_n)\times:\prod_1^n A_{\rm in}(k_j)\delta_m(k_j):\delta(k-\sum_1^n k_j)\prod_1^n dk_j.$$

Then

$$S_{k_1k_2;k_3k_4}=1+2\pi i(2\pi)^{-6}\prod_1^4\theta(k_j)\delta_m(k_j)\times[m^2-(k_3+k_4-k_1)^2]f^{(3)}(-k_1,k_3,k_4).$$

From this we see that when a suitable (m^2-k^2) factor is applied to $f^{(3)}$ and it is restricted to the mass shell it becomes the exact T-matrix element for elastic scattering. Without these operations, $f^{(3)}$ is the scattering function with three legs on the mass shell and one leg off. In general, $f^{(n)}$ is the truncated (connected) $n+1$ leg function with n legs on the mass shell and one off. After suitable restriction to the mass shell it becomes the connected part of the S matrix for a collision involving $n+1$ particles.

It should be emphasized that although positive and negative frequencies can be retained on a symmetric footing, so that some notion of crossing symmetry seems compatible with the N-quantum approach, the analytic continuation properties which constitute crossing symmetry in a given $f^{(n)}$ have neither been stated nor established. Finally, we remark that internal variables can be introduced in a simple way.

3. ILLUSTRATION ON THE $\mathcal{L}_I=gA^3$ MODEL

(a) Renormalized Equation of Motion

We use the Ward-Hurst-Thirring model[13] of a self-coupled neutral scalar field with $\mathcal{L}_I=gA^3$ to illustrate the N-quantum approximation method.[2] We choose this model because of its algebraic simplicity, despite difficulties which the model may have.[14] The equation of motion of this model is

$$(\Box+m^2)A(x)=g[A(x)^2-\langle A(x)^2\rangle_0]+\delta M^2A(x)+K(\Box+m^2)A(x)-gL[A(x)^2-\langle A(x)^2\rangle_0].\quad(2)$$

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[12] O. W. Greenberg and A. L. Licht, J. Math. Phys. **4**, 613 (1963) showed that a local quantum field which has a complete asymptotic field of mass m and whose $f^{(n)}$'s vanish for $n>N$ can lead to no scattering or reactions. D. W. Robinson (to be published) showed that if the (off-mass-shell) truncated vacuum expectation values of a local quantum field vanish for $n>N$ then the field is a generalized free field (and, of course, leads to no scattering or reactions). Neither of these results stands against the approximate fields discussed in the present article since the approximate fields are not local. For analogous reasons, the results due to A. S. Wightman and H. Epstein, Ann. Phys. (N.Y.) **11**, 201 (1960); H. Araki, R. Haag, and B. Schroer, Nuovo Cimento **19**, 90 (1961); K. Bardakci and E. C. G. Sudarshan, Nuovo Cimento **21**, 722 (1961) also do not stand against the approximation considered in the present article.

[13] J. C. Ward, Phys. Rev. **79**, 406 (1950); C. A. Hurst, Proc. Cambridge Phil. Soc. **48**, 625 (1952); W. Thirring, Helv. Phys. Acta **26**, 33 (1953); A. Petermann, *ibid*. **26**, 291, 731 (1953).

[14] G. Baym, Phys. Rev. **117**, 886 (1960) has given heuristic arguments to show that this model has no vacuum state. A. Jaffe (private communication) has shown that Baym's argument can be made rigorous provided the field $\pi(f)\equiv\int A(\mathbf{x},t)f(\mathbf{x})d\mathbf{x}$, tested in space only, exists and generates a unitary transformation. In low orders of perturbation theory these conditions are satisfied in two-, but not in four-dimensional space-time.

The counter terms[15] in Eq. (2) are introduced, from the mathematical point of view, to make the singular expression $A(x)^2$ finite. From a physical point of view, each counter term plays a specific role. The term with $\langle A(x)^2\rangle_0$ removes (divergent) tadpole graphs and thus allows $\langle A(x)\rangle_0$ to be set equal to zero. The δM^2 term removes the (divergent) self-mass and ensures that the m^2 in the equations is the observed mass. The counter terms with K and L ensure that the g in the equations is the residue of the vertex function at the pole at $k^2=m^2$ in the unphysical region. The formal relation of these counter terms to the usual δm^2, Z_1, and Z_3 can be seen by rewriting Eq. (2) and comparing it with the usual standard form:

$$\left(\Box+m^2-\frac{\delta M^2}{1-K}\right)A=\frac{1-L}{1-K}g(A^2-\langle A^2\rangle_0)$$

$$(\Box+m_0^2)A_u=g_0(A_u^2-\langle A_u^2\rangle_0)$$

$$A_u=(\surd Z_3)A\,,\quad g_0=Z_1Z_2^{-3/2}g\,,\quad m_0^2=m^2-\delta m^2\,,$$

where A_u and g_0 are the unrenormalized field and bare coupling constant, respectively. Then we find

$$\delta m^2=\delta M^2/(1-K)\,,\quad Z_1=1-L\,,\quad Z_3=1-K\,.$$

(b) First Go-Around

For a first go-around we consider a first approximation in which we keep only one nontrivial term in the in-field expansion of the Heisenberg field $\tilde{A}(x)$, and we use the most naive procedure to find equations for the unknown quantities, which in this case are the c-number coefficient of the nontrivial term and the mass renormalization constant. The ansatz for $\tilde{A}(x)$ is

$$\tilde{A}(x)=\tilde{A}_{\text{in}}(x)+\frac{1}{2}\int\tilde{f}^{(2)}(x-y_1,\,x-y_2)\times:\tilde{A}_{\text{in}}(y_1)\tilde{A}_{\text{in}}(y_2):dy_1dy_2\,.$$

