Systematics of Angular and Polarization Distributions in Three-Body Decays*

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A general method is described for the measurement of the polarization and alignment of a particle of arbitrary spin from the analysis of its three-body decays. This method provides a procedure for the determination of spin and parity of the decaying system which is independent of the dynamics of the decay process. The procedure is closely related to the one currently used for two-body reactions except that the normal to the decay plane replaces the center-of-mass momentum as an analyzer. The general formalism is developed and illustrated by two examples: three-pion decays and baryon–two-pion decays.

I. INTRODUCTION

The description of three interacting bodies is a well seasoned and familiar problem, the angular-momentum aspect of which has received a revived interest among particle physicists during the past few years.[1,2] Because increasing numbers of particles (or resonances) of high mass are being experimentally discovered which have appreciable three-body decay modes, it behooves us to examine the three-body problem from the standpoint of a decaying system. However, we do not consider the dynamics of the decay process, but merely make use of the consequences of rotational and inversion invariance. The treatment presented here is therefore completely general, exhibiting the kind of angular and polarization distributions which are consistent with a system of arbitrary spin decaying into three particles with spin. Such distributions, when compared with experiment, provide a possible determination of the spin and parity of the decaying particle and eventually a means to measure its polarization and alignment, quantities of great interest for the understanding of its production mechanism.[3] The method applied to three-body decays is closely related to the one currently used in the analysis of two-body decays except that the normal to the decay plane replaces the center-of-mass momentum as an analyzer of the polarization. Formulas giving the angular and polarization distributions in terms of the decaying particle density matrix are in fact written in a very similar form for both cases.

As is well known, the description of a three-body system requires five variables. A convenient choice of these variables consists of two energies and three angles. The two energies are taken to be the center-of-mass energy of two decay particles, whose domain of variation defines a Dalitz plot. The three angles can be chosen as those which define completely the orientation of the decay plane. In the treatment presented here we consider only the orientation of the decay plane and sum over all energy configurations, or, in some cases, separately over different regions of the Dalitz plot. In this sense, the distributions presented here are the complement of the Dalitz-plot distribution, where all angular configurations are averaged over, and where the three-body system is studied in terms of its energy distribution.[4]

The analysis of the energy distribution in terms of a Dalitz plot has the advantage of giving useful information even if the decaying particle is neither polarized nor aligned. Nevertheless, its practical interest is bound to the dominance of a very small number of independent amplitudes. In many cases the general analysis presented here, which does not rely on any dynamical assumptions governing the decay process, can be used to determine the spin and parity of a decaying state via its three-body decay alone. When the system has, in addition, a two-body decay mode, the combined analysis of both two- and three-body modes can be applied in unison in order to obtain improved and more accurate knowledge of the system's quantum numbers.[5] In all cases it can be used in order to get information about the production mechanisms by means of polarization and alignment analysis.

The angular distribution of the normal to the decay plane is readily obtained when three free relativistic particle states of well-defined angular momentum J and parity are constructed using the general projection method of Wigner.[6] The angular dependence of the decay amplitude is given as a linear combination of rotation-matrix elements corresponding to the $(2J+1)$-dimensional representation of the rotation group: $D_{m'm}^J(\alpha,\beta,\gamma)$. The arguments are three Euler angles, which can be chosen as the azimuthal and polar angles of the normal to the decay plane and a third angle γ, referring to a rotation of the decay plane around the

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[1] G. C. Wick, Ann. Phys. **18**, 65 (1961).

[2] R. Omnès, Phys. Rev. **134**, B1358 (1964); D. Branson, P. V. Landshoff, and J. C. Taylor, Phys. Rev. **132**, 902 (1963).

[3] See, for instance, J. D. Jackson, Rev. Mod. Phys. **37**, 484 (1965); S. M. Berman and S. D. Drell, Phys. Rev. **133**, B791 (1964).

[4] For a detailed discussion on the energy distribution of three pions see C. Zemach, Phys. Rev. **133**, B1201 (1964).

[5] For a recent compilation of resonance quantum numbers see, for example, A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **36**, 977 (1964).

[6] E. P. Wigner, *Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra* (Edwards Brothers, Inc., Ann Arbor, Michigan, 1954), Chap. 12.

normal. These angles then completely specify the orientation of the decay plane. This is a straightforward extension to three particles of a procedure already used to construct two-particle states.[7,1]

The extension to n-particle states has been worked out by Werle.[8] Present interest in three-body decays may, however, warrant the special treatment presented here.

The general formalism is presented in Sec. II, and a general expression for the angular distribution of the normal to the decay plane is given.[9,10] The simplifications due to parity conservation and possible identity of two of the particles are also discussed. The formalism is then applied in Sec. III to the problem of the decay into three spin-zero particles and in Sec. IV to the problem of the decay into two spin-zero and one spin-$\frac{1}{2}$ particle. The distribution of the polarization of the decay spin-$\frac{1}{2}$ particle is discussed in detail and we stress the analogy between the formulas obtained and the ones currently used for two-body decays into a spin-zero and a spin-$\frac{1}{2}$ particle. In both Secs. III and IV we also discuss decays into a pion and into a resonance which eventually decays into two pions or a pion and a hyperon, depending on its quantum numbers.

In addition to giving the general formulas, the simplest cases are explicitly treated. In Sec. III angular distributions are given for the decay of particles with spin $1^{\pm}$ and $2^{\pm}$ into three pions. In Sec. IV angular distribution of the normal to the decay plane, as well as polarization distributions for the daughter hyperon, is given for the decaying states having spin $\frac{1}{2}$ and spin $\frac{3}{2}$.

The $D_{m'm}{}^J$ functions required for explicit calculations with spins less than or equal to three are given in an Appendix.

II. GENERAL FORMALISM

Three-Particle States

A quantum state containing three free particles is completely defined by the momentum and polarization of each particle. Such a state may be constructed as the direct product of three one-particle states $|\mathbf{q}_i,\lambda_i\rangle$, where $\mathbf{q}_i$ and λ_i stand, respectively, for the momentum and helicity of the ith particle. To be precise, we could define the state $|\mathbf{q}_i,\lambda_i\rangle$ as in Ref. 7, namely

$$|\mathbf{q}_i,\lambda_i\rangle = R_{\varphi_i\theta_i 0}|\mathbf{Q}_i,\lambda_i\rangle, \tag{1}$$

where $|\mathbf{Q}_i,\lambda_i\rangle$ is an helicity state with eigenvalue λ_i

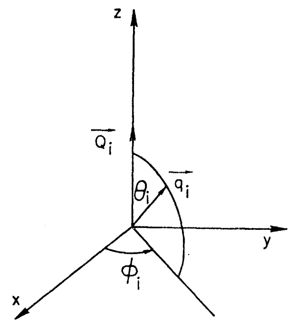

FIG. 1. The angles of rotation of the one-particle helicity state.

and momentum Q_i along the positive z axis ($|\mathbf{Q}_i| = |\mathbf{q}_i|$). The symbol $R_{\varphi_i\theta_i 0}$ stands for the rotation operator, with Euler angles φ_i, θ_i, 0. The quantities φ_i and θ_i are, respectively, the azimuthal and polar angles of $\mathbf{q}_i$ with respect to a fixed coordinate system x, y, z (Fig. 1). The helicity, i.e., the component of the total angular momentum of the particle along its momentum, is obviously invariant under rotation.

A three-particle state is written as

$$|\mathbf{q}_1,\lambda_1;\mathbf{q}_2,\lambda_2;\mathbf{q}_3,\lambda_3\rangle. \tag{2}$$

It is convenient to describe the decay in the center-of-mass system where

$$\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3=0. \tag{3}$$

The three momenta then form a triangle in a plane, the normal of which is defined as a unit vector along $\mathbf{q}_1\times\mathbf{q}_2$. The conservation of energy gives the further restriction

$$(q_1^2+m_1^2)^{1/2}+(q_2^2+m_2^2)^{1/2}+(q_3^2+m_3^2)^{1/2}=m_0, \tag{4}$$

where m_0 is the mass of the decaying particle.

A more convenient description of this state is in terms of a different set of quantum numbers which are the energies ω_1, ω_2, and ω_3 of the three particles restricted by (4)—and three Euler angles α, β, γ which specify the orientation of the momentum triangle in space (Fig. 2).

The rotation angles are defined by starting from a standard position where the triangle is in the x-y plane. As a convention we take $\mathbf{q}_1+\mathbf{q}_2$ along the x axis and the normal $\mathbf{q}_1\times\mathbf{q}_2$ along the z axis. These are our basic states and we define more conveniently the three helicity states by rotations around the z axis with angles φ_i ($0\leq\varphi_i<2\pi$), taking $\mathbf{Q}_i$ along the x axis. It follows that the polarization of each particle is described as usual with the conventional z and y axes, respectively, taken along the momentum and along the normal to the decay plane. The angles α and β are, respectively, chosen as the azimuthal and polar angles of the normal to the decay plane. The angle γ refers to a rotation around the normal and is illustrated in Fig. 2. All helicities remain unchanged through these three successive rotations. We then write a three-particle state thus defined as

$$|\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3;\alpha,\beta,\gamma\rangle. \tag{5}$$

[7] M. Jacob and G. C. Wick, Ann. Phys. 7, 404 (1959).

[8] J. Werle, Phys. Letters 4, 127 (1963); Nucl. Phys. 44, 579 (1963); 44, 579 (1963).

[9] The separation of angular variables by means of the Wigner projection method has been used in connection with the three-particle scattering problem by Omnès (Ref. 2) and also by Bhatia and Temkin (Ref. 10) in the study of the two-electron, fixed-nucleon problem. Even though the general procedure is perhaps well known, the practical application to three-body decays as a means for spin and parity determination appears to carry enough interest for the specialized and detailed discussion given here.

[10] A. K. Bhatia and A. Temkin, Rev. Mod. Phys. 36, 1050 (1964).

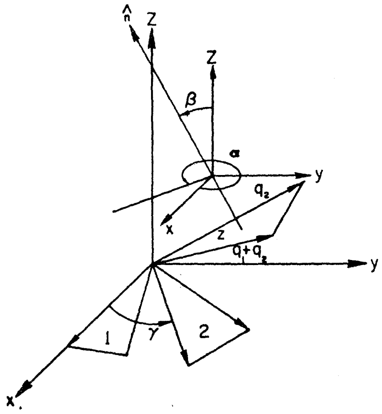

FIG. 2. The decay-plane configuration. Triangle 1 represents the decay plane in the standard position. Triangle 2 shows the plane after rotation of angle γ. Triangle 3 shows the decay plane in its actual position with its normal indicated by $\hat{n}$.

With the set of states (2) the density of final states $d\rho_F$ for the three-body decay is written as

$$d\rho_F=\frac{d^3\mathbf{q}_1d^3\mathbf{q}_2d^3\mathbf{q}_3\delta(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3)\delta(\omega_1+\omega_2+\omega_3-m_0)}{2\omega_12\omega_22\omega_3},$$

or after integration over $d^3\mathbf{q}_3$,

$$d\rho_F=\tfrac{1}{8}d\omega_1d\omega_2d\varphi_1d\cos\theta_1d\varphi_{12}d\cos\theta_{12}\delta\left(\cos\theta_{12}-\frac{((m_0-\omega_1-\omega_2)^2-q_1^2-q_2^2-m_3^2)}{2q_1q_2}\right),\quad(6)$$

where φ_{12} and θ_{12} are the azimuthal and polar angles of $\mathbf{q}_2$ with respect to $\mathbf{q}_1$. Integration with respect to $\cos\theta_{12}$, φ_{12}, $\cos\theta_1$, and φ_1 gives a density distribution in the ω_1, ω_2 plane. This is the Dalitz plot.

