Padé Approximant in Potential Scattering

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It is suggested that the Padé approximant be used as an approximation outside the radius of convergence of the Born series for scattering from a short-range potential free from a strong singularity. Following Weinberg's analysis of the Born series, one approximates the potential by a finite number of separable potentials in order to deal with a limited range of energy, and thus the associated K matrix can be evaluated without further approximation. When a number N of separable potentials are constructed by using the lowest $2N$ terms of the Born series, the result is identical to the (N,N) Padé approximant. If expanded formally into a power series in the potential strength, this approximation reproduces the original Born series exactly up to the $2N$th order; but without an expansion it takes a closed form for any finite potential strength, and therefore it is well defined outside the radius of convergence of the Born series. When the Weinberg eigenvalues and eigenfunctions on the energy shell are closely approximated with a suitably chosen N, the divergence difficulty of the conventional Born series can be overcome by the use of the Padé approximant, because the latter is correctly continued analytically outside the convergence radius of its Born series.

1. INTRODUCTION

SCATTERING theory is well founded for a local potential which is short-range and free from a strong singularity.[1] The latter properties can be formulated in terms of the existence of the first and the second moments of the potential

$$\int_0^\infty r^n|V(r)|dr<\infty\,, \quad \text{for} \quad n=1,2. \tag{1.1}$$

Such properties guarantee the validity of the adiabatic hypothesis for switching on a potential. They can be formulated in momentum space in such a manner that they are valid for nonlocal and/or energy-dependent potentials.[2] Individual terms of the Born series for such a potential are well defined for any finite strength of the potential, while the Born series may or may not converge. However, the Born series is available as the starting point of an analytical continuation even when it diverges.[3] Weinberg has investigated the properties of the eigenfunctions of the kernel of the Lippmann-Schwinger equation. The eigenfunctions are extremely useful for the discussion of the Born series.[4]

In this paper we shall derive an approximation method which is applicable even if the Born series diverges. Its usefulness outside the convergence radius of the Born series will be established by taking advantage of the analysis of the Born series carried out by Weinberg. The objective is to combine a number of the lowest order terms of the Born series in such a way that the convergence question of the Born series can be circumvented. It turns out that the use of Padé approximant considered by Chisholm[5] enables us to achieve our objective.

We shall restrict ourselves to the investigation of a partial wave in single-channel scattering. We shall deal with the standing-wave Green's function and take the principal value in integrating over momentum; this is because many relevant quantities become real in this formulation. The R matrix will be defined by the iteration of the kernel with the standing-wave Green's function. The nth-order term of its Born series will be denoted by R_n, which is defined by the recurrence formula

$$(k|R_{n+1}(E)|k') = \int_0^\infty dk''(k|V|k'')\frac{1}{k''^2-E}(k''|R_n(E)|k')\,, \tag{1.2}$$

where the matrix element of a generalized potential is denoted by $(k|V|k')$. Within the radius of convergence, the R matrix is defined by the sum of the Born series

$$(k|R(E)|k')=\sum_{n=1}^{\infty}(k|R_n(E)|k')\,, \tag{1.3}$$

and outside the radius of convergence an analytical continuation defines the R matrix for a stronger potential. We shall denote the momentum on the energy shell by k_0

$$E=k_0^2. \tag{1.4}$$

The R matrix on the energy shell will be called the K matrix, and we shall set

$$K_n(E)=(k_0|R_n(E)|k_0). \tag{1.5}$$

The tangent of the phase shift $\delta(k_0)$ is given by the

* Supported by the National Science Foundation.
[1] R. G. Newton, J. Math. Phys. 1, 319 (1960); further references are given in this review article.
[2] S. Tani, Bull. Am. Phys. Soc. 8, 301 (1963); report, 1964 (unpublished) which will be recapitulated in a forthcoming paper.
[3] A. Chen, S. Tani, and S. Borowitz, Phys. Rev. 137, B236 (1965), for the distorted wave approach and some of the related earlier works.
[4] S. Weinberg, Phys. Rev. 131, 440 (1963).
[5] J. S. R. Chisholm, J. Math. Phys. 4, 1506 (1963).

well-known formula

$$-\tan\delta(k_0)=(\pi/2k_0)K(E). \tag{1.6}$$

In Sec. 2 we shall recapitulate the formulas concerning the Padé approximant and also introduce the approximate values to the first N members of the set of Weinberg eigenvalues. In Sec. 3 we shall review the formalism suggested by Weinberg's work and exhibit the ideas underlying the Nth-rank approximation, in which we shall approximate the first N members of the set of Weinberg eigenfunctions by certain combinations of the lowest N terms of the Born series, (1.3). In Sec. 4 the details of the Nth-rank approximation is given. In Sec. 5, it is proved, by using the Fredholm theory, that the Nth-rank approximation is equal to the (N,N) Padé approximant introduced in Sec. 2. Some pathological case in which an approximation of a certain rank cannot be worked out, but which cannot be ruled out *a priori*, is discussed in the Appendix.

2. PADÉ APPROXIMANT

Baker, Gammel, and Wills[6] have proposed the use of the Padé approximant[7] in order to derive a meaningful result from a diverging series. Chisholm[5] has investigated the application of this method to the scattering problem. We shall go much further in this paper and display the power of this method in the scattering problem. In this section we shall recapitulate some formulas which will be useful later.

Since the potential is short-range and free from a strong singularity, there are bounds to individual terms in the Born series, which can be set in the form

$$|K_n(E)|<\tilde{M}^n\tilde{N}^{n-1}, \tag{2.1}$$

where both $\tilde{M}$ and $\tilde{N}$ are independent of energy.[2] Let us define a function Ψ by

$$\Psi(E;\lambda)=1+\sum_{n=1}^{\infty}\lambda^n K_n(E), \tag{2.2}$$

which is associated with the Born series for the potential λV. In view of (2.1), there is a nonvanishing radius of convergence for the series on the right-hand side of (2.2) such that, for

$$|\lambda|<\lambda_0, \tag{2.3}$$

the series converges absolutely and uniformly in both E and λ. The (N,N) Padé approximant to Ψ, to be denoted $\Psi_P{}^{[N]}$, is defined by

$$\Psi_P{}^{[N]}(E;\lambda)=N_P{}^{[N]}(E;\lambda)/D_P{}^{[N]}(E;\lambda), \tag{2.4}$$

where both the numerator and the denominator are algebraic functions of λ of the Nth order:

$$N_P{}^{[N]}(E;\lambda)=1+\sum_{m=1}^{N}\lambda^m a_m{}^{[N]}(E), \tag{2.5}$$

$$D_P{}^{[N]}(E;\lambda)=1+\sum_{m=1}^{N}\lambda^m b_m{}^{[N]}(E), \tag{2.6}$$

and the formal expansion of (2.4) into a power series of λ agrees with the right-hand side of (2.2) exactly up to the $2N$th order, or

$$\Psi(E;\lambda)D_P{}^{[N]}(E;\lambda)-N_P{}^{[N]}(E;\lambda) = (\text{terms of orders higher than } 2N \text{ in } \lambda). \tag{2.7}$$