Insertion of this ansatz in Eq. (1) followed by renormal ordering produces factors such as $\Delta^{(1)}(y_2-y_1')$ so that $\tilde{f}^{(2)}$ does not satisfy a differential equation. Since $\Delta^{(1)}$ and the other singular functions are simpler in Fourier-transformed variables, we work in momentum space, using as equation of motion,

$$\begin{aligned}(m^2-k^2)A(k)&=\tfrac{1}{2}g(1-L)\int([A(k_1),A(k_2)]_+-\langle[A(k_1),A(k_2)]_+\rangle_0)\\&\quad\times\delta(k-k_1-k_2)dk_1dk_2+\delta M^2A(k)\\&\quad+K(m^2-k^2)A(k)\,,\end{aligned}\qquad(3)$$

where

$$\int A(k_1)A(k_2)\delta(k-k_1-k_2)dk_1dk_2=\int A(k_2)A(k_1)\delta(k-k_1-k_2)dk_1dk_2$$

has been replaced by

$$\frac{1}{2}\int[A(k_1),A(k_2)]_+\delta(k-k_1-k_2)dk_1dk_2$$

in order to ensure symmetric treatment of the fields on the right-hand side. The anticommutator is $[A,B]_+=AB+BA$.

$$A(k)=A_{\text{in}}(k)\delta_m(k)+\frac{1}{2}\int f^{(2)}(k_1,k_2)\delta(k-k_1-k_2)\times:A_{\text{in}}(k_1)\delta_m(k_1)A_{\text{in}}(k_2)\delta_m(k_2):dk_1dk_2$$

as ansatz for $A(k)$. The notation in these equations is that given in Sec. 2(a). Insertion of the ansatz for $A(k)$ in Eq. (3), renormal ordering, and use of the independence of the normal-ordered products in momentum space leads to

$$0=(2\pi)^{-3}g(1-L)\int f^{(2)}(k,q)\delta_m(q)dq\,|_{k^2=m^2}+\delta M^2\qquad(4)$$

and

$$\begin{aligned}&[m^2-(k_1+k_2)^2]f^{(2)}(k_1,k_2)\\&=g(1-L)\left[2+(2\pi)^{-3}\int f^{(2)}(k_1,q)f^{(2)}(k_2,-q)\delta_m(q)dq\right]\\&\quad+\delta M^2f^{(2)}(k_1,k_2)+K[m^2-(k_1+k_2)^2]f^{(2)}(k_1,k_2)\end{aligned}\qquad(5)$$

which come from the coefficients of normal ordered products of one and two in-fields, respectively.

(c) Diagrams

We introduce diagrams to represent Eqs. (4) and (5) (and similar equations). For the time being, we do not give rules for computing numerical factors from the

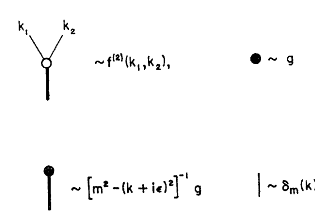
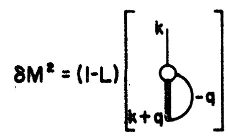
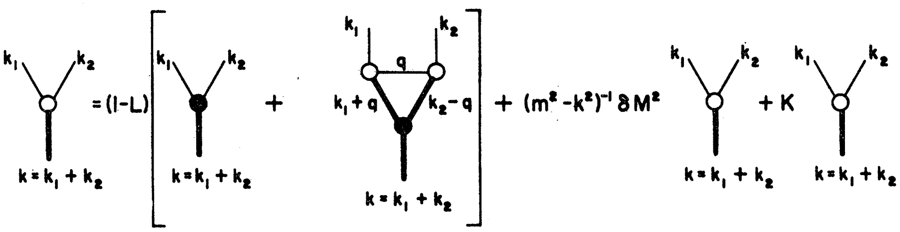


FIG. 1. Symbols used in diagrams. The heavy line is part of $f^{(2)}(k_1,k_2)$ when connected directly to the open circle; the heavy line is a retarded denominator, otherwise. The light line is $\delta_m(k)$ as an internal line; however, it indicates an in-field as an external line. Closed loops correspond to $\int dq$.

[15] These counter terms agree with those which K. Wilson (to be published) considered in a general analysis of the definition of products of field operators at the same space-time point. The remarks about the counter terms which we make here are well known.



(a) Eq. (4)



(b) Eq. (5)

FIG. 2. Diagrams for Eqs. (4) and (5).

diagrams. The basic symbols are identified in Fig. 1 and are used to write Eqs. (4) and (5) in Fig. 2. Heavy lines are associated with Heisenberg field variables and can be off the mass shell with a retarded boundary condition; light lines are associated with in-field variables and are always on the mass shell. To make the retarded boundary condition well defined, one end of each heavy line segment must be distinguished. For the graphs which will be discussed, the distinguished end is (1) the end with an open circle, if an open circle occurs at only one end, (2) the end with a coupling constant dot, if it occurs at only one end, (3) otherwise, the distinguished end is the one for which all lines have the same sense when the heavy line is traversed continuously.[16]