With the states (5), the density of states is obtained by replacing $d\varphi_1d\cos\theta_1d\varphi_{12}$ by $d\alpha d\cos\beta d\gamma$ in (6). The Jacobian determinant is equal to 1.[11]

In their center-of-mass system the three decay particles are in a state of well-defined angular momentum and, if we consider only decays via strong or electromagnetic interactions, also parity. The total angular momentum is equal to the spin j of the decaying particle. Such a state is written as

$$|\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3;jmM\rangle,\quad(7)$$

where m is the eigenvalue of the component of angular-momentum operator J along a fixed axis chosen as the z axis; M is the eigenvalue of angular momentum along the normal to the decay plane, which can be used together with the other observables J^2 and J_z to specify the state.[6]

The angular distribution of the normal to the decay plane, obtained for a pure state of definite m and M such as (7), is given by

$$\frac{dN}{d\Omega}=\int|A|^2d\gamma,\quad(8)$$

where $d\Omega=\sin\beta d\beta d\alpha$, and where

$$A=\langle\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3;\alpha\beta\gamma|\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3;jmM\rangle.$$

In order to continue further we need the relationship between a state of definite angular momentum such as (7) and a state described in terms of Euler angles. To achieve this we follow the procedure of Wigner[6] and write

$$|\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3;jmM\rangle=\int D_{mM}^{j*}(\alpha\beta\gamma)\times|\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3;\alpha\beta\gamma\rangle d\alpha\sin\beta d\beta d\gamma,\quad(9)$$

where the integration is performed over all rotations, namely

$$0\leq\alpha\leq2\pi,\quad0\leq\beta\leq\pi,\quad0\leq\gamma\leq2\pi.$$

As is well known, these angles can be defined as in Fig. 2 or, just as well, γ may be considered as the angle of the third rotation performed around the normal to the decay plane. As is easily checked using the group property of the D functions, (9) transforms under rotations as a state of total angular momentum j with z component m and with component M along the normal to the decay plane, a rotationally invariant quantity. The energy and helicity of each particle are invariant under rotations and their same eigenvalues appear on both sides of (9). It should be remarked that we do not obtain in this way the most convenient orthonormal set of states for three free particles as in the two-body problem.[7] Such states have been explicitly constructed by Wick[1] by coupling two particles together and then coupling the third one to the system constructed from the first two. A quantum state with eigenvalues j, m, and M will in general be described by a wave function of ω_1 and ω_2 which multiplies the angular wave function (10). The angular distribution of the normal which is obtained by integration over the Dalitz plot (11) will average over all configurations the final-state interaction of two of the decay particles in a particular angular-momentum state.

Using the angular-momentum eigenstate (9) we have

$$A=D_{mM}^{j*}(\alpha\beta\gamma).\quad(10)$$

A normalization coefficient could appear in (10). It is, however, independent of m and M and therefore irrelevant for our purposes.

[11] This result may be seen as follows. The integration indicated by (6) is over all possible directions of two vectors whose relative angle is fixed. But this integration may as well be considered as ranging over all possible rotations of a rigid body. In this case we may apply the well-known result that the differential element may be written as $dR=d\alpha d\gamma\sin\beta d\beta$, where α, β, and γ are the usual Euler angles. For a detailed derivation see Ref. 4.

The Normal to the Decay Plane as an Analyzer

Many of the formulas which are presented now are special cases of general relations for n-particle states given by Werle.[8] We derive here those expressions which are relevant for the special case of three-body decays.

We now turn to the decay of a particle of spin j whose state is not pure but rather a statistical mixture of states described by a density matrix $\rho_{mm'}$. The eigenvalues m and m' run from $-j$ to $+j$ in integer steps and refer to the z axis. The angular distribution of the normal to the decay plane can be obtained for each set of eigenvalues of the final particle helicities. On using (10), the angular distribution reads as

$$\left(\frac{dN}{d\Omega}\right)_{\lambda_1,\lambda_2,\lambda_3}=\sum_{M,M'}\sum_{m,m'}\rho_{mm'}\times\int D_{mM}^{j*}(\alpha\beta\gamma)D_{m'M'}^{j}(\alpha\beta\gamma)d\gamma\,\mathfrak{F}_{MM'},$$

where

$$\mathfrak{F}_{MM'}=\int d\omega_1 d\omega_2 F_M(\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3)\times F_{M'}^*(\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3).$$

The phenomenological decay amplitudes F_M which have been introduced are functions of rotationally invariant quantities only. They depend in general on M but not on m.

Since the γ dependence of a D function is simply a factor $e^{-iM\gamma}$, interference between different F_M amplitudes vanishes in the normal angular distribution when it is integrated over γ.

If everything else but the direction of the normal to the decay plane is summed over, a simple relation is obtained for the angular distribution of the normal:

$$dN/d\Omega=\sum_{m,m'}\rho_{mm'}\sum_M D_{mM}^{j*}(\alpha\beta 0)\times D_{m'M}^{j}(\alpha\beta 0)|R_M|^2, \quad (11)$$

where

$$|R_M|^2=2\pi\sum_{\lambda_1,\lambda_2,\lambda_3}\int d\omega_1 d\omega_2|F_M(\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3)|^2.$$

Equation (11) relates the angular distribution of the normal to the density matrix of the initial particle in terms of the $2j+1$ decay parameters R_M.

This also shows that the maximum number of independent decay amplitudes, as far as the orientation of the decay plane is considered, is actually $2j+1$ for each set of final helicities. Conservation of parity in the decay process further reduces this number, as will be shown later. This number of independent decay amplitudes is also equal to the maximum number of linearly independent tensors that can be built with the particle momenta—in terms of which the decay amplitudes can also be written.

In order to use (11), one may calculate the required D functions. Alternatively, use of the Clebsch-Gordan series allows (11) to be written as

$$dN/d\Omega=\sum_{mm'}\rho_{mm'}\sum_M\sum_l C(jjl|m',-m)\times C(jjl|M,-M)(-1)^{M-m}\left(\frac{4\pi}{2l+1}\right)^{1/2}\times Y_{m'-m}^{l*}(\beta,\alpha)|R_M|^2, \quad (12)$$

where we have introduced standard Clebsch-Gordan coefficients.[12]

The angular distribution is thus given by a sum of spherical harmonics with highest order $2j$. This generalizes the well-known theorem on the complexity of the angular distribution in two-body reactions to the case of three bodies in terms of the normal to the decay plane.

It is convenient to group together terms with opposite values of M and to write (11) as

$$dN/d\Omega=\sum_{M\geq 0}\{\sum_{mm'}(\mathrm{Re}\rho_{mm'}\cos(m-m')\alpha-\mathrm{Im}\rho_{mm'}\sin(m-m')\alpha)[R_M^+Z_{mm'}^{jM+}(\beta)+R_M^-Z_{mm'}^{jM-}(\beta)]\}, \quad (13)$$

where we have introduced the notations

$$Z_{mm'}^{jM\pm}(\beta)=d_{mM}^{j}(\beta)d_{m'M}^{j}(\beta)\pm d_{m-M}^{j}(\beta)d_{m'-M}^{j}(\beta)$$

and $R_M^\pm=\frac{1}{2}(|R_M|^2\pm|R_{-M}|^2)$; $R^+\geq 0$ and R^- may be either positive or negative. The D functions have been written[12] as

$$e^{-im'\alpha}d_{m'M}^{j}(\beta)e^{-iM\gamma}.$$

As follows from their definition and the relation

$$d_{m'm}^{j}(\beta)=(-1)^{j+m'}d_{m'-m}^{j}(\pi-\beta),$$

the Z functions satisfy the relation

$$Z_{mm'}^{jM\pm}(\beta)=\pm(-1)^{m-m'}Z_{mm'}^{jM\pm}(\pi-\beta).$$

If we invert the direction of the normal (which, in terms of the Euler angles, means the transformation $\alpha\to\pi+\alpha$, $\beta\to\pi-\beta$), then the angular function which goes with R_m^+ is unchanged, while the function which goes with R_M^- changes sign, as is obvious from (13). The normal direction is determined only up to a sign when two particles are identical and when the summation over all available energies is performed according to (11). In that case, all terms proportional to R_M^- will vanish identically. In order to keep the direction of the

[12] We follow the notations of M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957). We refer the reader to this book for the various relations among rotation-matrix elements used throughout this paper.

normal well defined, it is necessary to sum independently on parts of the Dalitz plot, for instance separately for $\omega_1>\omega_2$ and $\omega_1<\omega_2$.

We can further group together terms with opposite values of both m and m' and write the angular distribution of the normal as

$$dN/d\Omega=\sum_{M\geq 0}\tfrac{1}{2}\sum_{mm'}\big[\cos(m-m')\alpha(\mathrm{Re}\rho_{mm'}+(-1)^{m-m'}\,\mathrm{Re}\rho_{-m-m'}) -\sin(m-m')\alpha(\mathrm{Im}\rho_{mm'}-(-1)^{m-m'}\,\mathrm{Im}\rho_{-m-m'})\big]Z_{mm'}{}^{jM+}(\beta)R_M{}^{+} +\big[\cos(m-m')\alpha(\mathrm{Re}\rho_{mm'}-(-1)^{m-m'}\,\mathrm{Re}\rho_{-m-m'}) -\sin(m-m')\alpha(\mathrm{Im}\rho_{mm'}+(-1)^{m-m'}\,\mathrm{Im}\rho_{-m-m'})\big]Z_{mm'}{}^{jM-}(\beta)R_M{}^{-}. \quad (14)$$

Because of the Hermiticity of the density matrix and the definition of the Z functions, terms where m and m' are interchanged give the same contribution. As follows from their definition, $Z_{m-m}{}^{jM-}(\beta)\equiv 0$ for integer j, and $Z_{m-m}{}^{jM+}(\beta)\equiv 0$ for half-integer j.

Parity Conservation

If parity is conserved in the decay we have to replace (7) by an eigenstate of the parity operator with the proper eigenvalue. We therefore consider the action of the parity operator P on an angular-momentum eigenstate (9). We have

$$P|\omega_1\lambda_1,\omega_2\lambda_2,\omega_3\lambda_3;\,jmM\rangle=\int D_{mM}{}^{j*}(\alpha\beta\gamma)R_{\alpha\beta\gamma}P|\omega_1\lambda_1,\omega_2\lambda_2,\omega_3\lambda_3,0,0,0\rangle d\alpha\,\sin\beta d\beta d\gamma\,,$$

since the parity operator P commutes with the rotation operator. We now use the fact that the parity operation can be defined as the product of a reflection with respect to a plane and a rotation of angle π around a normal to that plane. The plane chosen is the decay plane of the reference state

$$|\omega_1\lambda_1,\omega_2\lambda_2,\omega_3\lambda_3,0,0,0\rangle\,,$$

i.e., the x-y plane (Fig. 2). We denote by Y the reflection operator with respect to that plane and write $P=e^{+i\pi J_z}Y$. The action of Y changes the sign of all helicities. In fact, the following relation holds[13]:

$$Y|\omega_1\lambda_1;\,\omega_2\lambda_2;\,\omega_3\lambda_3;\,0,0,0\rangle=\eta_1\eta_2\eta_3(-1)^{S_1-\lambda_1+S_2-\lambda_2+S_3-\lambda_3}|\omega_1-\lambda_1;\,\omega_2-\lambda_2;\,\omega_3-\lambda_3;\,0,0,0\rangle,$$

where S and η stand for the spin and intrinsic parity of each particle. It follows that

$$P|\omega_1\lambda_1;\,\omega_2\lambda_2;\,\omega_3\lambda_3;\,jmM\rangle=\eta_1\eta_2\eta_3(-1)^{S_1-\lambda_1+S_2-\lambda_2+S_3-\lambda_3}\times\int D_{mM}{}^{j*}(\alpha\beta\gamma)R_{\alpha\beta\gamma}e^{+i\pi J_z}|\omega_1-\lambda_1;\,\omega_2-\lambda_2;\,\omega_3-\lambda_3,\,0,\,0,\,0\rangle d\alpha\,\sin\beta d\beta d\gamma.$$

In order to express the state after the parity operation in terms of the original states (9), we use $R_{\alpha\beta\gamma}=e^{-i\alpha J_z}\times e^{-i\beta J_z}e^{-i\gamma J_z}$ and simply add $-\pi$ to the first rotation angle, thus replacing $D_{mM}{}^{j*}(\alpha\beta\gamma)$ by $(-1)^M D_{mM}{}^{j*}(\alpha,\beta,\gamma-\pi)$. In this manner one obtains[14]

$$P|\omega_1\lambda_1;\,\omega_2\lambda_2;\,\omega_3\lambda_3;\,jmM\rangle=(-1)^M(-1)^{S_1-\lambda_1+S_2-\lambda_2+S_3-\lambda_3}\eta_1\eta_2\eta_3|\omega_1-\lambda_1;\,\omega_2-\lambda_2;\,\omega_3-\lambda_3;\,jmM\rangle. \quad (15)$$

We write $(-1)^M$ for $e^{i\pi M}$. Applying (15) to a 3-pion state we find the relation

$$P|\omega_1,\omega_2,\omega_3;\,jmM\rangle=(-1)^{M+1}|\omega_1,\omega_2,\omega_3;\,jmM\rangle. \quad (16)$$

This yields an important result for 3-pion decays, namely that if the parity of the decaying particle is even (odd), only odd (even) values of M contribute.

[13] See Eq. (9′) of Ref. 7. All helicity states in this paper are defined with the "conventional" z axis along the particle's momentum and the "conventional" y axis normal to the decay plane.

[14] See also J. Werle, Nucl. Phys. **49**, 433 (1963). Our phase conventions are different.

For a one-baryon, two-pion state the appropriate eigenstates of parity are

$$(1/\sqrt{2})(|\omega_1,\lambda_1;\,\omega_2;\,\omega_3;\,jmM\rangle \pm(-1)^M|\omega_1,\,-\lambda_1;\,\omega_2;\,\omega_3;\,jmM\rangle). \quad (17)$$

Either parity case will give the same angular distribution, since states of different helicities are orthogonal.