Equation (2.7) means that the sum of the terms on the left-hand side must vanish when their orders in λ do not exceed $2N$. By substitution of (2.1), (2.5), and (2.6) on the left side of (2.7) and by collecting terms of the $(N+m)$th order, we find

$$K_{N+m}+\sum_{n=1}^{N} b_n{}^{[N]}K_{N+m-n}=0, \tag{2.8}$$

for $m=1, 2, \cdots, N$. The set of equations (2.8) constitute the N inhomogeneous linear equations for the N variables $b_n{}^{[N]}$ $(n=1, 2, \cdots, N)$ which can be solved if the determinant

$$\Delta^{[N]}\equiv\det\begin{vmatrix} K_1, & K_2, & \cdots, & K_N \\ K_2, & K_3, & \cdots, & K_{N+1} \\ \vdots & \vdots & & \\ K_N, & K_{N+1}, & \cdots, & K_{2N-1} \end{vmatrix} \tag{2.9}$$

does not vanish,

$$\Delta^{[N]}\neq 0. \tag{2.10}$$

We shall denote by $\Delta_\alpha{}^{[N]}$ the determinant which is derived from (2.9) by replacing its $(N-\alpha+1)$st column by the N row matrix

$$\begin{bmatrix} K_{N+1} \\ K_{N+2} \\ \vdots \\ K_{2N} \end{bmatrix}. \tag{2.11}$$

The solvability condition (2.10) will be analyzed in detail in the Appendix. All cases are covered by one of the three alternatives:

(i) $$\Delta^{[N]}\neq 0;$$

such a case will be called normal.[8]

(ii) $$\Delta^{[N]}=0,$$

and the case is abnormal; however, the rank of the matrix associated with the set of Eqs. (2.8) is equal to some number M which is smaller than N,

$$\text{rank of } (M_{mn}=K_{m+n-1})=M<N. \tag{2.10'}$$

[6] G. A. Baker, Jr., and J. L. Gammel, J. Math. Anal. Appl. **2**, 21 (1961); G. A. Baker, Jr., J. L. Gammel, and J. G. Wills, *ibid.* **2**, 405 (1961).

[7] H. S. Wall, *Analytic Theory of Continued Fractions* (D. Van Nostrand and Company Inc., New York, 1948), Chap. XX.

[8] Reference 7, pp. 379.

In this case we shall keep using the Mth-rank approximant; we shall state that the Nth rank is "reduced" to the Mth rank.

(iii) $$\Delta^{[N]}=0,$$

and the case is abnormal. Moreover, the linear equations do not hold because the rank of the matrix associated with the set of Eqs. (2.8) is equal to N. In this case we shall skip the Nth rank. If the formula (2.12) below is used in this case, at least one of the $b_\alpha{}^{[N]}$ diverges.

Accordingly, if some rank is skipped, when necessary, the Padé approximant is well defined for the remaining ranks. The solution of (2.8) is given by

$$b_\alpha{}^{[N]}=-\Delta_\alpha{}^{[N]}/\Delta^{[N]}, \tag{2.12}$$

for $\alpha=1, 2, \cdots, N$. By collecting terms of the mth order after substitution of (2.1), (2.5), and (2.6) on the left-hand side of (2.7) we find

$$a_m{}^{[N]}=K_m+\sum_{n=1}^{m-1} b_n{}^{[N]}K_{m-n}, \tag{2.13}$$

for $m=1, 2, \cdots, N$. Equations (2.12) and (2.13) define the Padé approximant uniquely when the first $2N$ terms of the Born series of the K matrix are given.

In passing we note that we can set

$$K_m(E)=\sum_{\nu=1}^{N}[\eta_\nu{}^{[N]}(E)]^{m-1}s_\nu{}^{[N]}(E), \tag{2.14}$$

for $m=1, 2, \cdots, 2N$, for there are $2N$ parameters in all [N parameters $\eta_\nu{}^{[N]}$ ($\nu=1, 2, \cdots, N$) and N parameters $S_\nu{}^{[N]}$ ($\nu=1, 2, \cdots, N$)], in order that the $2N$ quantities K_m ($m=1, 2, \cdots, 2N$) may be represented in terms of them. When (2.14) is substituted for each K_m which appears in the determinant $\Delta^{[N]}$, (2.9), it is given in the form

$$\Delta^{[N]}=(-1)^{(1/2)(N^2-3N+4)}\prod_{\lambda=1}^{N}s_\lambda{}^{[N]}\prod_{\mu>\nu}(\eta_\mu{}^{[N]}-\eta_\nu{}^{[N]})^2. \tag{2.15}$$

Similarly, we shall find

$$b_\alpha{}^{[N]}=(-1)^\alpha\sum_{c(\alpha)}\eta_{\lambda_1}{}^{[N]}\times\cdots\times\eta_{\lambda_\alpha}{}^{[N]}\equiv(-1)^\alpha P_\alpha{}^{[N]}, \tag{2.16}$$

where the summation over $c(\alpha)$ means to sum over all combinations of α different numbers $\lambda_1, \cdots, \lambda_\alpha$ chosen out of 1 through N; thus $P_\alpha{}^{[N]}$ stands for the sum of the α-linear functions of the N variables $\eta_\nu{}^{[N]}$ ($\nu=1, 2, \cdots, N$). When the parameters $\eta_\nu{}^{[N]}$ are used by taking into account of (2.16), the Padé denominator (2.6) takes the form

$$D_P{}^{[N]}(E;\lambda)=\prod_{\mu=1}^{N}[1-\lambda\eta_\mu{}^{[N]}(E)]. \tag{2.17}$$

Finally a comment will be made on the possibility of reduction of a rank, (ii) (2.10). Either when some one of $s_\nu{}^{[N]}$ vanishes or when some $\eta_\mu{}^{[N]}$ is equal to another $\eta_\nu{}^{[N]}$, the determinant $\Delta^{[N]}$ vanishes, as is clear from (2.15). Then the first $2N$ terms of the Born series can be put in the form (2.14) with a smaller number of parameters; this is what happens when N is reduced to a number smaller than that.