(d) Difficulties with the First Go-Around

To study Eqs. (4) and (5), we expand $f^{(2)}=\sum g^n f_n{}^{(2)}$ and $\delta M^2=\sum g^n \delta M_n{}^2$ in powers of g; all odd powers occur in $f^{(2)}$ and even powers starting with two occur in δM^2. In order g^3 there are two different logarithmically divergent terms in Eq. (5): $(m^2-k^2)^{-1}\delta M_2{}^2 f_1{}^{(2)}$ and a divergent triangle diagram T_1 (Fig. 3). The first divergent term occurs because the self-coupled term in the equation does not generate anything proportional to $f_1{}^{(2)}$ which can cancel the mass renormalization counter term which has the logarithmically divergent coefficient $\delta M_2{}^2$,

$$\delta M_2{}^2=\frac{2}{(2\pi)^3}\int \delta_m(q)dq/[(k+q+i\epsilon)^2-m^2]|_{k^2=m^2}.$$

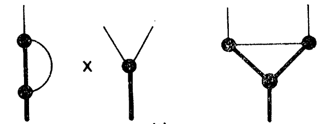
The second divergence[17] T_1 is surprising and rather disconcerting, since the usual Feynman triangle diagram in this model is convergent. Our divergent diagram T_1 differs from the corresponding Feynman diagram by having two retarded denominators and a mass-shell delta function as internal lines, rather than having Feynman denominators. An analytic expression for T_1 is

$$T_1=g^3(2\pi)^{-3}[m^2-(k_1+k_2)^2]^{-1}\int [m^2-(k_1+q)^2]^{-1} \times[m^2-(k_2-q)^2]^{-1}\delta_m(q)dq\,,$$

which, as power counting indicates, is logarithmically divergent (note that $q^2=m^2$ in the denominators). Comparison with the finite, order g^3 perturbation solution to the original Heisenberg operator equation of motion shows that terms which in our approach belong to $f^{(3)}$, the next term in the in-field expansion, play an essential role in removing these two divergences.[17] (See the Appendix for a summary of the perturbation solution to order g^3.) Thus to remove these divergences from our equations, we must add some contributions from $f^{(3)}$; however, since we still want to study the first N-quantum approximation to the $\mathfrak{L}_I=gA^3$ model, we do not want to introduce a really independent $f^{(3)}$ which would force us to the next approximation.

(e) Improved First Approximation

Although the final equation for $f^{(2)}$ in the improved first approximation could be written down directly, we want to expose the reasoning which led us to this



(a) $(m^2-k^2)^{-1}\ \delta M_2{}^2 f_1{}^{(2)}$ (b) T_1

FIG. 3. Divergent diagrams in first go-around.

[16] These graphs are similar to the "doubled graphs" introduced by F. J. Dyson, Phys. Rev. **82**, 428 (1951).

[17] We are indebted to Professor J. G. Taylor for pointing out that this divergent graph and the graphs which compensate its divergence were discussed by G. Källen, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1958), Vol. 5, Chap. 1.

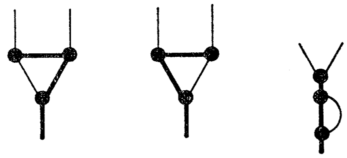


FIG. 4. Diagrams coming from $f^{(3)}$ which compensate divergences.

equation in order to allow the interested reader to assess to what extent this equation for $f^{(2)}$ is a natural consequence of the ideas of the N-quantum approximation.

The diagrams associated with the scattering function $f^{(3)}$ which are necessary to remove the two divergences which occurred in order g^3 in the first go-around are shown in Fig. 4. These diagrams are the lowest order contributions to $f^{(2)}$ coming from "that part of $f^{(3)}$ which is generated by $f^{(2)}$." We make the phrase in quotation marks precise in the following way: Add a term in $f^{(3)}$ to the ansatz for $A(k)$. Obtain new equations for δM^2, $f^{(2)}$, and $f^{(3)}$ by inserting the ansatz for $A(k)$ in the equation of motion and re-normal ordering, just as before. In each of these equations, drop all terms which contain more than one contraction (i.e., more than one mass-shell delta function in the integrand). In the equation for $f^{(3)}$, which has $[m^2-(\sum_1^3 k_j)^2]f^{(3)}$ on the left-hand side, drop all terms on the right-hand side which contain $f^{(3)}$. Then $f^{(3)}$ can be found algebraically from $f^{(2)}$ and the resulting $f^{(3)}$ is the desired "part of $f^{(3)}$ which is generated by $f^{(2)}$." The equation for δM^2 is the same [Eq. (4)] as before, the equations for $f^{(2)}$ and $f^{(3)}$ are[18]

$$[m^2-(k_1+k_2)^2]f^{(2)}(k_1,k_2)=g(1-L)\left\{2+(2\pi)^{-3}\int f^{(2)}(k_1,q)f^{(2)}(k_2,-q)\delta_m(q)dq+(2\pi)^{-3}\int f^{(3)}(k_1,k_2,-q)\delta_m(q)dq\right\}$$
$$+\delta M^2 f^{(2)}(k_1,k_2)+K[m^2-(k_1+k_2)^2]f^{(2)}(k_1,k_2) \quad (6)$$

$$[m^2-(\sum_1^3 k_j)^2]f^{(3)}(k_1,k_2,k_3)=2g[f^{(2)}(k_1,k_2)+f^{(2)}(k_2,k_3)+f^{(2)}(k_3,k_1)]. \quad (7)$$