One of the Momenta as an Analyzer

The basic quantum states (5) which we have introduced are labeled by Euler angles which refer to the direction of the normal. One could just as well consider these three angles as defining the direction of one of the three momenta, q_1 say, and a further rotation of q_2 around q_1. One can follow the same steps and obtain a formula identical to (11) for the angular distribution of one of the momenta. The functions R_M will, of course, be different. Equation (14) is still valid and gives the polarization of the decaying particle in terms of the distribution of one of the momenta. If the analysis in terms of the normal turns out to be a little easier to work through, it is due to the simple form in which parity conservation is expressed. For a three-pion decay, we simply have to eliminate either even or odd values of M. When the three Euler angles refer to one momentum it is found that (16) has to be replaced by the following relation:

$$P|\omega_1\omega_2\omega_3;\, jmM\rangle=(-1)^{j+M+1}|\omega_1\omega_2\omega_3;\, jm\,-M\rangle. \quad (15')$$

This approach is described in detail in Ref. 8. If the parity of the decaying particle is $(-1)^j$, the decay amplitudes R_M and R_{-M} are equal (opposite) if M is odd (even) and there is no $M=0$ amplitude. If the parity is $-(-1)^j$ the opposite assignment holds. For each M value, both parity states give the same angular distribution.

Identical Particles

The identity of two (or all three) particles will imply further relations among the decay amplitudes. In the examples considered in Secs. III and IV, for instance, they will apply when two π mesons have the same charge or are in an eigenstate of isotopic spin. If two identical particles are produced, the decay state has to be symmetrical (antisymmetrical) with respect to the exchange of the two particles according to their Bose-Einstein (Fermi-Dirac) statistics. In order to construct states with such a permutation property, we introduce a permutation operator P_{12} (exchange of particle 1 and 2 leaving 3 unchanged) and apply it on both sides of (9):

$$P_{12}|\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3;j,m,M\rangle=\int d\alpha\,\sin\beta d\beta d\gamma D_{mM}{}^{j}(\alpha\beta\gamma)P_{12}|\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3;\alpha\beta\gamma\rangle$$

$$=a\int d\alpha\,\sin\beta d\beta d\gamma D_{mM}{}^{j*}(\alpha\beta\gamma)|\omega_2\lambda_2;\omega_1\lambda_1;\omega_3\lambda_3;\alpha'\beta'\gamma'\rangle \quad (18)$$

with our phase conventions $a=e^{i\pi(\lambda_1+\lambda_2-\lambda_3)}$.

Note added in proof. The factor a in (18) is missing in an unpublished version of this work (Stanford Linear Accelerator Center Report No. 73). We wish to thank Professor J. Werle for calling this to our attention and for informing us of his general work which was inadvertently overlooked in the references given there.

The set of angles $\alpha'\beta'\gamma'$ which now appear in the ket vector no longer refers to the normal to the decay plane, since the direction of the normal is reversed when the two particles are interchanged. The angles referring to the normal are obtained through the transformation $\alpha'=\alpha+\pi$, $\beta'=\pi-\beta$, $\gamma'=2\pi-\gamma$. The rotation defined by the set of angles $\alpha+\pi$ and $\pi-\beta$ brings $\mathbf{q}_1+\mathbf{q}_2$ to a direction identical to the one obtained using α and β. A rotation of angle $2\pi-\gamma$ around the new normal then gives the same configuration as the one obtained with the set of angles α, β, and γ. Since we integrate over all rotation angles, we may replace the arguments of the D function and write (18) as

$$a\int d\alpha\,\sin\beta d\beta d\gamma D_{mM}{}^{j*}(\alpha-\pi,\,\pi-\beta,\,2\pi-\gamma)\times|\omega_2\lambda_2,\omega_1\lambda_1,\omega_3\lambda_3;\alpha\beta\gamma\rangle.$$

Transforming the D functions and using the definition of our state (9), we rewrite (18) as

$$\zeta|\omega_2\lambda_2,\,\omega_1\lambda_1,\,\omega_3\lambda_3;\,j,\,m,\,-M\rangle,$$

where $\zeta=(-1)^{j+2M+\lambda_1+\lambda_2-\lambda_3}=(-1)^{j+\lambda_1+\lambda_2+\lambda_3}$.

Since the decay states are symmetrical or antisymmetrical with respect to the exchange of the two particles, the amplitudes $F_M(\omega_1\lambda_1,\omega_2\lambda_2)$ will satisfy the relation

$$\zeta F_M(\omega_1\lambda_1,\omega_2\lambda_2)=\pm F_{-M}(\omega_2\lambda_2,\omega_1\lambda_1)\,, \quad (19)$$

where the sign is + for symmetrical and − for antisymmetrical decay states. When the identical particles are spin-zero mesons, the helicity indices are suppressed and we have in both cases

$$|F_M(\omega_1,\omega_2)|^2=|F_{-M}(\omega_2,\omega_1)|^2. \quad (20)$$

When integration over the whole Dalitz plot is performed according to (11), we find that opposite values of M give the same angular distribution for the normal to the decay plane, and therefore $R_M{}^-$ does not contribute.

III. DECAY INTO THREE SPINLESS PARTICLES

We now consider in more detail the decay of a particle of arbitrary integer spin j into three nonidentical pseudoscalar particles. At first we do not take into account any restrictions resulting from possible isotopic spin configurations.

The $2j+1$ *a priori* independent decay amplitudes are reduced by parity conservation according to (15) and we obtain the maximum number of independent amplitudes as shown in Table I. In the simplest cases we have one amplitude for 0^- and 1^-; two independent

TABLE I. The number of independent amplitudes describing the angular distribution of the three-pion decay of a spin j particle. The columns refer to the angular momentum j and the rows to the parity of the decaying particle.

	j even	j odd
Parity even	j	$j+1$
Parity odd	$j+1$	j

amplitudes for 1^+ and 2^+; three independent amplitudes for 2^- and 3^-, etc. This result may be obtained by other approaches, but not in such a simple way. We can, for example, exhibit sets of independent amplitudes written in terms of Cartesian tensors which, for the spin-1 and -2 cases, take the form

$$\begin{aligned}
&1^- \quad G\epsilon_{\mu\nu\rho\sigma}q_1{}^\nu q_2{}^\rho q_3{}^\sigma,\\
&1^+ \quad G_1(q_1+q_2)_\mu+G_2(q_1-q_2)_\mu,\\
&2^+ \quad (G_1(q_1+q_2)_\mu+G_2(q_1-q_2)_\mu)\epsilon_{\nu\rho\sigma\alpha}q_1{}^\rho q_2{}^\sigma q_3{}^\alpha,\\
&2^- \quad G_1(q_1{}^\mu q_1{}^\nu+q_2{}^\mu q_2{}^\nu)+G_2(q_1{}^\mu q_1{}^\nu-q_2{}^\mu q_2{}^\nu)\\
&\qquad\qquad +2G_3q_1{}^\mu q_2{}^\nu.
\end{aligned} \tag{21}$$

The G's which are the coefficients of the independent tensors are Lorentz-invariant quantities. They are assumed to be analytic functions of s, t, and u, the center-of-mass energies squared of the three particles taken two by two, i.e.,[15]

$$s=(q_3+q_1)^2,\quad u=(q_3+q_2)^2,\quad t=(q_1+q_2)^2. \tag{22}$$

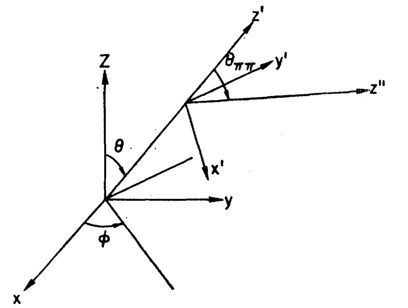

FIG. 3. ρ-π decay. The ρ momentum is taken along z'; the relative momentum of the decay pions is taken along z''.

The functions $|R_M|^2$ defined above will in general be linear combinations of products of two of the scalar invariants G_i with coefficients that are functions of s, t, or u.

Taking account of the conservation of parity, we next give the explicit expressions for the angular distribution of the normal to the decay plane. For the case of the decaying particle having spin and parity 1^- we have only the $M=0$ amplitude and $R_0{}^-=0$. The quantity $R_0{}^+$ is the common factor to the angular distribution, which, following (14), takes the form

$$\begin{aligned}
dN/d\Omega=R_0{}^+\{&(\rho_{11}+\rho_{-1-1})Z_{11}{}^{10}(\beta)+\rho_{00}Z_{00}{}^{10}(\beta)+2[\cos\alpha(\mathrm{Re}\rho_{10}-\mathrm{Re}\rho_{-10})-\sin\alpha(\mathrm{Im}\rho_{10}+\mathrm{Im}\rho_{-10})]Z_{10}{}^{10}(\beta)\\
&+2[\cos2\alpha(\mathrm{Re}\rho_{1-1})-\sin2\alpha(\mathrm{Im}\rho_{1-1})]Z_{1-1}{}^{10}(\beta)\}.
\end{aligned} \tag{23}$$

We readily get the Z functions from the table of d functions given in the Appendix and obtain

$$\begin{aligned}
dN/d\Omega=R_0{}^+\{&2\cos^2\beta\rho_{00}+\sin^2\beta(\rho_{11}+\rho_{-1-1})-2\sqrt{2}\sin\beta\cos\beta((\mathrm{Re}\rho_{10}-\mathrm{Re}\rho_{-10})\cos\alpha-(\mathrm{Im}\rho_{10}+\mathrm{Im}\rho_{-10})\sin\alpha)\\
&-2\sin^2\beta(\mathrm{Re}\rho_{1-1}\cos2\alpha-\mathrm{Im}\rho_{1-1}\sin2\alpha)\}.
\end{aligned} \tag{24}$$

This is a well-known result. The angular distribution determines six quantities (including the trace $\rho_{11}+\rho_{00}+\rho_{-1-1}$) of the spin-1 density matrix (this specifies the tensorial polarization), but leaves undetermined the three other terms (related to the vectorial polarization). The vectorial polarization is not determined because there is only one decay amplitude. The observation of the γ distribution would give nothing new.

We now turn to the pseudovector (1^+) case where there are two decay amplitudes corresponding to $M=\pm1$. The angular distribution is then a function of two terms, one proportional to $R_1{}^+$ and one proportional to $R_1{}^-$. It reads

$$\begin{aligned}
dN/d\Omega=R_1{}^+\{&(\rho_{11}+\rho_{-1-1})Z_{11}{}^{11+}(\beta)+\rho_{00}Z_{00}{}^{11+}(\beta)+2[\cos\alpha(\mathrm{Re}\rho_{10}-\mathrm{Re}\rho_{-10})-\sin\alpha(\mathrm{Im}\rho_{10}+\mathrm{Im}\rho_{-10})]Z_{10}{}^{11+}(\beta)\\
&+2[\cos2\alpha\,\mathrm{Re}\rho_{1-1}-\sin2\alpha\,\mathrm{Im}\rho_{1-1}]Z_{1-1}{}^{11+}(\beta)\}+R_1{}^-\{(\rho_{11}-\rho_{-1-1})Z_{11}{}^{11-}(\beta)\\
&+2[\cos\alpha(\mathrm{Re}\rho_{10}+\mathrm{Re}\rho_{-10})-\sin\alpha(\mathrm{Im}\rho_{10}-\mathrm{Im}\rho_{-10})]Z_{10}{}^{11-}(\beta)\}.
\end{aligned}$$

The Z functions are easily calculated, yielding the explicit expression

$$\begin{aligned}
dN/d\Omega=R_1{}^+\{&(\rho_{11}+\rho_{-1-1})\tfrac{1}{2}(1+\cos^2\beta)+\rho_{00}\sin^2\beta\\
&+\sqrt{2}\sin\beta\cos\beta((\mathrm{Re}\rho_{10}-\mathrm{Re}\rho_{-10})\cos\alpha-(\mathrm{Im}\rho_{10}+\mathrm{Im}\rho_{-10})\sin\alpha)+\sin^2\beta(\cos2\alpha\,\mathrm{Re}\rho_{1-1}-\sin2\alpha\,\mathrm{Im}\rho_{1-1})\\
&+R_1{}^-\{(\rho_{11}-\rho_{-1-1})\cos\beta+\sqrt{2}\sin\beta[\cos\alpha(\mathrm{Re}\rho_{10}+\mathrm{Re}\rho_{-10})-\sin\alpha(\mathrm{Im}\rho_{10}-\mathrm{Im}\rho_{-10})]\}.
\end{aligned} \tag{25}$$

Provided the two decay amplitudes $R_1{}^\pm$ are both different from zero, the density matrix can now be completely determined. One needs only the ratio of their absolute values.

[15] We use a metric such that $a\cdot b=a_0b_0-\mathbf{a}\cdot\mathbf{b}$.