3. WEINBERG EIGENVALUE PROBLEM WITH STANDING-WAVE GREEN'S FUNCTION

Weinberg has introduced an eigenfunction of the kernel of the Lippmann-Schwinger equation[4]

$$[k^2-E-i\epsilon]^{-1}\int dk'(k|V|k')\psi_\nu(k';E)=\eta_\nu(E)\psi_\nu(k;E). \tag{3.1}$$

We shall deal with the standing-wave Green's function and modify the above formula by setting

$$\psi_\nu(k;E)=[k^2-E]^{-1}\sigma_\nu(k;E), \tag{3.2}$$

and consider a source of such an eigenfunction denoted by $\sigma_\nu(k;E)$. The eigenvalue equation for the σ_ν reads

$$\int dk'(k|V|k')[k'^2-E]^{-1}\sigma_\nu(k';E)=\eta_\nu(E)\sigma_\nu(k;E). \tag{3.3}$$

As will be shown in another paper,[2] a kernel with a short range and mild potential is completely continuous,[9] and therefore a potential can be approximated by a converging sequence of separable potentials

$$(k|V|k')=\lim_{N\to\infty}(k|V^{[N]}|k'), \tag{3.4}$$

where

$$(k|V^{[N]}|k')=\sum_{m=1}^{N}u_m{}^{[N]}(k)v_m{}^{[N]}(k'). \tag{3.5}$$

Then the potential can be put in the "diagonal" form

$$(k|V|k')=\textstyle\sum_\nu\sigma_\nu(k;E)\sigma_\nu{}^\dagger(k';E), \tag{3.6}$$

on normalizing the eigenfunctions by

$$\int dk[k^2-E]^{-1}\sigma_\mu(k;E)\sigma_\nu{}^\dagger(k;E)=\eta_\mu(E)\delta_{\mu\nu}. \tag{3.7}$$

By substitution of (3.6) and (3.7) into (1.2), the mth-order Born expansion of the R matrix is given by the "diagonal" form

$$(k|R_m(E)|k')=\textstyle\sum_\nu[\eta_\nu(E)]^{m-1}\sigma_\nu(k;E)\sigma_\nu{}^\dagger(k';E). \tag{3.8}$$

When the Born series is summed, the R matrix is

[9] F. Riesz and B. Sz.-Nagy, *Functional Analysis* (Frederick Ungar Publishing Company, New York, 1955), Part II; F. Coester, Phys. Rev. **133**, B1516 (1964).

given by

$$(k|R(E)|k')=\sum_\nu [1-\eta_\nu(E)]^{-1}\sigma_\nu(k;E)\sigma_\nu^\dagger(k';E). \quad (3.9)$$

As is obvious from (3.8), the power of η_ν is raised by one when the order in the Born expansion is raised by one, and consequently in each subspace spanned by an individual eigenfunction the Born series is a geometric series

$$1+\eta_\nu(E)+[\eta_\nu(E)]^2+\cdots=[1-\eta_\nu(E)]^{-1}. \quad (3.10)$$

That is to say, the operator geometric series of the kernel is converted into the geometric series of the eigenvalues in the "diagonal" representation. In the sense that the right-hand side of (3.10) gives the result of a correct analytical continuation; it stands for the sum even when the absolute value of η_ν is larger than unity and the series on the left-hand side diverges. In the same sense the right-hand side of (3.9) represents the correct R matrix outside the radius of convergence of the Born series. A reader can find some counterparts of the formulas (3.6) through (3.9) in Weinberg's paper[10] if Eq. (3.2) is remembered.

When the eigenfunctions take their value on the energy shell the K matrix is given by

$$K(E)=\sum_\nu [1-\eta_\nu(E)]^{-1}s_\nu(E), \quad (3.11)$$

where we have set

$$s_\nu(E)=\sigma_\nu(k_0;E)\sigma_\nu^\dagger(k_0;E). \quad (3.12)$$

The mth-order Born expansion of the K matrix is given by

$$K_m(E)=\sum_\nu [\eta_\nu(E)]^{m-1}s_\nu(E). \quad (3.13)$$

When the sum is extended over only the first N members of the set of eigenfunctions, we can define the N component approximation by

$$K_m(E)\approx\sum_\nu [\eta_\nu(E)]^{m-1}s_\nu(E). \quad (3.14)$$

It is more convenient to allow the parameters to deviate by a small amount and set

$$s_\nu^{[N]}=s_\nu+\epsilon_\nu^{[N]}, \quad (3.15)$$

$$\eta_\nu^{[N]}=\eta_\nu+\delta_\nu^{[N]}, \quad (3.16)$$

in such a way that the equality can be restored for the first $2N$ orders

$$K_m=\sum_{\nu=1}^{N} [\eta_\nu^{[N]}]^{m-1}s_\nu^{[N]}, \quad (1\leq m\leq 2N), \quad (3.17)$$

which has been referred to as (2.14) in the last section, and certain approximations hold for higher orders:

$$K_m\approx\sum_{\nu=1}^{N} [\eta_\nu^{[N]}]^{m-1}s_\nu^{[N]}, \quad (m>2N). \quad (3.18)$$

[10] Reference 4, Sec. VI.

It will be shown in another paper[2] that the approximate parameters $s^{[N]}$ and $\eta^{[N]}$ converge to their respective limits as the rank N tends to infinity

$$\lim_{N\to\infty} \epsilon_\nu^{[N]}=0, \quad (3.19)$$

$$\lim_{N\to\infty} \delta_\nu^{[N]}=0. \quad (3.20)$$

If the ranks are all normal in forming Padé approximants, as is the case with a potential of definite sign, the approximate eigenvalues $\eta_\nu^{[N]}$ derived from (2.14) are all different from each other because of (2.15) and (2.10). Since they converge to their respective limit as the rank tends to infinity, (3.20), the Weinberg eigenvalues η_ν as well as their approximate values are not degenerate. Then we can arrange them in such a way that their absolute values form a decreasing sequence

$$|\eta_1|>|\eta_2|>|\eta_3|>\cdots. \quad (3.21)$$

The sequence (3.21) has no accumulation point other than zero. That means that the number of eigenvalues with absolute values larger than any finite number must be finite. If this is not the case, we shall find a contradiction to the existence of bounds of the form (2.1) to the nth-order Born expansion of the R matrix when n becomes very large, and therefore the statement must be true. If these properties of the spectrum are taken into account, we can readily understand that only a limited number of terms dominate in (3.8) and (3.13) as the order m in the Born expansion becomes very large. Thus, with a suitably large N in order that all the major components may be covered, the N component approximation as defined by (3.17) and (3.18) will be very reliable. We shall show in the next section that such an approach has additional attractive features. Suppose we have carried out the iteration of the R matrix up to the $2N$th order using the recurrence formula (1.2). We know then the K matrix up to the $2N$th order, and from Eqs. (2.9)–(2.16) we can derive the N approximate eigenvaues $\eta_\nu^{[N]}$. By following the prescriptions to be given in next section, we can derive a reasonable approximation to the eigenfunctions $\sigma_\nu(k:E)$ $(\nu=1, 2, \cdots, N)$, and consequently we have an approximate potential of the form (3.5). Since it is a sum of separable potentials, the K matrix associated with it can be derived without further approximation, and also the Fredholm theory can be worked out thoroughly; in fact, the numerator and the denominator of the latter agree with the numerator and the denominator, respectively, of the rational function obtained from the Padé approximant (2.4)–(2.6) by subtracting unity from it. On taking the limit of the infinite rank,[2] a transparent analysis of the analytic properties of the K matrix as well as the S matrix is feasible.