These equations with the diagram for δM^2 inserted are shown in graphical form in Fig. 5. The graphs for the equation for $f^{(3)}$ show in what way $f^{(3)}$ is "generated by $f^{(2)}$." When $f^{(3)}$ and δM^2 are substituted in Eq. (6), a single equation for $f^{(2)}$ results,

$$[m^2-(k_1+k_2)^2]f^{(2)}(k_1,k_2)=g(1-L)\left\{2+(2\pi)^{-3}\int f^{(2)}(k_1,q)f^{(2)}(k_2,-q)\right.$$
$$\left.\times\delta_m(q)dq+2g(2\pi)^{-3}\int[f^{(2)}(k_1,-q)+f^{(2)}(k_2,-q)][m^2-(k_1+k_2-q+i\epsilon)^2]^{-1}\delta_m(q)dq\right\}+g(1-L)(2\pi)^{-3}$$
$$\times\int\{2g[m^2-(k_1+k_2-q+i\epsilon)^2]^{-1}-f^{(2)}(k,-q)\}\delta_m(q)dqf^{(2)}(k_1,k_2)+K[m^2-(k_1+k_2)^2]f^{(2)}(k_1,k_2). \quad (8)$$

The term in Eq. (6) involving $f^{(3)}$ has been evaluated in terms of $f^{(2)}$ in Eq. (8); the graphs for this evaluation are shown in Fig. 6. In lowest order, these graphs are precisely the graphs (in Fig. 4) desired to make $f^{(2)}$ finite in order g^3, and they occur with the proper numerical coefficients. Equation (8) for $f^{(2)}$ with $f^{(3)}$ and δM^2 inserted and terms regrouped is shown in graphical form in Fig. 7.

In 3rd order there is a one-parameter family of ways in which the (order g^2) renormalization constants K_2 and L_2 can be chosen in order to keep the residue of the pole of $f^{(2)}$ at $(k_1+k_2)^2=m^2$ equal to its Born approximation value $-2g$. Figure 8 shows the order g^3 graphs. The two bracketed sets of graphs fall off faster[19] than $|k^2|^{-1-\epsilon}$, $\epsilon>0$, for large absolute value of $k^2=(k_1+k_2)^2$, so that these sets of graphs, considered as an approximation for $f^{(2)}$, will produce a finite result when inserted in the right-hand side of the equation illustrated in Fig. 7. However, the third-order counter term, whose coefficient is K_2-L_2, is proportional to the Born approximation and falls off at large k^2 as $|k^2|^{-1}$. What has happened is that the renormalization which preserved the low-energy, long-distance property of $f^{(2)}$ (that the coupling constant is g) has introduced an unfavorable high-energy, short-distance behavior in order g^3. The terms which go as $|k^2|^{-1}$ for large k^2 will give a logarithmic divergence when iteratively inserted in Eq. (8), Fig. 7. This divergence occurs because the graph (a) in Fig. 7 contains $f^{(2)}$ quadratically, while graphs (b) and (c) contain $f^{(2)}$ only linearly. Divergences occur in higher

[18] We have dropped the L counter term on the right-hand side of Eq. (7) because it contains no terms with contractions and thus requires no compensating counter term. Comparison with ordinary fifth-order perturbation theory confirms that this counter term should be dropped.

[19] M. Cassandro and M. Cini, Nuovo Cimento 34, 1719 (1964).

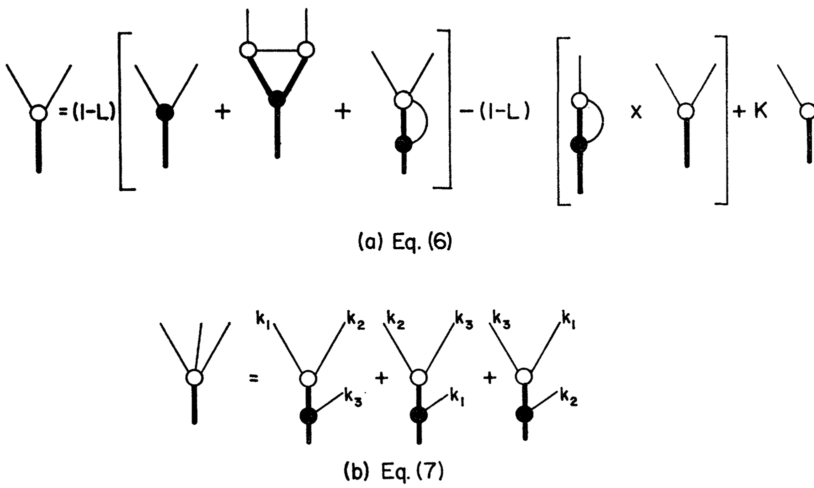


FIG. 5. Diagrams for Eqs. (6) and (7) with δM^2 inserted.

orders, just as they did in the first go-around, unless graphs (a), (b), and (c) occur in an equivalent way. Explicit calculation of perturbation theory to fifth order shows that the three triangle graphs in Fig. 8 always occur together as parts of fifth-order graphs. To ensure that these three graphs also occur together in the fifth and higher powers of g in the 1st N-quantum approximation, $f^{(2)}$ must occur quadratically in each of graphs (a), (b), and (c) of Fig. 7.