The angular distribution of the normal to the decay plane for a spin-2 particle is obtained in the same way. The pertinent d^2 functions are given in the Appendix. For the 2^+ case where there are two independent decay amplitudes we obtain for the normal angular distribution

$$\begin{aligned}
dN/d\Omega=R_1^+\{&(\rho_{22}+\rho_{-2-2})(\sin^2\beta)\tfrac{1}{2}(1+\cos^2\beta)+(\rho_{11}+\rho_{-1-1})\tfrac{1}{2}(\cos^2\beta+\cos^2 2\beta)\\
&+\rho_{00}\tfrac{3}{4}\sin^2 2\beta-(\cos\alpha\,\mathrm{Re}(\rho_{21}-\rho_{-2-1})-\sin\alpha\,\mathrm{Im}(\rho_{21}+\rho_{-2-1}))\sin 2\beta\cos^2\beta\\
&-(\cos 2\alpha\,\mathrm{Re}(\rho_{20}+\rho_{-20})-\sin\alpha\,\mathrm{Im}(\rho_{20}-\rho_{-20}))\tfrac{1}{2}(\tfrac{3}{2})^{1/2}\sin^2 2\beta\\
&+(\cos 2\alpha\,\mathrm{Re}\rho_{1-1}-\sin 2\alpha\,\mathrm{Im}\rho_{1-1})(\cos^2\beta-\cos^2 2\beta)\\
&+(\cos\alpha\,\mathrm{Re}(\rho_{10}-\rho_{-10})-\sin\alpha\,\mathrm{Im}(\rho_{10}+\rho_{-10}))\tfrac{1}{2}(\tfrac{3}{2})^{1/2}\sin 4\beta\\
&+(\cos 3\alpha\,\mathrm{Re}(\rho_{-21}-\rho_{2-1})-\sin 3\alpha\,\mathrm{Im}(\rho_{-21}+\rho_{2-1}))\sin 2\beta\sin^2\beta\\
&-(\cos 4\alpha\,\mathrm{Re}\rho_{2-2}-\sin 4\alpha\,\mathrm{Im}\rho_{2-2})\sin^4\beta\}\\
&+R_1^-\{(\rho_{22}-\rho_{-2-2})\sin^2\beta\cos\beta+(\rho_{11}-\rho_{-1-1})\cos\beta\cos 2\beta\\
&-(\cos\alpha\,\mathrm{Re}(\rho_{21}+\rho_{-2-1})-\sin\alpha\,\mathrm{Im}(\rho_{21}-\rho_{-2-1}))\sin\beta(3\cos^2\beta-1)\\
&-(\cos 2\alpha\,\mathrm{Re}(\rho_{20}-\rho_{-20})-\sin 2\alpha\,\mathrm{Im}(\rho_{20}+\rho_{-20}))(\tfrac{3}{2})^{1/2}\sin 2\beta\sin\beta\\
&+(\cos\alpha\,\mathrm{Re}(\rho_{10}+\rho_{-10})-\sin\alpha\,\mathrm{Im}(\rho_{10}-\rho_{-10}))(\tfrac{3}{2})^{1/2}\sin 2\beta\cos\beta\\
&-(\cos 3\alpha\,\mathrm{Re}(\rho_{-21}+\rho_{2-1})-\sin 3\alpha\,\mathrm{Im}(\rho_{-21}-\rho_{2-1}))\sin\beta\sin^2\beta\}\,. \qquad (26)
\end{aligned}$$

For 2^- we have three decay amplitudes corresponding respectively to $M=\pm 2$ and 0 and thus the decay distribution will be a three-parameter expression. We use (14) and the d^2 functions given in the Appendix and obtain

$$\begin{aligned}
dN/d\Omega=R_2^+\{&(\rho_{22}+\rho_{-2-2})[\tfrac{1}{8}\sin^4\beta+\cos^2\beta]+(\rho_{11}+\rho_{-1-1})\tfrac{1}{2}\sin^2\beta(1+\cos^2\beta)+\rho_{00}\tfrac{6}{8}\sin^4\beta\\
&+(\cos\alpha\,\mathrm{Re}(\rho_{21}-\rho_{-2-1})-\sin\alpha\,\mathrm{Im}(\rho_{21}+\rho_{-2-1}))\tfrac{1}{4}\sin 2\beta(3+\cos^2\beta)\\
&+(\cos 2\alpha\,\mathrm{Re}(\rho_{20}+\rho_{-20})-\sin 2\alpha\,\mathrm{Im}(\rho_{20}-\rho_{-20}))(\tfrac{1}{4}\sqrt{6})\sin^2\beta(1+\cos^2\beta)\\
&+(\cos 2\alpha\,\mathrm{Re}\rho_{1-1}-\sin 2\alpha\,\mathrm{Im}\rho_{1-1})\sin^4\beta\\
&+(\cos\alpha\,\mathrm{Re}(\rho_{10}-\rho_{-10})-\sin\alpha\,\mathrm{Im}(\rho_{10}+\rho_{-10}))(\tfrac{1}{4}\sqrt{6})\sin 2\beta\sin^2\beta\\
&-(\cos 3\alpha\,\mathrm{Re}(\rho_{-21}-\rho_{2-1})-\sin 3\alpha\,\mathrm{Im}(\rho_{-21}+\rho_{2-1}))\tfrac{1}{4}\sin 2\beta\sin^2\beta\\
&+(\cos 4\alpha\,\mathrm{Re}\rho_{2-2}-\sin 4\alpha\,\mathrm{Im}\rho_{2-2})\tfrac{1}{4}\sin^4\beta\}\\
&+R_2^-\{(\rho_{22}-\rho_{-2-2})\tfrac{1}{2}\cos\beta(1+\cos^2\beta)+(\rho_{11}-\rho_{-1-1})\sin^2\beta\cos\beta\\
&+(\cos\alpha\,\mathrm{Re}(\rho_{21}+\rho_{-2-1})-\sin\alpha\,\mathrm{Im}(\rho_{21}-\rho_{-2-1}))\tfrac{1}{2}\sin\beta(1+3\cos^2\beta)\\
&+(\cos 2\alpha\,\mathrm{Re}(\rho_{20}-\rho_{-20})-\sin 2\alpha\,\mathrm{Im}(\rho_{20}+\rho_{-20}))(\tfrac{1}{2}\sqrt{6})\sin^2\beta\cos\beta\\
&+(\cos\alpha\,\mathrm{Re}(\rho_{10}+\rho_{-10})-\sin\alpha\,\mathrm{Im}(\rho_{10}-\rho_{-10}))(\tfrac{1}{2}\sqrt{6})\sin^3\beta\\
&+(\cos 3\alpha\,\mathrm{Re}(\rho_{-21}+\rho_{2-1})-\sin 3\alpha\,\mathrm{Im}(\rho_{-21}-\rho_{2-1}))\tfrac{1}{2}\sin^3\beta\}\\
&+R_0\{(\rho_{22}+\rho_{-2-2})\tfrac{6}{8}\sin^4\beta+(\rho_{11}+\rho_{-1-1})\tfrac{3}{4}\sin^2 2\beta+\rho_{00}(2+\tfrac{9}{2}\sin^4\beta-6\sin^2\beta)\}\\
&-(\cos\alpha\,\mathrm{Re}(\rho_{21}-\rho_{-2-1})-\sin\alpha\,\mathrm{Im}(\rho_{21}+\rho_{-2-1}))\tfrac{3}{4}\sin 2\beta\sin^2\beta\\
&+(\cos 2\alpha\,\mathrm{Re}(\rho_{20}+\rho_{-20})-\sin 2\alpha\,\mathrm{Im}(\rho_{20}-\rho_{-20}))(\tfrac{1}{2}\sqrt{6})\sin^2\beta(3\cos^2\beta-1)\\
&-(\cos 2\alpha\,\mathrm{Re}\rho_{1-1}-\sin 2\alpha\,\mathrm{Im}\rho_{1-1})6\sin^2\beta\cos^2\beta\\
&-(\cos\alpha\,\mathrm{Re}(\rho_{10}-\rho_{-10})-\sin\alpha\,\mathrm{Im}(\rho_{10}+\rho_{-10}))(\tfrac{1}{2}\sqrt{6})\sin 2\beta(3\cos^2\beta-1)\\
&-(\cos 3\alpha\,\mathrm{Re}(\rho_{-21}-\rho_{2-1})-\sin 3\alpha\,\mathrm{Im}(\rho_{-21}+\rho_{2-1}))\tfrac{3}{2}\sin^2\beta\sin 2\beta\\
&+(\cos 4\alpha\,\mathrm{Re}\rho_{2-2}-\sin 4\alpha\,\mathrm{Im}\rho_{2-2})\tfrac{3}{2}\sin^4\beta\}\,. \qquad (27)
\end{aligned}$$

When two particles are identical, integrating over the Dalitz plot averages to zero those terms proportional to R^- and the resulting expressions reduce to those given by Dennery and Krzywicki.[16] It is, however, possible to average separately over parts of the Dalitz plot ($\omega_1>\omega_2$ and $\omega_2>\omega_1$, say) and thereby allow for nonzero contributions from terms proportional to R^-.

Should resonances with higher spin be observed, explicit angular distributions of the normal to the decay plane could be readily obtained from the Legendre polynomial of order j, $P_j(\cos\beta)$ using the following relations[12]:

$$\begin{aligned}
d_{m,m'\pm1}{}^j(\beta)&=((j\pm m'+1)(j\mp m'))^{-1/2}\{(-m/\sin\beta)+m'\cot\beta\mp(\partial/\partial\beta)\}d_{m'm}{}^j(\beta)\\
d_{00}{}^j(\beta)&=P_j(\cos\beta)\,;\quad d_{m'm}{}^j(\beta)=(-1)^{m'-m}d_{-m'-m}{}^j(\beta)=(-1)^{m'-m}d_{mm'}{}^j(\beta)\,. \qquad (28)
\end{aligned}$$

[16] P. Dennery and A. Krzywicki, Phys. Rev. **136**, B839 (1964).

Relations (24), (25), (26), and (27) are somewhat more complicated than necessary since they correspond to the most general density matrix. In many practical cases the production mechanism is such that the density matrix has many symmetries when referred to particular axes and many of the terms written in (24–27) will not appear. On the other hand an observation of the presence or absence of particular terms in (24–27) would give information on the production process.[3] In this respect we recall the relations which express parity conservation in a two-body production process, when the initial beam and target are not polarized. If the z axis to which the density matrix is referred is chosen normal to the production plane, parity conservation in the production process yields

$$\rho_{m'm}=0 \quad \text{if} \quad m-m' \text{ is odd}. \tag{29}$$

If the z axis is along the resonance momentum in the center-of-mass system, parity conservation in the production processs yields[17]

$$\rho_{m'm}=(-1)^{m'-m}\rho_{-m'-m}. \tag{30}$$

This last choice of density matrix has the advantage of being invariant under special Lorentz transformations along the resonance momentum, i.e., when one passes from the production c.m. system to the decay c.m. system.[18]

We now consider the implication of the identity of the π mesons. If two of the π mesons are identical, i.e., have the same charge or are in a state of well-defined isotopic spin, we have shown in the preceding section that

$$F_M(\omega_1,\omega_2,\omega_3)=\pm(-1)^j F_{-M}(\omega_2,\omega_1,\omega_3),$$

the sign being + or − according as the states are symmetrical or antisymmetrical with respect to the exchange of the two particles. It follows that

$$R_M^+=\tfrac{1}{2}(|F_M|^2+|F_{-M}|^2)$$

and

$$R_M^-=\tfrac{1}{2}(|F_M|^2-|F_{-M}|^2)$$

are, respectively, symmetric and antisymmetric functions of $(\omega_1-\omega_2)$ or of $(s-u)$. An antisymmetric function does not contribute when the distribution is integrated over the Dalitz plot (11). In order to observe terms proportional to R_M^{-2}, and determine all parts of the decaying particle's density matrix, it is necessary to define the normal to the decay plane according to the different energies of the two identical particles. As mentioned above this corresponds to summing twice, over the halves of the Dalitz plot with $\omega_1>\omega_2$ and $\omega_1<\omega_2$.

In many cases the symmetric function will be dominant, since the simplest symmetric function is 1, while the simplest antisymmetric one is $(s-u)/M^2$, where M is a phenomenological parameter with the dimension of a mass. In any simple model this mass would be of the order of the inverse range of the interaction. If the range is short, i.e., if vector mesons play a dominant role,[19] the average energy of each particle could be less than the inverse range (depending, of course, on how heavy the decay particle is) and the antisymmetric term would then be quenched by the centrifugal barrier effect as opposed to the dominant symmetric one.