4. THE NTH-RANK APPROXIMATION TO A GIVEN POTENTIAL

To derive an approximate potential of the form (3.5), we begin with the set of equations

$$(k|R_m|k')=\sum_{\nu=1}^{N}[\eta_\nu^{[N]}]^{m-1}f_\nu(k,k'), \tag{4.1}$$

for $m=1, 2, \cdots, N$, where the N parameters $\eta_\nu^{[N]}$ are derived from Eqs. (2.9)–(2.16). The form of the right-hand side of (4.1) is made analogous to the right-hand side of (3.8), with (3.17) kept in mind. In other words, the function $f_\nu(k,k')$ is introduced as an approximation to $\sigma_\nu(k:E)\sigma_\nu^\dagger(k':E)$ which appears on the right-hand side of (3.8)

$$f_\nu(k,k')\approx\sigma_\nu(k;E)\sigma_\nu^\dagger(k';E). \tag{4.2}$$

We shall define the matrix $\mathbf{A}$ by

$$A_{\mu\nu}=[\eta_\nu^{[N]}]^{\mu-1}, \quad (\mu,\nu=1, 2, \cdots, N), \tag{4.3}$$

and denote its inverse by $\mathbf{B}$, namely,

$$\sum_{\lambda=1}^{N}B_{\mu\lambda}A_{\lambda\nu}=\delta_{\mu\nu}. \tag{4.4}$$

Since the matrix $\mathbf{A}$ has a simple structure known as the Vandermonde matrix,[11] it is possible to write down the matrix element $B_{\mu\nu}$ explicitly in terms of the $\eta_\mu^{[N]}$. We shall find, for $\nu=N$

$$B_{\mu,N}=\prod_{\nu\neq\mu}{}'[\eta_\mu^{[N]}-\eta_\nu^{[N]}]^{-1}, \tag{4.5}$$

and for $\nu=1$,

$$B_{\mu,1}=\prod_{\nu\neq\mu}{}'[\eta_\mu^{[N]}-\eta_\nu^{[N]}]^{-1}\cdot\eta_\nu^{[N]}, \tag{4.6}$$

and generally for $B_{\mu,N-\alpha}$ with $\alpha=1, 2, \cdots, N-2$,

$$B_{\mu,N-\alpha}=(-1)^{N-\alpha}B_{\mu,N}\sum_{c(\alpha,\mu)}\eta_{\lambda_1}^{[N]}\times\cdots\times\eta_{\lambda_\alpha}^{[N]}, \tag{4.7}$$

where the sum over the combinations $c(\alpha,\mu)$ means to take all the combinations of α different numbers $\lambda_1, \cdots, \lambda_\alpha$ chosen out of the $N-1$ numbers 1 through N from which a particular number μ is dropped. Note that the inverse $\mathbf{B}$ is well defined when the rank N is normal, (2.10); this follows from the nondegeneracy of the $\eta_\mu^{[N]}$ as shown by (2.15). Since (4.1) can be rewritten as

$$(k|R_\mu|k')=\sum_{\nu=1}^{N}A_{\mu\nu}f_\nu(k,k'), \quad (\mu=1, 2, \cdots, N), \tag{4.8}$$

the function $f_\nu(k,k')$ will be obtained from

$$f_\nu(k,k')=\sum_{\mu=1}^{N}B_{\nu\mu}(k|R_\mu|k'), \quad (\nu=1, 2, \cdots, N). \tag{4.9}$$

[11] E.g., F. E. Hohn, *Elementary Matrix Algebra* (The Macmillan Company, New York, 1958).

As shown in (3.8) or (4.2), the "diagonal" representation of the potential consists of separable potentials. Since the separability of individual components is essential in order to carry out our analysis to the end, we shall use the $f_\mu(k,k')$, (4.9), to construct a separable potential. Besides (4.9), we shall define the adjoint $f_\mu^\dagger(k,k')$ by

$$f_\nu^\dagger(k,k')=\sum_{\mu=1}^{N}B_{\nu\mu}(k'|R_\mu|k). \tag{4.10}$$

On the energy shell the function $f_\mu(k_0,k_0)$ becomes equal to its adjoint $f_\mu^\dagger(k_0,k_0)$

$$f_\mu(k_0,k_0)=f_\mu^\dagger(k_0,k_0)=\sum_{\nu=1}^{N}B_{\mu\nu}K_\nu(E). \tag{4.11}$$

We may set

$$\begin{aligned}\sigma_\nu(k;E)\sigma_\nu^\dagger(k';E)&\approx f_\nu(k,k_0)f_\nu^\dagger(k',k_0)/f_\nu(k_0,k_0),\\&=[f_\nu(k,k_0)f_\nu(k_0,k_0)^{-1/2}]\\&\quad\times[f_\nu^\dagger(k',k_0)f_\nu^\dagger(k_0,k_0)^{-1/2}],\end{aligned} \tag{4.12}$$

and use the right-hand side of the approximate equality to construct the approximate Nth-rank potential; it is so "normalized" that its value on the energy shell is given by (4.11) and thus the value of our approximate potential is equal to that of the original potential on the energy shell because of (4.1) for $m=1$. To compare with (3.5), we shall set

$$u_\mu^{[N]}(k)=f_\mu(k,k_0), \tag{4.13}$$

$$v_\mu^{[N]}(k)=f_\mu^\dagger(k,k_0)/f_\mu(k_0,k_0). \tag{4.14}$$

Our next task is to derive the K matrix from the Nth-rank potential

$$\begin{aligned}(k|V^{[N]}|k')&=\sum_{\mu=1}^{N}u_\mu^{[N]}(k)v_\mu^{[N]}(k'),\\&=\sum_{\mu=1}^{N}f_\mu(k,k_0)f_\mu^\dagger(k',k_0)/f_\mu(k_0,k_0).\end{aligned} \tag{4.15}$$

