Exchange Effects in Ferromagnetic Resonance with Nonlocal Conductivity*

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A solution is obtained for the surface impedance of a ferromagnetic metal with a wave-number-dependent conductivity function, by an extension of the method of Reuter and Sondheimer. Computed curves are given for some specific cases. It is concluded that exchange effects are large under suitable low-temperature conditions, but there are too many unknown parameters to allow a very accurate value of the exchange constant to be deduced from resonance experiments. For extreme anomalous conductivity, exchange effects depend on the combination (exchange constant)×(conductivity)$^{2/3}$.

I. INTRODUCTION

WHEN ferromagnetic resonance is observed under conditions of very small skin depth, the "exchange stiffness" or tendency of neighboring spins to remain parallel, described by the phenomenological exchange constant A, becomes important. It was shown experimentally by Rado and Weertman[1,2] that under properly chosen conditions the effect is fairly large, and Ament and Rado[3] have given a theory, valid for the case of normal conductivity, relating the experimentally measurable surface impedance to the exchange stiffness and other parameters. This method is potentially a useful tool for investigation of ferromagnetic metals; but work has been hampered by the fact that at low temperatures, where the exchange effects become large due to a decreased skin depth, the conductivity becomes nonlocal and the existing theory is unable to interpret the data. In this paper, we give a theory valid for the case of nonlocal conductivity and for a general range of fields and frequencies.

The effects of exchange can easily be understood qualitatively. In the case where the applied field is parallel to the sample surface, the following wave-number-dependent dynamic permeability can be obtained by the methods of Ref. 4:

$$\tilde{\mu}(q)=\frac{\gamma^2(H_{\rm eff}+4\pi M_s+2Aq^2/M_s-i\lambda\omega/\gamma^2M_s)^2-\omega^2}{\gamma^2(H_{\rm eff}+4\pi M_s+2Aq^2/M_s-i\lambda\omega/\gamma^2M_s)(H_{\rm eff}+2Aq^2/M_s-i\lambda\omega/\gamma^2M_s)-\omega^2}, \tag{1}$$

where $\gamma=-g|e|/2mc$ and λ is a phenomenological damping constant. If the exchange constant A is negligibly small, the ordinary theory of the anomalous skin effect,[5] with diffuse scattering of the electrons at the surface, gives for the surface impedance

$$Z=-\frac{4\pi^2i\omega\tilde{\mu}}{c^2}\left[\int_0^\infty \ln\left(1-\frac{4\pi i\omega\tilde{\mu}\tilde{\sigma}(q)}{c^2q^2}\right)dq\right]^{-1} \tag{2}$$

which, in the cases of normal (N) and extreme anomalous (EA) conductivity, respectively, reduces to

$$Z_{\rm N}=(1-i)(2\pi\omega\tilde{\mu}/c^2\sigma_0)^{1/2} \tag{3a}$$

and

$$Z_{\rm EA}=(\sqrt{3}-3i)(\pi^2\omega^2\tilde{\mu}^2/4c^4C_F)^{1/3}, \tag{3b}$$

where the normal conductivity is σ_0 and the extreme anomalous conductivity is C_F/q. In these cases there will be resonance when the applied field satisfies $H_{\rm eff}(H_{\rm eff}+4\pi M_s)=(\omega/\gamma)^2$.

When the exchange constant becomes appreciable, it will have the effect of shifting the resonance to a field lower by the amount $A\langle q^2\rangle$, where the bracket means "appropriate average," and it will also broaden the resonance precisely because more than one value of q^2 contributes to the average. It is expected that $\langle q^2\rangle$ will be the inverse-square skin depth in order of magnitude, and that the broadening will be comparable with the shift. As the work of Ament and Rado showed, under some conditions the resonance linewidth is dominated by exchange broadening.

In the normal and extreme anomalous cases respectively, with exchange neglected, the skin depths are given by

$$\delta_{\rm N}\approx(c^2/2\pi\omega\sigma_0\tilde{\mu})^{1/2}$$

and

$$\delta_{\rm EA}\approx 2(c^2/4\pi\omega C_F\tilde{\mu})^{1/3}.$$

The dynamic permeability at resonance can be as large as 1000, so the skin depth is significantly less, and $\langle q^2\rangle$ greater, than they would be under the same conditions in nonferromagnetic materials. For this reason anoma-

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[1] G. T. Rado and J. R. Weertman, Phys. Rev. **94**, 1386 (1954).
[2] G. T. Rado and J. R. Weertman, J. Phys. Chem. Solids **11**, 315 (1959).
[3] W. S. Ament and G. T. Rado, Phys. Rev. **97**, 1558 (1955).
[4] C. Kittel, Phys. Rev. **73**, 155 (1948).
[5] G. E. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc. (London) **A195**, 336 (1948).

lous conductivity effects become important when the dc resistivity is still relatively high, i.e., at temperatures not far below room temperature in typical cases. On the other hand, when the temperature is reduced, the conductivity "bottoms out" sooner and in the extreme anomalous limit the mean conductivity $C_F/\langle q\rangle$ is smaller than it would be in the absence of magnetism; at resonance it is of the order of ten times the room-temperature dc conductivity in a typical case. The effect of anomalous conductivity, as compared to the case when the conductivity function is taken equal to the dc conductivity, is to reduce exchange effects.