Furthermore, when the decay amplitude is written in terms of Cartesian tensors such as (21), as is usually the case when dealing with a particular model, the antisymmetric term vanishes when the different tensor amplitudes have the same phase, i.e., are relatively real. This can be seen as follows: If the spin is j, the decay amplitude is written as a Cartesian tensor of order j. It is constructed with the two linearly independent vectors available, for instance $q=q_1-q_2$ and $p=q_1+q_2$, where q_1 and q_2 are the four-momenta of the two identical pions. The decay amplitude is a linear combination of monomial expression of the type

$$G_k p_{i_1} p_{i_2} \cdots p_{i_n} q_{i_{n+1}} \cdots q_{i_J}. \tag{31}$$

The density-matrix element constructed in tensor form $\rho_{i_1\cdots i_J,j_1\cdots j_J}$ contributes to the angular distribution a term

$$\sum_{kl} G_k G_l^* p_{i_1}\cdots q_{i_J} p_{j_1}\cdots q_{j_J}\rho_{i_1\cdots i_J,j_1\cdots j_J},$$

where the indices of the sets $\{i\}$ and $\{j\}$ running from 1 to 3 refer either to p or q components, depending on the subscript $k, l, \cdots$. We can apply the Hermitian property of the density matrix to write the decay distribution as

$$\tfrac{1}{2}\sum_{kl}\{\mathrm{Re}[G_kG_l^*][(p_{i_1}\cdots q_{i_J}p_{j_1}\cdots q_{j_J})+(p_{j_1}\cdots q_{j_J})(p_{i_1}\cdots q_{i_J})]\,\mathrm{Re}\rho_{i_1\cdots i_J,j_1\cdots j_J}$$
$$-\mathrm{Im}[G_kG_l^*][(p_{i_1}\cdots q_{i_J}p_{j_1}\cdots q_{j_J})-(p_{j_1}\cdots q_{j_J})(p_{i_1}\cdots q_{i_J})]\,\mathrm{Im}\rho_{i_1\cdots i_J,j_1\cdots j_J}\}.$$

Using the fact that the whole decay amplitude is symmetrical with respect to the exchange of the two identical particles we have that if G_k is symmetrical (antisymmetrical), the associated tensor contains a component of q an even (odd) number of times. Inspection then shows that odd powers of components of

[17] We use the $(\varphi,\theta,0)$ representation of Ref. 7.

[18] This follows from the property that the generator of a Lorentz transformation along a particular axis M_{i4} commutes with the component of the angular momentum along that axis, i.e., $[M_{jK},M_{i4}]=0$.

[19] See, for example, M. Gell-Mann, D. Sharp, and W. Wagner, Phys. Rev. Letters 8, 71 (1962).

the normal to the decay plane, i.e., terms of the form $n_k = p_i q_j - q_i p_j$, which correspond to terms linear in $\cos\beta$ or $\sin\beta$ in the angular distribution, are obtained only in the terms proportional to $\mathrm{Im}\{G_k, G_l^*\}$.

In order to determine fully the decaying particle's density matrix, we see that it is necessary to have amplitudes of different phases. This is necessarily the case in three-pion decays when a two-pion resonance (the ρ meson) can actually be produced.

To illustrate this point, we consider the decay of a pseudovector particle A into a $\rho\pi$ state with the subsequent decay of the ρ into two pions (Fig. 3). To be more specific we consider a $A^{\mp} \to \pi^{\pm} + \pi^{\mp} + \pi^{\mp}$ decay. We introduce the unsymmetrized $A\rho\pi$ decay amplitude as

$$g_1 \epsilon_A \cdot \epsilon_\rho + g_2 (\epsilon_A \cdot q_2)(\epsilon_\rho \cdot q_2)$$

and a $\rho\pi\pi$ decay amplitude

$$g \epsilon_\rho (q - q_1).$$

q_1 and q_2 are the momenta of the two identical pions, and ϵ_A and ϵ_ρ, respectively, stand for the linear polarization vectors of the A and ρ mesons. The $A \to 3\pi$ decay amplitude can be expressed, after the proper symmetrization, as

$$\epsilon_A \cdot g \left\{ \frac{g_1(q-q_1) + g_2 q_2 q_2 \cdot (q-q_1)}{(q+q_1)^2 - m_\rho^2} + \frac{g_1(q-q_2) + g_2 q_1 q_1 \cdot (q-q_2)}{(q+q_2)^2 - m_\rho^2} \right\}.$$

This last expression is of the form

$$\epsilon_A \cdot (G_1(s,t,u)(q_1+q_2) + G_2(s,t,u)(q_1-q_2)), \quad (32)$$

where G_1 (G_2) is a symmetrical (antisymmetrical) function with respect to the exchange of s and u. In (32) the mass of the ρ is actually complex and we write m_ρ^2 as $M_\rho^2 + 2iM_\rho\Gamma_\rho$ where M_ρ and Γ_ρ are the ρ mass and width. In terms of the coupling constants g_1 and g_2 one finds for the interference term the covariant expression

$$\begin{aligned} \mathrm{Im}\{G_1^* G_2\} = &(2M_\rho \Gamma_\rho (s-u) / |(s-m_\rho^2)(u-m_\rho^2)|) \\ &\times \{g_2^2[(K\cdot q)^2 - (K\cdot p - 2q_1\cdot q_2 - \mu^2)^2] \\ &+ 3g_1^2 + 2g_1 g_2 (K\cdot p - 2q_1 \cdot q_2 - \mu^2)\}, \end{aligned}$$

where $K = q + q_1 + q_2$ is the A-meson momentum. The term R_1^- in Eq. (25) is proportional to the interference term $\mathrm{Im}(G_1 G_2^*)$. The interference term will be non-negligible on the ρ bands as compared with a symmetric $|G_1^+|^2$ term, except on that part of the ρ bands which actually cross over within the Dalitz plot. The non-crossover ρ bands contain the events useful for determining the vectorial polarization of the A particle.

Vector-Meson-Pion Decay

Since meson resonances appear to play a dominant role in elementary-particle interactions, a three-meson decay may often be considered as two successive two-body decays, two of the mesons being the decay products of a meson resonance produced together with the third one. Decays of this type have been already observed,[5] and we now consider in some detail an example of such a process (Fig. 3).

To illustrate the argument we consider a parity-conserving decay where the intermediate two-meson resonance is a vector meson and where the initial decaying state has a definite angular momentum. In order to construct a state of well-defined parity we use the result of applying the parity operator to a two-body helicity state given by Eq. (41) of Ref. 7, i.e.,

$$P|jm\lambda\rangle = \eta_1\eta_2(-1)^{j-s_1-s_2}|jm, -\lambda\rangle. \quad (33)$$

Therefore a decay state of well-defined parity can be expressed as

$$\sum_{\lambda \geq 0} F_\lambda (1/\sqrt{2})(|jm\lambda\rangle + \epsilon(-1)^j |jm, -\lambda\rangle), \quad (34)$$

where j is the spin of the parent decaying particle, m its component on a fixed axis, λ the helicity of the vector meson, and ϵ the relative parity of the vector meson and parent decaying particle. The sum in (34) extends over only two values of λ, $\lambda = 1$ (or -1) and 0.

It follows from (33) that for either choice of parity a vector-meson helicity of ± 1 is allowed, while the helicity 0 state is allowed only when $\epsilon = (-1)^j$. If the vector meson is a ρ (negative parity), the helicity state $\lambda = 0$ is allowed for the assignments $1^+, 2^-, 3^+, \cdots$, for the parent decaying particle. Turning now to the two-spinless-particle decay mode of the vector meson, we see that states with $\lambda = \pm 1$ and 0 give different angular distributions. When the angular distribution is referred to the vector-meson line of flight as a polar axis and averaged azimuthally, one finds, respectively, for the cases $\lambda = \pm 1$ and $\lambda = 0$ (in the vector-meson rest frame) angular distributions of the form

$$\sin^2\theta_{\pi\pi} \quad \text{or} \quad \cos^2\theta_{\pi\pi}.$$

This is true independently of the parent decaying particle's state of polarization or alignment.

A $\cos^2\theta_{\pi\pi}$ term allows for the occurrence of events with the three mesons along the same line in the parent decaying particle's rest frame, and its occurrence would show that the relative parity to the vector meson is $(-1)^j$. Taking into account the negative parity of the ρ meson yields a parity $(-1)^{j+1}$ for the parent particle decaying into an intermediate $\rho\pi$ state.[20]

To complete this discussion we give in (35) the angular distribution obtained from (34). The method for arriving at this expression follows the derivation of Eq. (38) given below.

[20] For a more detailed discussion of sequential decays see S. U. Chung, University of California Radiation Laboratory Report No. 11899 (unpublished); J. Button-Shafer, UCRL Report No. 11903 (unpublished); S. M. Berman and M. Jacob, Stanford Linear Accelerator Center Report No. 43 (unpublished).

The angular distribution of the vector meson in the parent-meson rest frame is then

$$I(\theta,\varphi)=\frac{1}{4}\sum_{mm'}\{\cos(m-m')\varphi\,\mathrm{Re}[\rho_{mm'}+(-1)^{m-m'}\rho_{-m-m'}]-\sin(m-m')\varphi\,\mathrm{Im}[\rho_{mm'}-(-1)^{m-m'}\rho_{-m-m'}]\}$$
$$\times[|F_1|^2Z_{mm'}{}^{j1+}(\theta)+2|F_0|^2Z_{mm'}{}^{j0+}(\theta)].\quad(35)$$

IV. ISOBAR TWO- AND THREE-BODY DECAYS

We consider next the decay of a particle of arbitrary half-integer spin j into a spin-$\frac{1}{2}$ hyperon and two π mesons. Parity is assumed to be conserved in the decay and hence the decay state corresponding to a pure spin state $J_z=m$ is written, according to (17), as

$$\sum_M F_M(|j,m,M\tfrac{1}{2}\rangle+\epsilon(-1)^M|j,m,M,-\tfrac{1}{2}\rangle),\quad(36)$$

where ϵ stands for the parity of the decaying particle, relative to the decay baryon. M takes all half-integer values such that $-j\leq M\leq j$.

Since all M values may appear in the expression obtained for the angular distribution of the normal to the decay plane, this distribution will appear slightly more complicated than the one obtained in the 3π case. Nevertheless, the *a priori* unknown parameters—the $2j+1$ decay amplitudes and the density matrix elements which describe the polarization and alignment of the decaying particle—also predict the polarization state of the daughter hyperon. Its density matrix can in turn be fully determined from the knowledge of the decay asymmetries.

Since this approach using the helicity formalism generalizes the derivation of well-known relations for two-body decays to three-body decays, we first briefly introduce our method for the two-body case. Many of these results are already known[21] but have not been given in the same concise and simple form presented here. Furthermore, in many practical cases two-body and three-body decays occur with similar branching ratios ($Y_1^*\to\Lambda\pi$, $Y_1^*\to\Lambda\pi\pi$, and $Y_0^*\to\Sigma+\pi$, $Y_0^*\to\Lambda\pi\pi\cdots$) and it may be useful to have the various decay distributions compiled together, since both cases refer to the same set of density matrices.

Consider now the parity-conserving two-body decay of a particle into a hyperon and a pseudoscalar meson. From Eq. (33) we find that parity conservation implies that the decay state corresponding to a pure spin state ($J_z=m$) takes the form[7]

$$(1/\sqrt{2})(|jm,\tfrac{1}{2}\rangle+\epsilon(-1)^{j+\frac{1}{2}}|jm,-\tfrac{1}{2}\rangle).\quad(37)$$

There is only one amplitude associated with a parity-conserving decay. It follows from (37) that the angular distribution and the longitudinal polarization of the decay hyperon depend on the angular momentum, and on the polarization state of the decaying particle, but not on the relative parity ϵ. However, the transverse polarization, which is an interference term between the two helicity states, changes sign with ϵ, which is a well-known result.[22] We take the z axis and the hyperon momentum (in the isobar rest frame) to define a decay plane, and will consider the polarization vector of the final hyperon in this plane (Fig. 4). Our phase conventions for two-body decays are those of Ref. 7. The polarization is described with the conventional z and y axes, respectively, taken along the z' and $\mathbf{z}\times\mathbf{z}'$ axes.

From (37) one readily finds the angular distribution of the daughter hyperon

$$I(\theta,\varphi)=|F|^2\times\frac{1}{2}\sum_{mm'}\rho_{mm'}\{D_{m\frac{1}{2}}{}^{j}(\varphi,\theta,0)D_{m\frac{1}{2}}{}^{j*}(\varphi,\theta,0)+D_{m'-\frac{1}{2}}{}^{j}(\varphi,\theta,0)D_{m-\frac{1}{2}}{}^{j*}(\varphi,\theta,0)\},$$
$$=\frac{1}{2}\sum_{mm'}(\mathrm{Re}\rho_{mm'}\cos(m-m')\varphi-\mathrm{Im}\rho_{mm'}\sin(m-m')\varphi)Z_{m'm}{}^{j\frac{1}{2}+}(\theta),$$
$$=\frac{1}{4}\sum_{mm'}\{\cos(m-m')\varphi\,\mathrm{Re}(\rho_{mm'}+(-1)^{m-m'}\rho_{-m-m'})$$
$$-\sin(m-m')\varphi\,\mathrm{Im}(\rho_{mm'}+(-1)^{m-m'}\rho_{-m-m'})\}Z_{m'm}{}^{j\frac{1}{2}+}(\theta).\quad(38)$$

For the longitudinal polarization, i.e., the expectation value of the helicity, we have merely to replace $Z^+(\theta)$ by $Z^-(\theta)$ and thus we obtain

$$p_L\times I(\theta,\varphi)=\frac{1}{4}|F|^2\sum_{mm'}\{\cos(m-m')\varphi\,\mathrm{Re}(\rho_{mm'}-(-1)^{m-m'}\rho_{-m-m'})$$
$$-\sin(m-m')\varphi\,\mathrm{Im}(\rho_{mm'}+(-1)^{m-m'}\rho_{-m-m'})\}Z_{m'm}{}^{j\frac{1}{2}-}(\theta).\quad(39)$$

The longitudinal polarization given by (39) vanishes if the isobar is not polarized. Even for a polarized isobar the longitudinal polarization of the hyperon is zero when averaged over the angular distribution, since $Z^{\frac{1}{2}-}(\pi-\theta)=-Z^{\frac{1}{2}-}(\theta)$.