On substituting (4.15) in the recurrence formula (1.2), the nth-order term of the R matrix will be given by

$$(k|R_n^{[N]}|k')=\sum_{\mu,\nu=1}^{N}u_\mu^{[N]}(k)[\mathbf{H}^{n-1}]_{\mu\nu}v_\nu^{[N]}(k'), \tag{4.16}$$

where the matrix element $H_{\mu\nu}$ is defined by

$$H_{\mu\nu}=\int dk[k^2-E]^{-1}v_\mu^{[N]}(k)u_\nu^{[N]}(k). \tag{4.17}$$

We remind ourselves here of the formula

$$\begin{aligned}\int dk(k_0|R_\kappa|k)[k^2-E]^{-1}(k|R_\lambda|k_0)\\=(k_0|R_{\kappa+\lambda}|k_0)\equiv K_{\kappa+\lambda}(E),\end{aligned} \tag{4.18}$$

which is derived straightforwardly from the recurrence formula (1.2) and (1.5). After the substitution of (4.12) and (4.13) and the use of (4.18) the right-hand side of (4.17) will be transformed to

$$H_{\mu\nu}=\sum_{\kappa,\lambda=1}^{N} B_{\mu\kappa}B_{\nu\lambda}K_{\kappa+\lambda}/\sum_{\lambda=1}^{N} B_{\mu\lambda}K_{\lambda}. \tag{4.19}$$

The denominator on the right-hand side of (4.19) will be evaluated first. From (4.5)–(4.7), it follows that

$$\sum_{\lambda=1}^{N} B_{\mu\lambda}K_{\lambda}=B_{\mu 1}K_1+\sum_{\lambda=1}^{N-2} B_{\mu,N-\lambda}K_{N-\lambda}$$

$$=B_{\mu N}\sum_{\beta=1}^{N-1} K_{N-\beta}(-1)^{\beta}$$

$$\times\sum_{c(\beta,\mu)} \eta_{\lambda_1}{}^{[N]}\times\cdots\times\eta_{\lambda_\beta}{}^{[N]}. \tag{4.20}$$

On using (3.17) on the right-hand side above, we shall find, after some algebra,

$$\sum_{\lambda=1}^{N} B_{\mu\lambda}K_{\lambda}=B_{\mu N}\sum_{\alpha=1}^{N} s_{\alpha}\{\sum_{\beta=1}^{N-1}[(-1)^{\beta}(\eta_{\alpha}{}^{[N]})^{N-\beta-1}$$

$$\times\sum_{c(\beta,\mu)} \eta_{\lambda_1}{}^{[N]}\times\cdots\times\eta_{\lambda_\beta}{}^{[N]}]\}$$

$$=B_{\mu N}\sum_{\alpha=1}^{N} s_{\alpha}\prod_{\beta\neq\mu}{}'(\eta_{\alpha}{}^{[N]}-\eta_{\beta}{}^{[N]}). \tag{4.21}$$

In forming the products in the last member with all possible β, some one of its values is equal to α unless $\alpha=\mu$; that means that only the term $\alpha=\mu$ will be left on the right side and thus we have

$$\sum_{\lambda=1}^{N} B_{\mu\lambda}K_{\lambda}=B_{\mu N}s_{\mu}{}^{[N]}\prod_{\beta\neq\mu}{}'(\eta_{\mu}{}^{[N]}-\eta_{\beta}{}^{[N]})=s_{\mu}{}^{[N]}, \tag{4.22}$$

where we have used (4.5) in passing to the last member. As can be seen by referring to (3.17), when $[\eta_{\alpha}{}^{[N]}]^{\lambda}s_{\alpha}{}^{[N]}$ are substituted for all the $s_{\alpha}{}^{[N]}$ ($\alpha=1, 2, \cdots, N$) in the κth-order term K_{κ}, the latter will be transformed into the $(\lambda+\kappa)$th-order term $K_{\kappa+\lambda}$. By noting this, the evaluation of the sum

$$\sum_{\kappa=1}^{N} B_{\mu\kappa}K_{\kappa+\lambda},$$

which appears in the numerator of (4.19), can be carried out in the way similar to Eqs. (4.20)–(4.22) for the sum

$$\sum_{\kappa=1}^{N} B_{\mu\kappa}K_{\kappa}.$$

Then we find

$$\sum_{\kappa=1}^{N} B_{\mu\kappa}K_{\kappa+\lambda}=s_{\mu}{}^{[N]}(\eta_{\mu}{}^{[N]})^{\lambda}, \tag{4.23}$$

and the right-hand side of (4.19) can be transformed into

$$H_{\mu\nu}=\sum_{\lambda=1}^{N} B_{\nu\lambda}(\eta_{\mu}{}^{[N]})^{\lambda}, \tag{4.24}$$

using (4.22) and (4.23). Referring to (4.3), we can set

$$H_{\mu\nu}=\sum_{\lambda=1}^{N} B_{\nu\lambda}A_{\lambda\mu}\eta_{\mu}{}^{[N]}, \tag{4.25}$$

and in view of (4.4) we have the result

$$H_{\mu\nu}=\eta_{\mu}{}^{[N]}\delta_{\mu\nu}. \tag{4.26}$$

Therefore the matrix **H** is diagonal, which means that the Nth-rank potential (4.15) is already in the "diagonal" form and that the parameters $\eta_{\mu}{}^{[N]}$ introduced at the end of Sec. 2 are actually Weinberg eigenvalues for the Nth-rank potential. Accordingly Eq. (4.16) will be simplified as

$$(k|R_n{}^{[N]}|k')=\sum_{\mu=1}^{N}(\eta_{\mu}{}^{[N]})^{n-1}u_{\mu}{}^{[N]}(k)v_{\mu}{}^{[N]}(k'). \tag{4.27}$$

Taking the energy shell value of (4.27) and summing the Born series, the approximate K matrix is given by

$$K^{[N]}=\sum_{\mu=1}^{N}(1-\eta_{\mu}{}^{[N]})^{-1}s_{\mu}{}^{[N]}, \tag{4.28}$$

where Eqs. (4.11), (4.14), and (4.22) have been taken into account to write down the right-hand side.

5. EQUIVALENCE OF THE NTH-RANK APPROXIMATION TO THE (N,N) PADÉ APPROXIMANT; FREDHOLM THEORY

We shall show in this section that the approximate K matrix (4.28) derived from the Nth-rank approximation (4.15) of the potential is equal to the (N,N) Padé approximant (2.4) in which we set λ equal to unity. Let us recall that, if the Fredholm theory is applied to the derivation of the K matrix for the Nth-rank potential (4.15), both the numerator and the denominator are terminated by the Nth order in the potential strength; that is, they are algebraic function of the Nth order of the potential strength. Thus the Fredholm theory gives the same kind of rational function of the potential strength as the Padé approximant exhibited in Eqs. (2.4)–(2.6), provided we subtract unity from the latter because the right-hand side of (2.2) differs from the Born series for the K matrix by unity.