In order to evaluate these effects of exchange, one must know the wave-number spectra of the fields, i.e., their Fourier transforms. One cannot use the results of the usual anomalous skin-effect theory because the wave-number spectrum is itself strongly modified by exchange effects in the critical region near resonance. Estimates can be obtained in various ways,[6] but an accurate solution when exchange effects are large requires that the problem be solved self-consistently, i.e., that one solve the problem of the anomalous skin effect where the permeability, as well as the conductivity, depends upon wave number and frequency. In Sec. II, we obtain in implicit form a solution of the problem by an extension of the method of Reuter and Sondheimer. In the subsequent sections, we discuss in more detail the physics involved and examine numerically the properties of our solution in various cases of interest.

[The effect we are considering must be distinguished from the exchange narrowing which has been derived by Van Vleck for paramagnetic resonance[7] and which is expected to occur for ferromagnetic resonance also.[8] In that case one finds that the size of the damping term in the magnetic equation (4), when calculated quantum mechanically, is reduced by exchange effects. In the present calculation, on the other hand, we take the damping term as given and investigate exchange effects coming directly from the second term of (4).]

II. THE FORMAL SOLUTION

We consider the metal to fill the region $y>0$, with a dc magnetic field H_{eff} applied in the z direction. Anisotropy fields and static demagnetizing fields are included in H_{eff}. We use the time dependence $e^{-i\omega t}$ and represent the perturbing time-dependent parts of the fields by small letters. The Landau-Lifshitz equation for the precession of the magnetization is[9,10]

$$d\mathbf{M}/dt=\gamma(\mathbf{M}\times\mathbf{H})+(2A\gamma/M_s^2)(\mathbf{M}\times\nabla^2\mathbf{M})-(\lambda/\gamma M_s^2)(\mathbf{M}\times d\mathbf{M}/dt), \quad (4)$$

where

$$\mathbf{M}=M_s\hat{z}+m_x(y)\hat{x}+m_y(y)\hat{y}, \quad (4a)$$

$$\mathbf{H}=H_{\text{eff}}\hat{z}+h_x(y)\hat{x}+h_y(y)\hat{y}, \quad (4b)$$

and A is the exchange stiffness constant, $\gamma=-g|e|/2mc$ is the magnetomechanical ratio, λ the phenomenological magnetic relaxation constant, and M_s is the saturation magnetization.

We consider microwaves impinging from the negative direction, with the electric field in the z direction, and the h field in the x direction. The fields will vary significantly in the y direction only.

To proceed, we assume "diffuse reflection" at the surface $y=0$ and write the Maxwell curl equations for $y>0$, which are

$$\nabla\times\mathbf{h}(y)=\left(\frac{4\pi}{c}\right)\int_0^\infty\sigma(y-y')\mathbf{e}(y')dy', \quad (5)$$

$$\nabla\times\mathbf{e}(y)=(i\omega/c)(\mathbf{h}+4\pi\mathbf{m}). \quad (6)$$

The diffuse-reflection condition is represented by the lower limit of the integral in (5), according to which the current is generated by the fields seen by the electrons along their trajectories. We have assumed that all trajectories are straight lines, and that all memory of its previous history is lost by an electron when it strikes the surface. Further, we suppose that the conductivity is a scalar. These assumptions will be supported in Sec. III.

Equations (4)–(6) form a coupled set of integral-differential equations of the Wiener-Hopf type. They may be solved by converting the equations to a generalized Hilbert problem, a method which is very similar to that of Reuter and Sondheimer. Although the solution is the main contribution of the present work, we relegate it to the Appendix because of its formal character. We shall, however, at this stage, write down the result and discuss it.