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[21] N. Byers and S. Fenster, Phys. Rev. Letters **11**, 52 (1963).
[22] R. Gatto and H. Stapp, Phys. Rev. **121**, 1553 (1961).

The transverse polarization, in the zz' plane, reads

$$p_T\times I(\theta,\varphi)=\epsilon(-1)^{j+\frac{1}{2}}(|F|^2/2)\sum_{mm'}\rho_{mm'}\{D_{m'\frac{1}{2}}{}^j(\varphi,\theta,0)D_{m-\frac{1}{2}}{}^{j*}(\varphi,\theta,0)+D_{m'-\frac{1}{2}}{}^j(\varphi,\theta,0)D_{m\frac{1}{2}}{}^{j*}(\varphi,\theta,0)\}\,,$$

$$=\epsilon(-1)^{j+\frac{1}{2}}(|F|^2/2)\sum_{mm'}(\mathrm{Re}\rho_{mm'}\cos(m-m')\varphi-\mathrm{Im}\rho_{mm'}\sin(m-m')\varphi)X_{m'm}{}^j(\theta)\,,$$

where

$$X_{m'm}{}^j(\theta)=d_{m'\frac{1}{2}}{}^j(\theta)d_{m-\frac{1}{2}}{}^j(\theta)+d_{m'-\frac{1}{2}}{}^j(\theta)d_{m\frac{1}{2}}{}^j(\theta)\,.$$

With the relation $X_{-m'-m}{}^j(\beta)=(-1)^{m'+m}X_{m'm}{}^j(\beta)$ we write

$$p_T\times I(\theta,\varphi)=\epsilon(-1)^{j+\frac{1}{2}}(|F|^2/4)\sum_{mm'}\{\cos(m-m')\varphi\,\mathrm{Re}(\rho_{mm'}+(-1)^{m+m'}\rho_{-m-m'})$$
$$-\sin(m-m')\varphi\,\mathrm{Im}(\rho_{mm'}-(-1)^{m+m'}\rho_{-m-m'})\}X_{m'm}{}^j(\theta)\,. \quad (40)$$

Examination of Eq. (40) shows that the transverse polarization also vanishes if the isobar is not polarized, and furthermore, that if the azimuthal angle φ is not observed, only diagonal terms of the density matrix contribute.

The simplicity of the method is related to the fact that the ratio of the helicity amplitudes does not change when transformed from the isobar rest system to the rest frame of the hyperon.

For a specific illustration, we give the above decay distributions obtained for the decay of a spin-$\frac{1}{2}$ and of a spin-$\frac{3}{2}$ isobar. The $Z^\pm$ and X functions are obtained from the values of the d function given in the Appendix. In order to give relatively simple expressions we average over the φ angle. The effect of any other density-matrix elements whose contributions have been averaged out can be obtained in a straightforward way if this azimuthal average is not performed.

For $j=\frac{1}{2}$ we have the well-known results

$$\begin{aligned}I(\theta)&=\int_0^{2\pi}I(\theta,\varphi)d\varphi=2\pi|F|^2\tfrac{1}{4}\,,\\ p_L\times I(\theta)&=2\pi|F|^2\tfrac{1}{4}(\rho_{\frac{1}{2}\frac{1}{2}}-\rho_{-\frac{1}{2}-\frac{1}{2}})\cos\theta\,,\\ p_T\times I(\theta)&=2\pi|F|^2\tfrac{1}{4}\epsilon(\rho_{\frac{1}{2}\frac{1}{2}}-\rho_{-\frac{1}{2}-\frac{1}{2}})\sin\theta\,.\end{aligned}\quad (41)$$

In the $j=\frac{3}{2}$ case, it reads

$$\begin{aligned}I(\theta)&=2\pi|F|^2\tfrac{1}{16}\{(\rho_{\frac{1}{2}\frac{1}{2}}+\rho_{-\frac{1}{2}-\frac{1}{2}})(1+3\cos^2\theta)\\&\qquad+(\rho_{\frac{3}{2}\frac{3}{2}}+\rho_{-\frac{3}{2}-\frac{3}{2}})3\sin^2\theta\}\,,\\ p_LI(\theta)&=\frac{2\pi}{16}|F|^2\{(\rho_{\frac{1}{2}\frac{1}{2}}-\rho_{-\frac{1}{2}-\frac{1}{2}})(9\cos^2\theta-5)\\&\qquad+3(\rho_{\frac{3}{2}\frac{3}{2}}-\rho_{-\frac{3}{2}-\frac{3}{2}})\sin^2\theta\}\cos\theta\,,\\ p_TI(\theta)&=-(2\pi/16)|F|^2\epsilon\{(\rho_{\frac{1}{2}\frac{1}{2}}-\rho_{-\frac{1}{2}-\frac{1}{2}})(9\cos^2\theta-1)\\&\qquad+3(\rho_{\frac{3}{2}\frac{3}{2}}-\rho_{-\frac{3}{2}-\frac{3}{2}})\sin^2\theta\}\sin\theta\,.\end{aligned}\quad (42)$$

At this point we may easily derive a useful result. From (39) and (40) we get the ratio of the expectation values of $\langle p_L(\theta,\varphi)I(\theta,\varphi)\cos\theta\rangle$ and $\langle p_T(\theta,\varphi)I(\theta,\varphi)\sin\theta\rangle$, where the angular bracket means average over all directions. We use the Clebsch-Gordan series expansion (12) together with the orthogonality property of the D functions. We find

$$\langle p_L(\theta,\varphi)I(\theta,\varphi)\cos\theta\rangle=\tfrac{1}{3}(|F|^2/2)\textstyle\sum_m(-1)^{m-\frac{1}{2}}\mathrm{Re}(\rho_{mm}-\rho_{-m-m})\times C(jj1|m,-m)C(jj1|\tfrac{1}{2},-\tfrac{1}{2})$$

and

$$\langle p_T(\theta,\varphi)I(\theta,\varphi)\sin\theta\rangle=(|F|^2/2)\tfrac{1}{3}\sqrt{2}\epsilon(-1)^{j+\frac{1}{2}}\textstyle\sum_m(-1)^{m+\frac{1}{2}}\times\mathrm{Re}(\rho_{mm}-\rho_{-m-m})C(jj1|m,-m)C(jj1|\tfrac{1}{2}\tfrac{1}{2})\,.$$

It follows that

$$R_1=\frac{\langle p_L(\theta,\varphi)I(\theta,\varphi)\cos\theta\rangle}{\langle p_T(\theta,\varphi)I(\theta,\varphi)\sin\theta\rangle}=-\epsilon(-1)^{j+\frac{1}{2}}\frac{C(jj1|\frac{1}{2}-\frac{1}{2})}{\sqrt{2}C(jj1|\frac{1}{2}\frac{1}{2})}\,.$$

The ratio of the two Clebsch-Gordan coefficients is readily obtained and we find

$$R_1=\epsilon(-1)^{j-\frac{1}{2}}[1/(2j+1)]\,.$$

It should be stressed, however, that the two quantities which appear in this ratio are both proportional to the parent-particle polarization. This result can be generalized to higher moments of the type illustrated below with the restriction that l be odd.[21] For example, we can calculate the ratios

$$R_l=\frac{\langle p_L(\theta,\varphi)I(\theta,\varphi)P_l(\cos\theta)\rangle}{\langle p_T(\theta,\varphi)I(\theta,\varphi)\mathcal{P}_l{}^1(\theta)\rangle}\,,$$

where

$$\mathcal{P}_l{}^m(\theta)=e^{-im\varphi}Y_l{}^m(\theta,\varphi)\left(\frac{4\pi}{2l+1}\right)^{1/2}\,.$$

In a similar way we find for the average longitudinal polarization

$$\langle p_L(\theta,\varphi)I(\theta,\varphi)P_l(\cos\theta)\rangle=1/(2l+1)\times(|F|^2/2)\textstyle\sum_m(-1)^{m-\frac{1}{2}}\mathrm{Re}(\rho_{mm}-\rho_{-m-m})\times C(jjl|m,-m)C(jjl|\tfrac{1}{2},-\tfrac{1}{2})\,,$$

which vanishes for even l, and for the average transverse

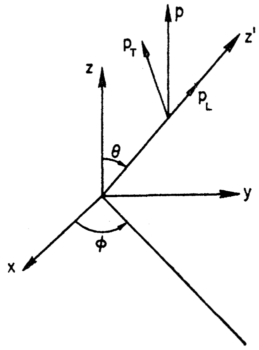

FIG. 4. Hyperon-pion decay. The decay-hyperon momentum is taken along z'. The vectors p_T, p_L, and p are the transverse, longitudinal, and total polarizations of the decay hyperon, respectively.

polarization

$$\langle p_T(\theta,\varphi)I(\theta,\varphi)\mathcal{P}_l^1(\theta)\rangle=\epsilon((-1)^{j+\frac{1}{2}}/[2l+1])\times(|F|^2/2)\sum_m(-1)^{m+\frac{1}{2}}\,\mathrm{Re}(\rho_{mm}-\rho_{-m-m})\times C(jjl|m,-m)C(jjl|\tfrac{1}{2},\tfrac{1}{2}).$$

It then follows that[21] R_l can be expressed as the ratio of Clebsch-Gordan coefficients

$$R_l=\epsilon(-1)^{j-\frac{1}{2}}\frac{C(jjl|\frac{1}{2},-\frac{1}{2})}{C(jjl|\frac{1}{2},\frac{1}{2})}=\epsilon(-1)^{j-\frac{1}{2}}\frac{(l(l+1))^{1/2}}{2j+1}.$$

Similar relations can also be obtained in the same simple way when the Legendre polynomials and Legendre functions are replaced by D functions. (One always obtains the ratio of two Clebsch-Gordan coefficients, but off-diagonal density-matrix elements are introduced.)

We now turn to the three-body decay into a spin-$\frac{1}{2}$ hyperon and two pseudoscalar mesons. The angular distribution of the normal to the decay plane is given by (13) and (14). This is a simple generalization of (38) where the normal to the decay plane replaces the momentum as an analyzer of the decaying-particle polarization. However, for a three-body decay into two spin-0 mesons and a spin-$\frac{1}{2}$ hyperon, there are in general $2j+1$ independent amplitudes, instead of one as in (38). The $2j+1$ decay amplitudes F_M are, in general, unknown functions of the invariant scalars s, t, and u. However, the kind of angular functions which arise in the normal angular distribution do not depend on the explicit form of the F_M but only on the parameter M. Just as in the case of 3π decays, if some of the decay products are in a fixed isospin state, then there can be some additional relations among the amplitudes F_M. For example, the two π mesons will be in a state of well-defined isotopic spin for the decay $Y_1^*(1660)\to\Lambda\ 2\pi$ (branching ratio 0.23), and for the decay $Y_0^*(1520)\to\Lambda\ 2\pi$ (branching ratio $0.16)\cdots$. The decay amplitudes F_M with opposite values of M are then related by (19) and, just as in the case treated above for the three-pion decay, the R_M^- amplitudes will vanish when summed over all energy configurations.