Thus a new vertex, illustrated in Fig. 9, which has not yet been defined in the N-quantum approximation, must be introduced. To see how this can be done, we look more closely at graph (c) of Fig. 7 in Fig. 10. The formula for it is

$$2g[m^2-(k_1+k_2)^2]^{-1}\int dq \times\delta_m(q)[m^2-(k_1+k_2-q+i\epsilon)^2]^{-1}f^{(2)}(k_2,-q).$$

The leg with k_2-q is included in $f^{(2)}(k_2,-q)$; the leg with k_1+k_2-q together with the vertex which this leg makes with k_1 and k_2-q are represented by $2g[m^2-(k_1+k_2-q+i\epsilon)^2]^{-1}$. Since Lorentz invariance implies that $f^{(2)}$ is a function only of the sum of its arguments, we can replace $f^{(2)}(k_1,k_2)$ by $f(k_1+k_2)$, and we can introduce a second factor of $f^{(2)}$ into graph (c) by replacing $2g[m^2-(k_1+k_2-q+i\epsilon)^2]^{-1}$ by $f(k_1+k_2-q+i\epsilon)$.[20] A second factor of $f^{(2)}$ is introduced into graph 7(b) in an analogous way. The introduction of this new vertex introduces some contributions from higher $f^{(n)}$'s into the equation for $f^{(2)}$, just as the earlier addition of graphs (b) and (c) in Fig. 7 introduced some contributions from $f^{(3)}$.

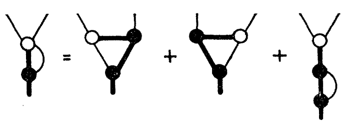


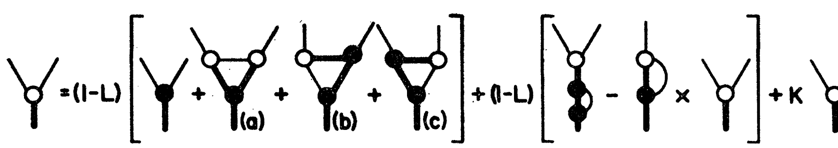
FIG. 6. Diagrams for the term involving $f^{(3)}$ in Eq. (6).

Next we study the two self-mass diagrams in the bracket in Fig. 7. The first term is linear in f, while the second is quadratic. Just as before, we introduce a second factor of f to make the first graph quadratic. The new equation for f is shown graphically in Fig. 11; in analytic form the equation is

$$[m^2-(k_1+k_2)^2]f(k_1+k_2)=g(1-L)\left\{2+(2\pi)^{-3}\int dq\,\delta_m(q)\{f(k_1+q)f(k_2-q)+f(k_1+k_2-q)[f(k_1-q)+f(k_2-q)]\}\right\}$$
$$+(2\pi)^{-3}g(1-L)\int dq\,\delta_m(q)[f(k_1+k_2-q)-f(k_1+k_2-q)|_{(k_1+k_2)^2=m^2}]f(k_1+k_2)+K[m^2-(k_1+k_2)^2]f(k_1+k_2).\quad(9)$$

This is our final equation; however, we want to put it in better form before demonstrating that iterating the

FIG. 7. Diagrams for Eq. (8).



[20] Professor K. Symanzik pointed out that this method of introducing the new vertex amounts to assuming a constant extrapolation of the leg carrying energy-momentum k_2-q [see Fig. 10(b)] off the mass shell.

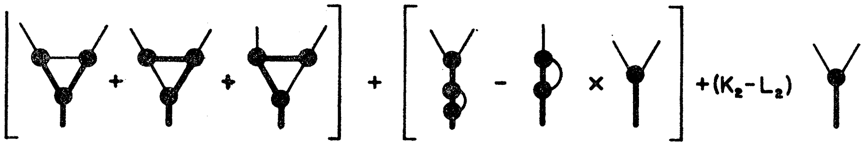


FIG. 8. Diagrams for order g^3 terms.

equation any finite number of times starting from the Born approximation yields finite results. We first determine K to accomplish coupling-constant renormalization. The renormalization condition is

$$\lim_{(k_1+k_2)^2\to m^2}[m^2-(k_1+k_2)^2]f(k_1+k_2)=2g.$$

Applying this to the right-hand side of Eq. (9), we find

$$K=L+(1-L)\left\{(2\pi)^{-3}g\frac{\partial}{\partial k^2}\int dq\,\delta_m(q)f(k-q)\right.$$
$$\left.-\tfrac{1}{2}(2\pi)^{-3}\int dq\,\delta_m(q)[f(k_1+q)f(k_2-q)+f(k_1+k_2-q)[f(k_1-q)+f(k_2-q)]]\right\}\Bigg|_{(k_1+k_2)^2=k^2=m^2}.$$

We insert K in Eq. (9), and at the same time separate the Born term from f via

$$f(k)=f_0(k)+\varphi(k),\quad f_0(k)=2g[m^2-(k+i\epsilon)^2]^{-1},$$

and cancel a common factor of $(1-L)$. The resulting equation for φ is

$$\begin{aligned}(m^2-k^2)\varphi(k)=&\Phi\{k;f_0+\varphi\}+\mathfrak{M}\{k;f_0+\varphi\}\\&+(2g)^{-1}[\mathfrak{M}\{k;f_0+\varphi\}-F_0\{f_0+\varphi\}]\\&\times(m^2-k^2)\varphi(k).\end{aligned}\quad(10)$$

The new symbols introduced in Eq. (10) are

$$F\{k_1+k_2;\chi\}=(2\pi)^{-3}g\times\int[\chi(k_1+q)\chi(k_2-q)+\chi(k_1+k_2-q)\times\{\chi(k_1-q)+\chi(k_2-q)\}]\delta_m(q)dq,$$

$$F_0\{\chi\}=F\{k;\chi\}|_{k^2=m^2},$$

$$\Phi\{k;\chi\}=F\{k;\chi\}-F_0\{\chi\},$$

$$M\{k;\chi\}=(2\pi)^{-3}g^2\int\chi(k-q)\delta_m(q)dq,$$

$$M_0\{\chi\}=M\{k;\chi\}|_{k^2=m^2},$$

$$M_1\{\chi\}=\frac{\partial}{\partial k^2}M\{k;\chi\}|_{k^2=m^2},$$

$$\mathfrak{M}\{k;\chi\}=2(m^2-k^2)^{-1}\times[M\{k;\chi\}-M_0\{\chi\}-M_1\{\chi\}(k^2-m^2)].$$

The boundary conditions of all functions are fixed by adding $+i\epsilon$ to their arguments.