Since the Fredholm theory is available in the ordinary form for a partial-wave problem, we can evaluate various terms of the Fredholm denominator by using the relations given in last section. Starting with the

formula for the Fredholm denominator

$$D_F^{[N]}=1-\int dk[k^2-E]^{-1}(k|V^{[N]}|k)+(2!)^{-1}\int dkdk'[k^2-E]^{-1}[k'^2-E]^{-1}$$
$$\times\{(k|V^{[N]}|k)(k'|V^{[N]}|k')-(k|V^{[N]}|k')(k'|V^{[N]}|k)\}-\cdots, \quad (5.1)$$

we shall find a concise form by using Eqs. (4.15), (4.17), and (4.26); the right-hand side of (5.1) can be transformed into

$$1-\sum_{\mu=1}^{N}\int dk[k^2-E]^{-1}v_\mu^{[N]}(k)u_\mu^{[N]}(k)+(2!)^{-1}\int dkdk'[k^2-E]^{-1}[k'^2-E]^{-1}$$
$$\times\sum_{\mu,\nu=1}^{N}\{u_\mu^{[N]}(k)v_\mu^{[N]}(j)u_\nu^{[N]}(k')v_\nu^{[N]}(k')-u_\mu^{[N]}(k)v_\mu^{[N]}(k')u_\nu^{[N]}(k')v_\nu^{[N]}(k)\}-\cdots,$$
$$=1-\sum_{\mu=1}^{N}\eta_\mu^{[N]}+(2!)^{-1}\{(\sum_{\mu=1}^{N}\eta_\mu^{[N]})^2-\sum_{\mu=1}^{N}(\eta_\mu^{[N]})^2\}-\cdots,$$
$$=1-\sum_{\mu=1}^{N}\eta_\mu^{[N]}+\sum_{\mu>\nu}\eta_\mu^{[N]}\eta_\nu^{[N]}-\cdots. \quad (5.2)$$

Therefore we established the formula

$$D_F^{[N]}=\prod_{\nu=1}^{N}(1-\eta_\nu^{[N]}). \quad (5.3)$$

Similarly, starting with the formula for the Fredholm numerator

$$N_F^{[N]}=(k_0|V^{[N]}|k_0)-\int dk[k^2-E]^{-1}\{(k_0|V^{[N]}|k_0)(k|V^{[N]}|k)-(k_0|V^{[N]}|k)(k|V^{[N]}|k_0)\}$$
$$+(2!)^{-1}\int dkdk'[k^2-E]^{-1}[k'^2-E]^{-1}\{(k_0|V^{[N]}|k_0)(k|V^{[N]}|k)(k'|V^{[N]}|k')$$
$$+2(k_0|V^{[N]}|k)(k|V^{[N]}|k')(k'|V^{[N]}|k_0)-(k_0|V^{[N]}|k_0)(k|V^{[N]}|k')(k'|V^{[N]}|k)$$
$$-2(k_0|V^{[N]}|k)(k|V^{[N]}|k_0)(k'|V^{[N]}|k')\}-\cdots, \quad (5.4)$$

we shall find the result

$$N_F^{[N]}=\sum_{\nu=1}^{N}s_\nu^{[N]}(1-\sum_{\mu\neq\nu}{}'\eta_\mu^{[N]}+\sum_{\lambda\neq\nu,\mu\neq\nu}{}''\eta_\lambda^{[N]}\eta_\mu^{[N]}-\cdots), \quad (5.5)$$

by using Eqs. (4.11), (4.15), (4.17), (4.22), and (4.26). Thus we can establish the result

$$N_F^{[N]}=\sum_{\nu=1}^{N}s_\nu^{[N]}\prod_{\mu\neq\nu}{}'(1-\eta_\mu^{[N]}). \quad (5.6)$$

On combining (5.3) with (5.6), the result from the Fredholm theory reads

$$K_F^{[N]}=N_F^{[N]}/D_F^{[N]}=\sum_{\nu=1}^{N}(1-\eta_\nu^{[N]})^{-1}s_\nu^{[N]}. \quad (5.7)$$

Namely, the result from the Fredholm theory is exactly the same as the results from the Nth-rank approximation (4.28), and thus we set

$$K^{[N]}=K_F^{[N]}. \quad (5.8)$$

On the other hand, according to (2.17), the Padé denominator, when the λ is set equal to unity, is equal to Fredholm denominator (5.3)

$$D_P^{[N]}(E;\lambda=1)=D_F^{[N]}. \quad (5.9)$$

When we subtract unity from the Padé approximant defined by Eqs. (2.2)–(2.6) and set the λ equal to unity, it is a rational function of the potential strength whose formal Born series agrees with that for the $K^{[N]}$

$$K^{[N]}(E)=\sum_{m=1}^{\infty}\sum_{\mu=1}^{N}s_\mu^{[N]}(\eta_\mu^{[N]})^{m-1}, \quad (5.10)$$

at least up to the $2N$th order. The same statement applies of the result from the Fredholm theory because of (5.8), and moreover, the denominators of the two rational functions agree with each other as shown by (5.9). Therefore, the numerators of the two rational functions, for which N parameters are available each,

must agree with each other. Thus we can set

$$N_P{}^{[N]}(E;\lambda=1)-D_P{}^{[N]}(E;\lambda=1)=N_F{}^{[N]}(E). \quad (5.11)$$

Finally we have established the equality

$$\begin{aligned} K^{[N]}(E)&=\sum_{\mu=1}^{N}(1-\eta_\mu{}^{[N]})^{-1}s_\mu{}^{[N]} \\ &=K_F{}^{[N]}(E)=\Psi_F{}^{[N]}(E;\lambda=1)-1 \\ &=\sum_{n=1}^{N}(1+\sum_{m=1}^{N}b_m{}^{[N]})^{-1} \\ &\quad\times[K_n+\sum_{l=1}^{n-1}K_{n-l}b_l{}^{[N]}-b_n{}^{[N]}], \end{aligned} \quad (5.12)$$

where the last member is derived from (2.4) by using (2.5), (2.6), and (2.15). Our result has proved that the rather involved derivation of the Nth-rank approximate potential exhibited in last section can be circumvented in practice by using the Padé approximant, which results from the solution of the set of linear equations (2.8). On the other hand, the Nth-rank approximation is based on a close approximation to the set of the Weinberg eigenfunctions and eigenvalues, and the accuracy of the analytical continuation when the Born series diverges can be readily estimated. Thus the fact that the Padé approximant is equal to the result from the Nth-rank approximation is a warrant for the reliability of the former method outside the radius of convergence of the Born series.