As an illustration of the general formula (14), we give the angular distribution of the normal obtained when the parent particle has angular momentum $\frac{3}{2}$. In order to make the resultant expression more compact we define the 12 quantities

$$C_1=\rho_{\frac{3}{2}\frac{3}{2}}+\rho_{-\frac{3}{2}-\frac{3}{2}},$$
$$C_2=\rho_{\frac{1}{2}\frac{1}{2}}+\rho_{-\frac{1}{2}-\frac{1}{2}},$$
$$C_3(\alpha)=\cos\alpha\,\mathrm{Re}(\rho_{\frac{3}{2}\frac{1}{2}}-\rho_{-\frac{3}{2}-\frac{1}{2}})-\sin\alpha\,\mathrm{Im}(\rho_{\frac{3}{2}\frac{1}{2}}+\rho_{-\frac{3}{2}-\frac{1}{2}}),$$
$$C_4(\alpha)=\cos2\alpha\,\mathrm{Re}(\rho_{\frac{3}{2}-\frac{1}{2}}+\rho_{-\frac{3}{2}\frac{1}{2}})-\sin2\alpha\,\mathrm{Im}(\rho_{\frac{3}{2}-\frac{1}{2}}-\rho_{-\frac{3}{2}\frac{1}{2}}),$$
$$C_5(\alpha)=\cos3\alpha\,\mathrm{Re}(\rho_{\frac{3}{2}-\frac{3}{2}}-\rho_{-\frac{3}{2}\frac{3}{2}})-\sin3\alpha\,\mathrm{Im}(\rho_{\frac{3}{2}-\frac{3}{2}}+\rho_{-\frac{3}{2}\frac{3}{2}}),$$
$$C_6(\alpha)=\cos\alpha\,\mathrm{Re}(\rho_{\frac{1}{2}-\frac{1}{2}}-\rho_{-\frac{1}{2}\frac{1}{2}})-\sin\alpha\,\mathrm{Im}(\rho_{\frac{1}{2}-\frac{1}{2}}+\rho_{-\frac{1}{2}\frac{1}{2}}),$$

and

$$C_1'=\rho_{\frac{3}{2}\frac{3}{2}}-\rho_{-\frac{3}{2}-\frac{3}{2}},$$
$$C_2'=\rho_{\frac{1}{2}\frac{1}{2}}-\rho_{-\frac{1}{2}-\frac{1}{2}},$$
$$C_3'(\alpha)=\cos\alpha\,\mathrm{Re}(\rho_{\frac{3}{2}\frac{1}{2}}+\rho_{-\frac{3}{2}-\frac{1}{2}})-\sin\alpha\,\mathrm{Im}(\rho_{\frac{3}{2}\frac{1}{2}}-\rho_{-\frac{3}{2}-\frac{1}{2}}),$$
$$C_4'(\alpha)=\cos2\alpha\,\mathrm{Re}(\rho_{\frac{3}{2}-\frac{1}{2}}-\rho_{-\frac{3}{2}\frac{1}{2}})-\sin2\alpha\,\mathrm{Im}(\rho_{\frac{3}{2}-\frac{1}{2}}+\rho_{-\frac{3}{2}\frac{1}{2}}),$$
$$C_5'(\alpha)=\cos3\alpha\,\mathrm{Re}(\rho_{\frac{3}{2}-\frac{3}{2}}+\rho_{-\frac{3}{2}\frac{3}{2}})-\sin3\alpha\,\mathrm{Im}(\rho_{\frac{3}{2}-\frac{3}{2}}-\rho_{-\frac{3}{2}\frac{3}{2}}),$$
$$C_6'(\alpha)=\cos\alpha\,\mathrm{Re}(\rho_{\frac{1}{2}-\frac{1}{2}}+\rho_{-\frac{1}{2}\frac{1}{2}})-\sin\alpha\,\mathrm{Im}(\rho_{\frac{1}{2}-\frac{1}{2}}-\rho_{-\frac{1}{2}\frac{1}{2}}).$$

In terms of these quantities, the angular distribution of the normal may be expressed as

$$\begin{aligned}dN/d\Omega=&\{C_1\tfrac{1}{4}((1+3\cos^2\beta)R_{\frac{3}{2}}^{+}+3\sin^2\beta R_{\frac{1}{2}}^{+})+C_2\tfrac{1}{4}(3\sin^2\beta R_{\frac{3}{2}}^{+}+(1+3\cos^2\beta)R_{\frac{1}{2}}^{+})\\&+(\sqrt{3}/2)C_3(\alpha)\sin2\beta(R_{\frac{3}{2}}^{+}-R_{\frac{1}{2}}^{+})+(\sqrt{3}/2)C_4(\alpha)\sin^2\beta(R_{\frac{3}{2}}^{+}-R_{\frac{1}{2}}^{+})\}\\&+\{C_1'\tfrac{1}{4}\cos\beta((\cos^2\beta+3)R_{\frac{3}{2}}^{-}+3\sin^2\beta R_{\frac{1}{2}}^{-})+C_2'\tfrac{3}{4}\cos\beta[\sin^2\beta R_{\frac{3}{2}}^{-}+(3\cos^2\beta-(5/3))R_{\frac{1}{2}}^{-}]\\&+(\sqrt{3}/2)C_3'(\alpha)\sin\beta((1+\cos^2\beta)R_{\frac{3}{2}}^{-}+(1-3\cos^2\beta)R_{\frac{1}{2}}^{-})+(\sqrt{3}/2)C_4'(\alpha)\cos\beta\sin^2\beta(R_{\frac{3}{2}}^{-}-3R_{\frac{1}{2}}^{-})\\&+\tfrac{1}{4}C_5'(\alpha)\sin\beta\sin^2\beta(R_{\frac{3}{2}}^{-}-3R_{\frac{1}{2}}^{-})+\tfrac{1}{4}C_6'(\alpha)\sin\beta((9\cos^2\beta-1)R_{\frac{1}{2}}^{-}+3\sin^2\beta R_{\frac{3}{2}}^{-})\}.\quad(43)\end{aligned}$$

Analysis of the three-body decay in terms of Eq. (43) would provide 16 different functions of α and β which can in principle fully determine the density matrix of the decaying particle.

We now turn to the polarization of the daughter hyperon. As follows from the way we decomposed the parity operation, where the z and y axes were defined to be along the hyperon momentum and along the normal to the decay plane, respectively, the state

$$(1/\sqrt{2})(|j,m,M,\tfrac{1}{2}\rangle+\epsilon(-1)^M|j,m,M,-\tfrac{1}{2}\rangle)$$

is an eigenstate of the spin component of the hyperon normal to the decay plane, with eigenvalue $\epsilon(-1)^{M-\frac{1}{2}}$. As usual, this polarization is defined in the hyperon rest system.

It follows from (36) that the expectation value of the polarization of the hyperon, normal to the decay plane, can be easily expressed in terms of the parent particle's density matrix. The polarization is defined as the expectation value of $\boldsymbol{\sigma}\cdot\hat{n}$, where $\hat{n}$ is a unit vector along the normal to the decay plane. In terms of the parent particle's density matrix $\rho_{mm'}$, the distribution of transverse polarization along the normal can be expressed as

$$p_T(dN/d\Omega)=\epsilon\sum_M(-1)^{M-\frac{1}{2}}|F_M|^2\sum_{mm'}\rho_{mm'}\times\{D_{m'M}{}^j(\alpha\beta0)D_{mM}{}^{j*}(\alpha\beta0)\}. \quad (44)$$

Just as for the angular distribution of the normal, we regroup terms with opposite values of M and obtain

$$\begin{aligned}p_T(dN/d\Omega)=\epsilon\sum_{M>0}(-1)^{M-\frac{1}{2}}\{R_M{}^+\sum_{mm'}[\mathrm{Re}\rho_{mm'}\cos(m-m')\alpha-\mathrm{Im}\rho_{mm'}\sin(m-m')\alpha]Z_{m'm}{}^{jM-}(\beta)\\+R_M{}^-\sum_{mm'}[\mathrm{Re}\rho_{mm'}\cos(m-m')\alpha+\mathrm{Im}\rho_{mm'}\sin(m-m')\alpha]Z_{m'm}{}^{jM+}(\beta)\}\\=\epsilon\sum_{M>0}(-1)^{M-\frac{1}{2}}\tfrac{1}{2}\{\sum_{mm'}[\mathrm{Re}(\rho_{mm'}+(-1)^{m+m'}\rho_{-m-m'})\cos(m-m')\alpha\\-\mathrm{Im}(\rho_{mm'}-(-1)^{m+m'}\rho_{-m-m'})\sin(m-m')\alpha]Z_{m'm}{}^{jM-}(\beta)R_M{}^+\\+[\mathrm{Re}(\rho_{mm'}-(-1)^{m+m'}\rho_{-m-m'})\cos(m-m')\alpha\\-\mathrm{Im}(\rho_{mm'}+(-1)^{m-m'}\rho_{-m-m'})\sin(m-m')\alpha]Z_{m'm}{}^{jM+}(\beta)R_M{}^-\}. \quad (45)\end{aligned}$$

In order to illustrate this general relation, we consider the case where the parent particle has angular momentum $\frac{3}{2}$. Employing the $Z^\pm$ functions already obtained for the normal angular distribution, we find

$$\begin{aligned}p_T(dN/d\Omega)=-\epsilon\{\tfrac{1}{4}C_1'\cos\beta[(\cos^2\beta+3)R_{\frac{3}{2}}{}^+-3\sin^2\beta\, R_{\frac{1}{2}}{}^+]+\tfrac{3}{4}C_2'\cos\beta[\sin^2\beta\, R_{\frac{3}{2}}{}^+-(3\cos^2\beta-(5/3))R_{\frac{1}{2}}{}^+]\\+\tfrac{1}{2}\sqrt{3}C_3'(\alpha)\sin\beta[(1+\cos^2\beta)R_{\frac{3}{2}}{}^++(3\cos^2\beta-1)R_{\frac{1}{2}}{}^+]+\tfrac{1}{2}\sqrt{3}C_4'(\alpha)\cos\beta\sin^2\beta(R_{\frac{3}{2}}{}^++3R_{\frac{1}{2}}{}^+)\\+\tfrac{1}{4}C_5'(\alpha)\sin\beta\sin^2\beta(R_{\frac{3}{2}}{}^++3R_{\frac{1}{2}}{}^+)+\tfrac{1}{4}C_6'(\alpha)\sin\beta[3\sin^2\beta\, R_{\frac{3}{2}}{}^+-(9\cos^2\beta-1)R_{\frac{1}{2}}{}^+]\\+C_1\tfrac{1}{4}(1+3\cos^2\beta)(R_{\frac{3}{2}}{}^--3\sin^2\beta\, R_{\frac{1}{2}}{}^-)+C_2\tfrac{1}{4}[3\sin^2\beta\, R_{\frac{3}{2}}{}^--(1+3\cos^2\beta)R_{\frac{1}{2}}{}^-]\\+\tfrac{1}{2}\sqrt{3}C_3(\alpha)\sin2\beta(R_{\frac{3}{2}}{}^-+R_{\frac{1}{2}}{}^-)+\tfrac{1}{2}\sqrt{3}C_4(\alpha)\sin^2\beta(R_{\frac{3}{2}}{}^-+R_{\frac{1}{2}}{}^-)\}. \quad (46)\end{aligned}$$

Equations (43) and (46) can be used to determine the spin and parity of the decaying isobar by fitting to the three-body data, or at least can be used to impose further consistency requirements when the two-body decay data are simultaneously analyzed in terms of (38), (39), and (40).

For example (46) when applied to the Y_1^* (1660) data should yield expressions of the same sign for the Λ and Σ, employing averages over both the Dalitz plot and the azimuthal angle of the normal if the Λ and Σ particles have the same parity. This comparison could be considered as an independent determination of the $\Sigma\Lambda$ relative parity, and generalizes to three-body decays—a result already known for two-body decays.[23]

If desired, the expectation value of the hyperon polarization along any other direction is readily obtained from (36). However, the polarization normal to the decay plane is the only component of polarization which does not vanish when an average is performed over γ.

Isobar-Pion Decay

Since a three-body decay of a high-mass isobar may proceed through an intermediate isobar-pion decay, we now consider, as in the case of the three-pion decay, two successive parity-conserving two-body decays[20] eventually producing a final three-body state of one spin-$\frac{1}{2}$ baryon and two spinless mesons. We restrict the arguments below to exclude any possible overlapping isobar bands, thus eliminating any possible ambiguities as to the kind of two-body decay. The decays

$$N^*(1688)\rightarrow N^*(1238)+\pi$$

and

$$\Xi^*(1810)\rightarrow\Xi^*(1530)+\pi$$

provide two such examples.[5] In both cases, one of the daughter particles is a decuplet member with angular momentum $\frac{3}{2}^+$. For the first step of this two-step process, parity conservation implies two independent decay amplitudes. Assuming that the intermediate particles are a spin-$\frac{3}{2}$ particle and a pseudoscalar particle, we find that the intermediate decay state corresponding to a pure spin state of the initial particle can be expressed as

$$(1/\sqrt{2})\{F_{3/2}(|j,m,\tfrac{3}{2}\rangle+\epsilon(-1)^{j-\frac{1}{2}}|j,m,-\tfrac{3}{2}\rangle)+F_{1/2}(|j,m,\tfrac{1}{2}\rangle+\epsilon(-1)^{j-\frac{1}{2}}|j,m,-\tfrac{1}{2}\rangle)\}, \quad (47)$$

where j is the angular momentum of the parent isobar and ϵ is the relative parity of the parent and daughter isobars. For the special case of the parent isobar having

[23] Ph. Meyer, J. Prentki, and Y. Yamagouchi, Phys. Rev. Letters 5, 442 (1960).