Because Φ and $\mathfrak{M}$ are subtracted functions, and because the other term in Eq. (10) contains a factor (m^2-k^2), it is clear that an iterative solution of Eq. (10) for φ, starting from the g^3 term, remains free of a pole at $k^2=m^2$. In addition, from the limit of Eq. (10) as $k^2\to m^2$, it follows that φ can never have such a pole provided $2g+F_0\{f_0+\varphi\}\neq 0$.

If we define the "bare" coupling constant g_0 by

$$g_0=-\tfrac{1}{2}\lim_{k^2\to\infty}k^2f(k),$$

then

$$g_0=g[1-g^{-1}M_1\{f_0+\varphi\}-(2g)^{-1}F_0\{f_0+\varphi\}]^{-1},$$

provided the relevant limits behave as indicated by perturbation theory.

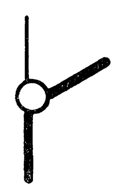


FIG. 9. New vertex.

(f) Finiteness and Renormalizability of the Improved First Approximation in All Powers of g

Here we will show that the perturbation solution to our improved first approximation Eq. (10) is finite in all orders. Our procedure will be to show that each iteration of Eq. (10) for φ, starting with the order g^3 term, produces a finite result. We will also show that Eq. (10) is renormalized, i.e., that φ has neither (a) a second-order nor (b) a simple pole at $k^2=m^2$. Condition (a) means

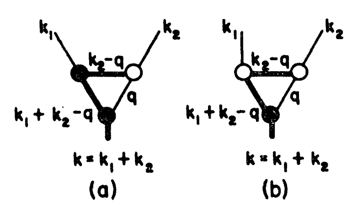
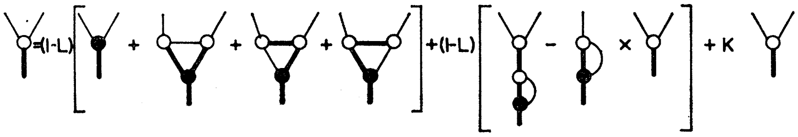


FIG. 10. (a) Diagram (c) of Fig. 7; (b) same diagram with the new vertex introduced.

FIG. 11. Diagrams for Eq. (9).



that the mass in Eq. (10) is the observed mass, and condition (b) means that the residue of the pole in f at $k^2=m^2$ is contained entirely in the first Born approximation term.

The order g^3 solution (which we call φ_1 for "first iteration") to Eq. (10) is

$$\varphi_1(k)=(m^2-k)^{-1}\Phi\{k;f_0\}.$$

The function $F\{k;f_0\}$ is finite and varies as $(k^2)^{-1}\times[\ln(k^2/m^2)]^2$ for large $|k^2|$.[19] Therefore the dominant high $|k^2|$ dependence of φ_1 comes from the term $F_0\{f_0\}$ in $\Phi\{k;f_0\}$, and φ_1 varies as $(k^2)^{-1}$ for large $|k^2|$, which is the same high $|k^2|$ dependence as the Born approximation. For the next iteration of Eq. (10), all terms on the right-hand side contribute. Again, the dominant term at high $|k^2|$ has the same dependence as the Born approximation which we showed above gives a finite result when inserted in the functionals F and $\mathfrak{M}$. Therefore the second and all succeeding iterations yield finite results, and vary as $(k^2)^{-1}$ for large $|k^2|$. Thus the perturbation expansion of Eq. (10) is finite term by term.

4. CONCLUDING REMARKS

(a) Summary

The N-quantum approximation, which can be considered to be the field-operator version of the Tamm-Dancoff method, uses the normal-ordered expansion of the Heisenberg field operators in an irreducible set of in-fields (including those for bound states) to find approximate operator solutions to the equations of motion of a given theory.

In other words, one looks for approximate solutions of the equations of motion which lie in the (assumed irreducible) algebra of in-field operators. For simplicity, one initially considers only normal-ordered terms with degree $N+1$ in the Nth approximation. Among the properties of the N-quantum approach which lend some support to the hope that this approach will yield useful results in strong interaction physics are manifest covariance, symmetric treatment of positive and negative frequencies, and use of truncated functions. The rationale of the approximation is that terms in the expansion of the Heisenberg fields which contain high-degree normal-ordered products of in-fields can be neglected. This amounts to neglecting connected graphs with many external lines; i.e., to assuming that connected graphs with few external lines dominate. However, some contributions from higher connected graphs must be included in order to ensure that the resulting equations can be renormalized. Indeed, the main technical difficulty in applying the N-quantum approach to a given theory in a given approximation is to find a way of terminating the expansion so that the approximate equations have finite and renormalized solutions. In this article, we studied this difficulty in the first approximation, where only one nontrivial term is kept in the expansion of the Heisenberg field, for the model of a neutral scalar field with $\mathfrak{L}_I=gA^3$. We exposed the arguments which led from the "first go-around" in which the perturbation solutions to the equations [Eqs. (4) and (5)] are neither finite nor renormalized, to the "improved first approximation" Eq. (10) in which both these criteria are satisfied. This final equation for the vertex function minus the Born approximation is a quadratic integral equation in one variable. We have not studied the existence or properties of solutions to this equation because it does not correspond to a real physical problem. However, if solutions exist, it seems likely that numerical results could be obtained in a reasonable amount of machine time. Our main purpose was to show that in this example there is a method of termination of the in-field expansion which gives equations which have formal power-series solutions which are finite and renormalized in each power of the coupling constant. This was shown in Subsection 3(e).