6. CONCLUDING REMARKS

The convergence of the Nth-rank approximation to the exact K matrix as the rank increases without limit will be proved in a separate paper.[2] It follows from the condition that the potential is free from a long-range tail and a strong singularity. Under this condition the kernel of the Lippmann-Schwinger equation and its iteration have a bounded norm. It can be seen then that the theory of Hilbert space[9] can be applied when the energy is negative real (the bound-state problem), and that the kernel is completely continuous for negative energies, or the potential can be approximated by a converging series of separable potentials. The sequence of separable potentials obtained from the set of bound states can be used for investigating the positive real energies. By taking the limit as the number of separable potentials increases without limit, we can establish the existence of Weinberg eigenfunctions for positive real energies and also the convergence of the Padé approximant in the limit of infinite rank. Thus the validity of the method proposed by Coester[12] can be established explicitly. On the other hand, the generalization of the method of the symmetrization of the kernel[13] to nonlocal and energy-dependent potentials does not appear obvious, although similar results follow from both methods eventually.

Accordingly, we may state that with a suitably large N the terms of the lowest $2N$ orders of the Born series can be rearranged in such a way that the most relevant members of the set of Weinberg eigenfunctions and eigenvalues are closely approximated. In practice the same result follows by deriving the (N,N) Padé approximant. Such a result is presented in a closed form regardless of the potential strength, and is correctly continued analytically outside the radius of convergence of the Born series. Thus by the use of a Padé approximant of a suitable rank we can circumvent the convergence problem of the Born series. Recently, several similar attempts have been made to rearrange the simple Born series in such a way as to obtain a result applicable in a wider range of potential strength by Wellner,[14] Brysk,[15] and others. Perhaps, it is the unique advantage of the Padé approximant that it is simple, is based on the use of the Weinberg eigenfunctions, and keeps a close relationship with the Fredholm theory.

A list of physical problems to which the method of Padé approximant or of continued fraction was applied is given in Lovelace and Masson's paper[16] on the computation of a Regge pole. As for more recent works which appeared after that reference, we note Fried and Eberly's work[17] on the scattering of very low-frequency photons and Baker's work[18] on the susceptibility of the three-dimensional Heisenberg ferromagnet. As shown in this paper, potential scattering is a problem in which this method is highly successful.

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APPENDIX: ABNORMAL RANK

In Sec. 2, the possibility of an abnormal case has been mentioned: cf. (2.10)–(2.10′), (ii)–(iii). Such an abnormality may persist through a number of ranks, but we shall discuss the case where the preceding rank is normal and then go over to a general case.

Suppose the Nth rank is abnormal, while the $(N-1)$st

[12] F. Coester, Phys. Rev. **133**, B1516 (1964).

[13] M. Scadron, S. Weinberg, and J. Wright, Phys. Rev. **135**, B202 (1964); the same method has also been used by K. Meetz, J. Math. Phys. **3**, 690 (1962); J. Schwinger, Proc. Natl. Acad. Sci. U. S. **47**, 122 (1961); A. Grossman and T. T. Wu, J. Math. Phys. **2**, 710 (1961); H. Rollnik, Z. Physik **145**, 639 (1956).

[14] M. Wellner, Phys. Rev. **132**, 1848 (1963).

[15] H. Brysk, Phys. Rev. **133**, B1625 (1964); related earlier works are referred to in this paper.

[16] D. Lovelace and D. Masson, Nuovo Cimento **26**, 472 (1962)

[17] Z. Fried and J. H. Eberly, Phys. Rev. **136**, B871 (1964).

[18] G. A. Baker, Jr., Phys. Rev. **136**, A1376 (1964).

rank is normal

$$\Delta^{[N]}(K_1,\cdots,K_{2N-1})=0, \tag{A1}$$

$$\Delta^{[N-1]}(K_1,\cdots,K_{2N-3})\neq 0. \tag{A2}$$

When the $(N-1)$st-rank representation

$$K_m=K_m{}^{[N-1]}\equiv\sum_{\nu=1}^{N-1}(\eta_\nu{}^{[N-1]})^{m-1}S_\nu{}^{[N-1]},$$

$$(1\leq m\leq 2N-2), \tag{A3}$$

is used for the lowest $2N-2$ terms of the Born series in $\Delta^{[N]}$, (A1), it follows that the $(2N-1)$st-order term K_{2N-1} can also be represented by the $(N-1)$st-rank form

$$K_{2N-1}=\sum_{\nu=1}^{N-1}(\eta_\nu{}^{[N-1]})^{2N-2}S_\nu{}^{[N-1]}, \tag{A4}$$

which will be abbreviated as

$$K_{2N-1}=K_{2N-1}{}^{[N-1]}. \tag{A5}$$

Hereafter a superscript signifies the type of the representation. If the $(N-1)$st representations for the lowest $(2N-1)$ terms of the Born series are used, we shall find that the $\Delta^{[N+1]}$ is independent of K_{2N+1} and satisfies the equation

$$\Delta^{[N+1]}(K_1,\cdots,K_{2N},K_{2N+1})=[\Delta_1{}^{[N]}(K_1,\cdots,K_{2N})]^2[\Delta^{[N-1]}(K_1,\cdots,K_{2N-3})]^{-1},$$

$$(\text{when } \Delta^{[N]}=0). \tag{A6}$$

Then, the abnormal case (A1) will be divided into two subclasses depending on whether

(a) $$\Delta_1{}^{[N]}(K_1,\cdots,K_{2N})=0 \tag{A7}$$

or

(b) $$\Delta_1{}^{[N]}(K_1,\cdots,K_{2N})\neq 0. \tag{A8}$$

The case (a): On using the $(N-1)$st-rank representation for K_1 through K_{2N-1}, (A3)–(A5), in (A7), we shall find that the $2N$th-order term K_{2N} can also be put in the $(N-1)$st-rank form

$$K_{2N}=K_{2N}{}^{[N-1]}. \tag{A9}$$

Thus in view of Eqs. (A5) and (A9), the Nth rank can be reduced to the $(N-1)$st rank.

The case (b): If (A8) holds, the $(N+1)$st rank is normal

$$\Delta^{[N+1]}\neq 0, \tag{A10}$$

because of (A2) and (A6). On the other hand, when (2.12) is formally used, it will be found that $b_1{}^{[N]}$ diverges. Hence we should skip the Nth rank and go over to the $(N+1)$st rank.