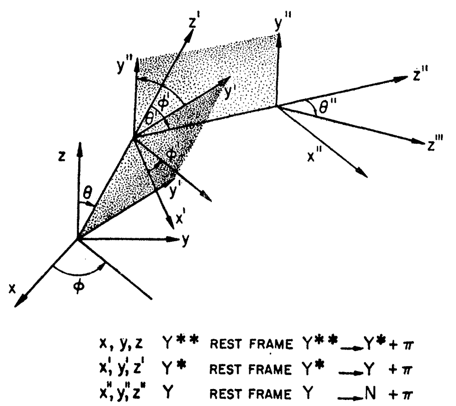

FIG. 5. The two-stage decay $Y^{**}\to Y^{*}+\pi$, $Y^{*}\to Y+\pi$. The coordinate system (x,y,z) is the rest frame associated with the Y^{**}. The Y^{*} momentum is along z', and (x',y',z') is the rest frame associated with the Y^{*}. The Y momentum is along z''. In addition, the coordinate system (x'',y'',z'') in the Y rest frame, used for the analysis of the final hyperon polarization, is indicated on the figure, where z''' is along the nucleon in the analyzing decay $Y\to N+\pi$. Note that the direction of the y axis remains invariant between any two successive frames of reference.

spin $\frac{1}{2}$ there is only one decay amplitude, and $F_{3/2}$ will not appear in (47).

The density matrix ρ' of the daughter isobar can be expressed in terms of the parent density matrix ρ as

$$\rho_{\mu\nu}'=F_\mu F_\nu^* \sum_{mm'} e^{i(m-m')\varphi} d_{m\mu}{}^j(\theta) d_{m'\nu}{}^j(\theta)\rho_{mm'}. \quad (48)$$

The density matrix ρ' is defined in terms of a coordinate system derived from the initial coordinate system in the parent-isobar rest frame by a rotation through angles θ and φ, the polar and azimuthal angles of the momentum of the daughter isobar in the parent-isobar system (Fig. 5). Parity conservation as expressed by equations of the form (47) then implies that for an unpolarized parent particle

$$\rho_{-\mu-\nu}'=\rho_{\mu\nu}'(-1)^{\mu-\nu}. \quad (49)$$

As follows from the transformation property of the helicity amplitudes under Lorentz transformation, the density matrix ρ' is the same in either the rest frame of the parent isobar or the rest frame of the daughter isobar. We note also that Eqs. (48) and (49) are valid for any spin of the daughter isobar.

If the daughter isobar subsequently has a two-body decay, its density matrix given by (48) may now be used directly in (38), (39), and (40) to express the resultant angular distributions. In particular, for the case of the daughter isobar having spin $\frac{3}{2}$, the density matrix (48) can be substituted directly in (42). The results obtained in the beginning of Sec. IV pertaining to two-body decays can now be applied directly to the daughter-isobar decay, notably the theorem on the ratio of transverse to longitudinal polarization.

The succession of reference frames used in the analysis of such a two-step process, followed by the eventual isobar decay into $Y+\pi$, is shown on Fig. 5.

It is perhaps by this last example of the two-stage decay that the simplicity of a method using helicity states is clearly demonstrated. The more traditional treatment would require recoupling coefficients to describe the second stage of the decay in terms of the parameters describing the first stage, a complication avoided in this presentation.

APPENDIX

We list together the d functions which are useful for the analysis of the decay of particles of spin less than or equal to 3. Not all the d functions are given. The missing ones are easily obtained using the simple symmetry relations

$$d_{m'm}{}^j(\beta)=(-1)^{m-m'}d_{-m'-m}{}^j(\beta),$$
$$d_{m'm}{}^j(\beta)=(-1)^{m-m'}d_{mm'}{}^j(\beta).$$

Several recurrent relations useful for the calculation of the d functions are given in the appendix of Ref. 7. More relations are given in Refs. 12 and 24.

The relevant d are now listed below.

Spin $\frac{1}{2}$:

$$d_{\frac{1}{2}\frac{1}{2}}(\beta)=\cos\tfrac{1}{2}\beta,\quad d_{-\frac{1}{2}\frac{1}{2}}(\beta)=\sin\tfrac{1}{2}\beta.$$

Spin 1:

$$d_{11}(\beta)=\tfrac{1}{2}(1+\cos\beta),\quad d_{01}(\beta)=(\sin\beta)/\sqrt{2},$$
$$d_{1-1}(\beta)=\tfrac{1}{2}(1-\cos\beta),\quad d_{00}(\beta)=\cos\beta.$$

Spin $\frac{3}{2}$:

$$d_{\frac{3}{2}\frac{3}{2}}(\beta)=\tfrac{1}{2}(1+\cos\beta)\cos\tfrac{1}{2}\beta,\quad d_{\frac{3}{2}\frac{1}{2}}(\beta)=-\tfrac{1}{2}\sqrt{3}(1+\cos\beta)\sin\tfrac{1}{2}\beta,$$
$$d_{\frac{3}{2}-\frac{1}{2}}(\beta)=\tfrac{1}{2}\sqrt{3}(1-\cos\beta)\cos\tfrac{1}{2}\beta,\quad d_{\frac{3}{2}-\frac{3}{2}}(\beta)=-\tfrac{1}{2}(1-\cos\beta)\sin\tfrac{1}{2}\beta,$$
$$d_{\frac{1}{2}\frac{1}{2}}(\beta)=\tfrac{1}{2}(3\cos\beta-1)\cos\tfrac{1}{2}\beta,\quad d_{\frac{1}{2}-\frac{1}{2}}(\beta)=-\tfrac{1}{2}(1+3\cos\beta)\sin\tfrac{1}{2}\beta.$$

Spin 2:

$$d_{22}(\beta)=\tfrac{1}{4}(1+\cos\beta)^2,\quad d_{21}(\beta)=-\tfrac{1}{2}(1+\cos\beta)\sin\beta,$$
$$d_{20}(\beta)=(\sqrt{6}/4)\sin^2\beta,\quad d_{2-1}(\beta)=-\tfrac{1}{2}(1-\cos\beta)\sin\beta,$$
$$d_{2-2}(\beta)=\tfrac{1}{4}(1-\cos\beta)^2,\quad d_{11}(\beta)=\tfrac{1}{2}(1+\cos\beta)(2\cos\beta-1),$$
$$d_{10}(\beta)=-(\tfrac{3}{2})^{1/2}\sin\beta\cos\beta,\quad d_{1-1}(\beta)=\tfrac{1}{2}(1-\cos\beta)(2\cos\beta+1),$$
$$d_{00}(\beta)=\tfrac{1}{2}(3\cos^2\beta-1).$$

[24] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

Spin $\frac{5}{2}$:

$$d_{\frac{5}{2}\frac{5}{2}}(\beta)=\tfrac{1}{4}(1+\cos\beta)^2\cos\tfrac{1}{2}\beta\,,\qquad d_{\frac{5}{2}\frac{3}{2}}(\beta)=-(5)^{1/2}\tfrac{1}{4}(1+\cos\beta)^2\sin\tfrac{1}{2}\beta\,,$$

$$d_{\frac{5}{2}\frac{1}{2}}(\beta)=(10^{1/2}/4)\sin^2\beta\cos\tfrac{1}{2}\beta\,,\qquad d_{\frac{5}{2}-\frac{1}{2}}(\beta)=-(10^{1/2}/4)\sin^2\beta\sin\tfrac{1}{2}\beta\,,$$

$$d_{\frac{5}{2}-\frac{3}{2}}(\beta)=5^{1/2}\tfrac{1}{4}(1-\cos\beta)^2\cos\tfrac{1}{2}\beta\,,\qquad d_{\frac{5}{2}-\frac{5}{2}}(\beta)=-\tfrac{1}{4}(1+\cos\beta)^2\sin\tfrac{1}{2}\beta\,,$$

$$d_{\frac{3}{2}\frac{3}{2}}(\beta)=\tfrac{1}{2}(5\cos\beta-3)\cos^3\tfrac{1}{2}\beta\,,\qquad d_{\frac{3}{2}\frac{1}{2}}(\beta)=(1/\sqrt{2})(-(5\cos\beta-1))\cos^2\tfrac{1}{2}\beta\sin\tfrac{1}{2}\beta\,,$$

$$d_{\frac{3}{2}-\frac{1}{2}}(\beta)=(1/\sqrt{2})(1+5\cos\beta)\sin^2\tfrac{1}{2}\beta\cos\tfrac{1}{2}\beta\,,\qquad d_{\frac{3}{2}-\frac{3}{2}}(\beta)=-\tfrac{1}{2}(5\cos\beta+3)\sin^3\tfrac{1}{2}\beta\,,$$

$$d_{\frac{1}{2}\frac{1}{2}}(\beta)=\tfrac{1}{2}(5\cos^2\beta-2\cos\beta-1)\cos\tfrac{1}{2}\beta\,,\qquad d_{\frac{1}{2}-\frac{1}{2}}(\beta)=-\tfrac{1}{2}(5\cos^2\beta+2\cos\beta-1)\sin\tfrac{1}{2}\beta\,.$$

Spin 3:

$$d_{33}(\beta)=\tfrac{1}{8}(1+\cos\beta)^3\,,\qquad d_{32}(\beta)=-(6^{1/2}/8)\sin\beta(1+\cos\beta)^2\,,$$

$$d_{31}(\beta)=(15^{1/2}/8)\sin^2\beta(1+\cos\beta)\,,\qquad d_{30}(\beta)=-(5^{1/2}/4)\sin^3\beta\,,$$

$$d_{3-1}(\beta)=(15^{1/2}/8)\sin^2\beta(1-\cos\beta)\,,\qquad d_{3-2}(\beta)=-(6^{1/2}/8)\sin\beta(1-\cos\beta)^2\,,$$

$$d_{3-3}(\beta)=\tfrac{1}{8}(1-\cos\beta)^3\,,\qquad d_{22}(\beta)=\tfrac{1}{4}(1+\cos\beta)^2(3\cos\beta-2)\,,$$

$$d_{21}(\beta)=-(5^{1/2}/4\sqrt{2})\sin\beta(3\cos^2\beta+2\cos\beta-1)\,,\qquad d_{20}(\beta)=(15^{1/2}/2\sqrt{2})\cos\beta\sin^2\beta\,,$$

$$d_{2-1}(\beta)=(5^{1/2}/4\sqrt{2})\sin\beta(3\cos^2\beta-2\cos\beta-1)\,,\qquad d_{2-2}(\beta)=\tfrac{1}{4}(1-\cos\beta)^2(3\cos\beta+2)\,,$$

$$d_{11}(\beta)=\tfrac{1}{8}(1+\cos\beta)(15\cos^2\beta-10\cos\beta-1)\,,\qquad d_{10}(\beta)=-(\sqrt{3}/4)\sin\beta(5\cos^2\beta-1)\,,$$

$$d_{1-1}(\beta)=\tfrac{1}{8}(1-\cos\beta)(15\cos^2\beta+10\cos\beta-1)\,,\qquad d_{00}(\beta)=(5\cos^3\beta-3\cos\beta)/2\,.$$

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Introduction to the *N*-Quantum Approximation in Quantum Field Theory*

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The N-quantum approximation is designed to find approximate operator solutions of theories characterized by a specific Hamiltonian. The Heisenberg field operators of the theory are approximated by finite-degree normal-ordered expansions in an irreducible set of in-fields. The c-number functions which are the coefficients of these expansions are the unknown quantities in the approximation. The approximation assumes that the dominant contributions to the vertex function, scattering function, and other low-order functions come from functions of similar low order. The c-number functions correspond to the connected graphs with a given number of external lines. Thus in graphical language the approximation assumes that the connected graphs with few external lines dominate. The N-quantum approximation is manifestly covariant, treats positive and negative frequencies in a symmetric way, allows a calculation of several different physical processes simultaneously, allows incorporation of bound states, and requires extrapolation off the mass shell in fewer variables than the usual Green's function approaches. After describing the N-quantum approximation, it is shown to be compatible with renormalization theory in first order of the approximation in the model with $\mathcal{L}_I=gA^3$. It should be emphasized that all powers of the coupling constant occur in first order of the N-quantum approximation in this model. A quadratic integral equation is obtained for the vertex function, and it is shown that the vertex function satisfies the renormalization criteria that the particles in the theory have a given observed mass, and that the vertex function has a given coupling constant as the residue of a pole $(m^2-k^2)^{-1}$ in the unphysical region. It is also shown that the power-series-expansion solution is finite term by term in all orders of the coupling constant.

1. INTRODUCTION

QUANTUM electrodynamics is the only quantum field theory which provides a quantitative description of relativistic particle interactions. Even this theory has rather restricted scope, since it applies only to purely electromagnetic interactions and is not valid when effects due to strong and weak interactions enter, for example, at high energies. After a number of unsuccessful attempts to treat specific theories of strong interactions without using perturbation theory, these attempts largely have been abandoned, and interest has shifted to approaches in which various general requirements, such as relativistic invariance, spectrum, locality,

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