(b) Outlook for Higher Approximations

On the basis of the improved first approximation discussed in Subsection 3(e), we guess how to make higher approximations which are finite and renormalized. The transition from the first go-around to the improved first approximation, illustrated in the transition between Fig. 2 and Eqs. (4) and (5), and Fig. 11 and Eqs. (9) and (10) required the addition of new triangle graphs to the one present originally in such a way that the entire set of triangle graphs had the same symmetry under interchange of retarded (heavy) and mass-shell (light) internal lines as perturbation theory. In addition, a mass renormalization graph had to be added to cancel the divergent part of the mass renormalization counter term. Both the triangle graphs and the mass renormalization graph which were added came from a well-defined "part of the scattering function generated by the vertex function" with the single, but crucial, change that the added terms must contain the unknown function (the vertex function) in the same degree (quadratically) as the original terms. In order to introduce the vertex function quadratically, we had to define a new vertex Fig. 9 in which the unknown function has *two* legs off the mass shell and one leg on the mass shell.[20]

With this example in mind, we sketch the transition from the first go-around to the improved version for higher approximations. The first go-around for the nth approximation is obtained by keeping the first $n+1$ terms in the in-field expansion of the Heisenberg field, inserting the ansatz in the questions of motion, renormal ordering, dropping all terms with greater than $n+1$ in-fields, and equating to zero the c-number coefficient of each normal-ordered term. The resulting graphs will have neither the proper symmetry under interchange of retarded and mass-shell internal lines nor the proper association of mass renormalization graphs. The first step in adding additional terms to these equations in order to make them finite and renormalized is to add contributions to the equations for $f^{(j)}$, $j=2, 3,\cdots,n+1$, coming from "that part of the $f^{(j)}$, $j>n+1$, generated by the lower $f^{(j)}$" (with this phrase in quotation marks defined in analogy with its definition for the first approximation) until the retarded and internal lines have the same symmetry and the mass renormalization graphs are associated in the same way as in the lowest relevant order of perturbation theory. The first graphs added should have the same topology as the graphs present originally. For example, one of the terms in the equation for $f^{(3)}$ in the first go-around for the second approximation in $\mathfrak{L}_I=gA^3$ theory is shown in Fig. 12(a). The terms of Figs. 12(b), 12(c), which come from the part of $f^{(4)}$ generated from $f^{(3)}$ should be added. Similarly, the original mass renormalization graph of Fig. 13(a) should be augmented by the graph of 13(b) which comes from $f^{(4)}$. The second step is to add new vertices to make all topologically identical graphs homogeneous in the unknown functions. In our example, this second step replaces the graphs of Figs. 12(b), 12(c) and 13(b) by those of 12(d), 12(e) and 13(c), respectively. The procedure of adding additional graphs coming from the higher f's should be continued until equations with finite and renormalized formal power series are obtained. It seems plausible that the procedure sketched will yield such equations; however, we emphasize that nothing has been proved for the higher approximations.

(c) Critique

(a) Because the A^3 model studied is super-renormalizable, the problem of finding a finite, renormalized approximation may not reflect the full difficulties of an ordinary renormalizable theory. Other interactions which yield ordinary renormalizable theories should be

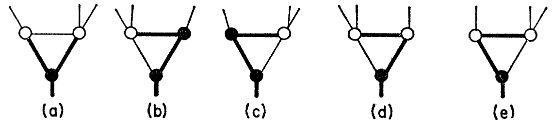


FIG. 12. (a) A diagram in the equation for $f^{(3)}$ in the first go-around for the second approximation. (b), (c) Diagrams coming from the part of $f^{(4)}$ generated by $f^{(3)}$. (d), (e) Diagrams (b) and (c) with new vertices introduced.

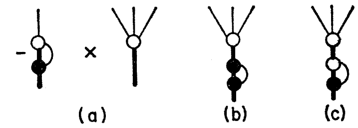


FIG. 13. (a) Mass-renormalization counter-term diagram for $f^{(3)}$. (b) Compensating mass-renormalization diagram coming from the part of $f^{(4)}$ generated by $f^{(3)}$. (c) Diagram (b) with new vertex introduced.

studied, or the A^3 model should be studied in six-dimensional space-time where it is no longer super-renormalizable.[2]

(b) Even for the first approximation, the equation for the vertex function is a quadratic integral equation. In higher approximations the complication of the system of nonlinear integral equations increases rapidly. The difficulty of finding good solutions of the nonlinear equations is the main problem in the practical application of the method described above. However, even in low-order approximations, the method sums rather complicated sets of parts of Feynman diagrams. Further terms can be dropped in order to obtain tractable equations. The experience of S-matrix calculations, in which the most important graphs seem to be those with few internal lines and as low internal masses as possible, can suggest which graphs to keep.

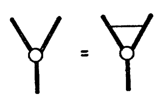


FIG. 14. Diagram of a Bethe-Salpeter equation for the vertex function.