Turning to a general case, we shall first note that the reduction may persist through a number of ranks continuously. Suppose all the M ranks beyond N, the $(N+1)$st through the $(N+M)$th, are reduced to N. Then we shall consider the matrix of the dimension $(N+M+L)\times(N+M+L+1)$,

$$\begin{vmatrix} K_1, & K_2, & \cdots, & K_N, & \cdots, & K_{N+M+L}, & K_{N+M+L+1} \\ K_2, & K_3, & \cdots, & \vdots & & \vdots & \vdots \\ \vdots & & & \vdots & & \vdots & \vdots \\ K_N, & & \cdots, & K_{2N-1}, & & \vdots & \vdots \\ \vdots & & & \vdots & & \vdots & \vdots \\ K_{N+M}, & & \cdots, & & & \vdots & \vdots \\ \vdots & & & \vdots & & \vdots & \vdots \\ K_{N+M+L}, & & \cdots, & K_{2N+M+L-1}, & \cdots, & K_{2N+2M+2L-1}, & K_{2N+2M+2L} \end{vmatrix}, \tag{A11}$$

associated with the set of linear equations (2.8) of the $(N+M+L)$th rank, which is larger than the $(N+M)$th by L. Since the $(N+1)$st rank through the $(N+M)$th rank are reduced to the Nth, the Nth-rank representation is applicable up to the $(2N+2M)$th order in the Born series

$$K_m=K_m{}^{[N]}, \quad (1\leq m\leq 2N+2M). \tag{A12}$$

There will be three subclasses, then:

(α) The Nth rank representation is not applicable any more at the $(2N+2M+1)$st order:

$$K_{2N+2M+1}\neq K_{2N+2M+1}{}^{[N]}. \tag{A13}$$

In this case we shall set

$$L=M+1 \tag{A14}$$

in (A11). The $(N+1)$st column reads then

$$\begin{bmatrix} K_{N+1} \\ K_{N+2} \\ \vdots \\ K_{2N+2M} \\ K_{2N+2M+1} \end{bmatrix}, \tag{A15}$$

where the Nth-rank representation cannot be applied to the member at the bottom, while all the preceding N columns are filled with members to which the Nth-rank representation applies. Thus the determinant $\Delta^{[N+2M+1]}$

can be transformed into the form

$$\Delta^{[N+2M+1]}=\begin{vmatrix} K_1, & \cdots, & K_N, & 0, & 0, & \cdots, & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ K_N, & \cdots, & K_{2N-1}, & 0, & 0, & \cdots, & 0 \\ K_{N+1}, & \cdots, & K_{2N}, & 0, & 0, & \cdots, & \delta_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ K_{N+2M}, & \cdots, & K_{2N+2M-1}, & 0, & \delta_1, & \cdots, & \delta_{2M} \\ K_{N+2M+1}, & \cdots, & K_{2N+2M}, & \delta_1, & \delta_2, & \cdots, & \delta_{2M+1} \end{vmatrix}, \tag{A16}$$

where we have set

$$K_{2N+2M+m}=K_{2N+2M+m}{}^{[N]}+\delta_m, \quad (1\leq m\leq 2M+1). \tag{A17}$$

Thus we obtain

$$\Delta^{[N+2M+1]}=(-1)^{N+M}\cdot\delta_1{}^{2M+1}\Delta^{[N]}\neq 0 \tag{A18}$$

because of (A13), which means $\delta_1\neq 0$. On the other hand, by taking off the last row and the last column successively from a determinant of the form (A16), we shall easily see that for the $(N+M+1)$st rank through the $(N+2M)$th rank the determinants vanish

$$\Delta^{[N+M+m]}=0, \quad (1\leq m\leq M-1). \tag{A19}$$

Thus we shall see that the $(N+2M+1)$st rank is normal, while all the ranks in the middle, the $(N+M+1)$st through the $(N+2m)$th, are abnormal. If we fictitiously set

$$K_{2N+2M-m}=K_{2N+2M-m}{}^{[N]}+\epsilon_{M-m}, \quad (0\leq m\leq M-1), \tag{A20}$$

all the $\epsilon_n(1\leq n\leq M)$ must vanish, because the Nth-rank representation applies to those orders of the Born series. If (2.12) is formally applied to these abnormal ranks, the parameters $b_\alpha{}^{[N]}$ are ill defined. But on scrutinizing the limit as the ϵ_n tend to zero, we have to conclude that all these abnormal ranks ought to be skipped, which will be shown as follows. First, we shall find that for the $(N+M+1)$st rank the $b_M{}^{[N+M+1]}$ diverges, because from (2.12) we shall find

$$b_M{}^{[N+M+1]}=(\delta_1-\epsilon_2/\epsilon_1), \tag{A21}$$

and the right-hand side diverges on taking the limit as both ϵ_1 and ϵ_2 tend to zero. Similarly, everyone of the following sequence

$$b_{M-1}{}^{[N+M+2]}, \quad b_{M-2}{}^{[N+M+3]}, \quad \cdots, \quad b_1{}^{[N+2M]}, \tag{A22}$$

will be found to diverge. In conclusion, in the case (α) we ought to skip all the ranks, the $(N+M+1)$st through the $(N+2M)$th, while the $(N+2M+1)$st rank is normal.

(β) The Nth rank representation applies to the $(2N+2M+1)$st order of the Born series, but it does not apply to the $(2N+2M+2)$nd order:

$$K_{2N+2M+1}=K_{2N+2M+1}{}^{[N]}, \tag{A23}$$

$$K_{2N+2M+2}\neq K_{2N+2M+2}{}^{[N]}. \tag{A24}$$

In this case we shall set

$$L=M+2 \tag{A25}$$

in (A11), and the whole argument as applied to the case (α) can be repeated exactly in the same manner except that the matrix (A11) has a dimension larger than in the case (α) by one. The set of diverging members, (A21), (A22), will be replaced by

$$b_{M+1}{}^{[N+M+1]}, \quad b_M{}^{[N+M]}, \quad \cdots, \quad b_1{}^{[N+2M+1]}. \tag{A26}$$

Thus, we ought to skip all the ranks, the $(N+M+1)$st through the $(N+2M+1)$st, while the $(N+2M+2)$nd rank will be normal.

(γ) The Nth-rank representation applies to both the $(2N+2M+1)$st-order and the $(2N+2M+2)$nd-order terms of the Born series

$$K_{2N+2M+1}=K_{2N+2M+1}{}^{[N]}, \tag{A27}$$

$$K_{2N+2M+2}=K_{2N+2M+2}{}^{[N]}. \tag{A28}$$

In this case the $(N+M+1)$st rank ought to be reduced to the Nth rank for the $(M+1)$st time. Then the whole argument of the general case should be repeated with $(M+1)$ replacing M.

Thus, we have established the statement in the text: "If some rank is skipped, when necessary, the Padé approximant is well defined for the rest of ranks." If the original potential is of a finite rank, say, N, then the ranks beyond N are all reduced to N. The recovery of a normal rank after a number of the reduced ranks follows the rule exhibited in (α) and (β) above.