(c) It is not clear for which physical situations the method is best suited. This is a very serious lack of information. It seems difficult to make rigorous estimates of the importance of the higher $f^{(n)}$'s. Probably experience in specific calculations as compared with experimental results will help the most to decide if and when the method is applicable.

(d) It seems possible that some kind of crossing symmetry holds for the $f^{(n)}$'s since positive and negative frequencies are treated symmetrically. However, crossing symmetry has not been defined, and the analyticity properties of the approximate $f^{(n)}$'s have not been studied.

(e) The equal-time canonical commutation relations, re-expressed in terms of in-fields using the ansatz for the Heisenberg fields, yield sum rules on the $f^{(n)}$. In a model of the electron, proton, hydrogen system,[5] these sum rules would not be satisfied without bound-state contributions. The corresponding sum rules for the A^3 model should be studied in connection with possible bound states in this model. In addition, the vacuum expectation value of the equal-time canonical rules yields a definition of K which is independent of L (via Lehmann's formula for Z_3^{-1}).

(f) It is worth remarking that although the graphs of the N-quantum approximation have some superficial resemblence to those which would follow from a Bethe-Salpeter or an S-matrix approach, in fact the N-quan-

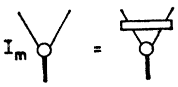


FIG. 15. Diagram of an S-matrix equation for the vertex function. The rectangular box is the elastic scattering amplitude.

tum approximation is entirely different. For example, a Bethe-Salpeter approach might yield the equation shown in Fig. 14 for the vertex function. The subset of graphs summed by this linear equation is entirely different from that summed by the nonlinear first N-quantum approximation. The same remarks apply to the equation shown in Fig. 15 which comes from S-matrix theory using generalized unitarity.

ACKNOWLEDGMENTS

We thank Dr. A. Dragt, Professor S. Drell, Professor H. Lehmann, Professor J. Sucher, Professor K. Symanzik, and Professor W. Zimmermann for helpful conversations. We thank Professor Robert Oppenheimer for his warm hospitality at the Institute for Advanced Study. Finally, we thank the Alfred P. Sloan Foundation for its generous support which made possible the leave of absence during which this work was finished.

APPENDIX: PERTURBATION THEORY TO THIRD ORDER

We condense our notation by defining

$$:k:\equiv A_{\text{in}}(k)\delta_m(k)\,,\quad :k_1k_2\cdots k_n:=:\prod_1^n A_{\text{in}}(k_j)\delta_m(k_j):.$$

Then the in-field expansion of the Heisenberg field takes the form

$$A(k)=:k:+\sum_{n=2}^{\infty}\frac{1}{n!}\int\prod_1^n dk_j\delta(k-\sum_1^n k_j)\times :k_1\cdots k_n:f^{(n)}(k_1,\cdots,k_n)\,.$$

For the perturbation theory, we expand each $f^{(n)}$ (and also the renormalization constants) in power series in g, $f^{(n)}=\sum g^m f_m{}^{(n)}$. We use the abbreviations

$$R_m(k+i\epsilon)\equiv[m^2-(k+i\epsilon)^2]^{-1}\,,\quad \delta_m(k)\equiv\delta(k^2-m^2)\,,$$
$$\{\chi(k_1,\cdots,k_n)\}_{\text{sym}}\equiv(n!)^{-1}\sum_P\chi(k_{\mu_1},\cdots,k_{\mu_n})\,,$$

where the sum runs over all $n!$ permutations

$$(k_1,\cdots,k_n)\rightarrow(k_{\mu_1},\cdots,k_{\mu_n})\,.$$

We list the nonvanishing $f_m{}^{(n)}$ and renormalization constants up to third order, $m=3$. The zero-order term in $A(k)$ is $:k:$.

Order 1:

$$f_1{}^{(2)}(k_1,k_2)=2R_m(k_1+k_2)\,.$$

Order 2:

$$f_2{}^{(3)}(k_1,k_2,k_3)=12\{R_m(\sum_1^3 k_j)R_m(k_2+k_3)\}_{\text{sym}}$$

$$\delta M_2{}^2=-\Delta_2(k)|_{k^2=m^2}\,,$$

$$\Delta_2(k)=2(2\pi)^{-3}\int dq\,\delta_m(q)R_m(k-q)\,.$$

Order 3:

$$f_3{}^{(4)}(k_1,\cdots,k_4)=4!R_m(\sum_1^4 k_j)\{4R_m(\sum_2^4 k_j)R_m(k_3+k_4)+R_m(k_1+k_2)R_m(k_3+k_4)\}_{\text{sym}}\,,$$

$$f_3{}^{(2)}(k_1,k_2)=2R_m(k_1+k_2)\times\Big[R_m(k_1+k_2)\Big(\Delta_2(k)-\Delta_2(k)|_{k^2=m^2}-\frac{\partial\Delta_2(k)}{\partial k^2}\Big|_{k^2=m^2}(k^2-m^2)\Big)+\varphi_3{}^{(2)}(k_1,k_2)-\varphi_3{}^{(2)}(k_1,k_2)|_{k^2=m^2}\Big]\,,$$

$$\varphi_3{}^{(2)}(k_1,k_2)=2(2\pi)^{-3}\Big\{\int dq\,\delta_m(q)R_m(k_2-q)\times[2R_m(k_1+k_2-q)+R_m(k_1+q)]\Big\}_{\text{sym}}$$

$$K_2-L_2=\left[\frac{\partial\Delta_2(k)}{\partial k^2}-\varphi_3{}^{(2)}(k_1,k_2)\right]\Bigg|_{k^2=m^2=(k_1+k_2)^2}.$$