The functions characterizing the metal are

$$\tilde{\sigma}(q)=\int_{-\infty}^{\infty}\sigma(y)e^{-iqy}dy \quad (7)$$

and $\tilde{\mu}(q)$, given by Eq. (1). The conductivity $\tilde{\sigma}(q)$ becomes the ordinary static conductivity σ_0 for small q and takes the form C_F/q in the extreme anomalous case, $|q|\gg 1/v_F\tau_{\text{elec}}$. The dynamic permeability $\tilde{\mu}(q)$ relates the dynamic magnetic field $\tilde{h}_x(q)$ to the dynamic induction $\tilde{b}_x(q)$,

$$\tilde{b}_x(q)=\tilde{h}_x(q)+4\pi\tilde{m}_x(q)=\tilde{\mu}(q)\tilde{h}_x(q) \quad (8)$$

in the idealized case in which there is a single Fourier component present. We have eliminated the y components of the fields by one of Maxwell's equations, which in this case is

$$b_y(y)=h_y(y)+4\pi m_y(y)=0. \quad (9)$$

The problem presented is to find the correct wave-

[6] G. T. Rado, J. Appl. Phys. **29**, 330 (1958).
[7] J. H. Van Vleck, Phys. Rev. **74**, 1168 (1948).
[8] J. H. Van Vleck, Phys. Rev. **78**, 266 (1950).
[9] L. D. Landau and E. Lifshitz, Physik. Z. Sowjetunion **8**, 153 (1935).
[10] T. L. Gilbert, Phys. Rev. **100**, 1243 (1955).

number spectrum of the fields. In the case of normal conductivity, the integral part of the equation becomes trivial, and discrete (complex) values of q are the only ones allowed. The correct weighting of the several values becomes a purely algebraic problem. In the present case we proceed by Fourier transform, with the surface terms being interpreted as fictitious sources of current and magnetization. By defining the fields to be zero in the unphysical $y<0$ region, we can replace the lower limit in (5) by $-\infty$ and introduce Fourier transforms. After the usual partial integration, which gives surface terms, we have, from (4) and (9),

$$\tilde{b}_x(q)=\tilde{\mu}(q)\tilde{h}_x(q)-\frac{2A\gamma iq}{M_s}\frac{(H_{\rm eff}+4\pi M_s+2Aq^2/M_s-i\lambda\omega/\gamma^2M_s)4\pi\gamma m_x(0)+i\omega 4\pi m_y(0)}{\gamma^2(H_{\rm eff}+4\pi M_s+2Aq^2/M_s-i\lambda\omega/\gamma^2M_s)(H_{\rm eff}+2Aq^2/M_s-i\lambda\omega/\gamma^2M_s)-\omega^2}, \tag{10}$$

where we have already dropped surface terms proportional to $dm_x(0)/dy$ and $dm_y(0)/dy$, which vanish by the "magnetic" boundary condition (see Sec. III).

Let us define the roots q_i, $i=1, 2, 3, 4$, such that

$$\tilde{\mu}(q)=(q^2-q_3^2)(q^2-q_4^2)/(q^2-q_1^2)(q^2-q_2^2), \tag{11}$$

where $\mathrm{Im}q_i\gtrless 0$.

Further, let

$$g(q)=1-4\pi i\omega c^{-2}q^{-2}\tilde{\sigma}(q)\tilde{\mu}(q). \tag{12}$$

Then we show in the Appendix that the five equations (13) and (14) determine the values of the fields at the surface:

$$\left\{\left(1+\frac{B'q+C'}{(q-q_1)(q-q_2)}\right)X(q)-1\right\}e_z(0)=\frac{M_s\omega q}{2A\gamma c}\frac{\gamma(H_{\rm eff}+4\pi M_s+2Aq^2/M_s-i\lambda\omega/\gamma^2M_s)4\pi m_x(0)+i\omega 4\pi m_y(0)}{(q^2-q_1^2)(q^2-q_2^2)},$$

$$(q=\pm q_3, \pm q_4); \tag{13}$$

and

$$e_z(0)(B'+iI/\pi)=(\omega/c)[h_x(0)+4\pi m_x(0)], \tag{14}$$

where

$$X(q)=\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}dq'\frac{\ln g(q')}{q'-q}\right\} \tag{15}$$

and

$$I=\int_0^{\infty}\ln g(q')dq'=\lim_{q\to\infty}\pi iq[1-X(q)]. \tag{16}$$

Taking $e_z(0)$ as given, the five equations (13) and (14) are to be solved for the five unknowns B', C', $m_x(0)$, $m_y(0)$, and $h_x(0)$, of which the first two are integration constants and the last three are values of the time-dependent fields at the surface of the metal. Of particular interest is $h_x(0)$, since the ratio $(4\pi/c)e_z(0)/h_x(0)$ is the surface impedance Z.

We have not found it possible to significantly simplify this set of equations. The range of exchange effects is $R=(A/2\pi M_s^2)^{1/2}\approx 10^{-6}$ cm and in typical cases $(R/\delta_0)^2\approx 0.01$, where δ_0 is the skin depth with the permeability set equal to unity. Thus the exchange stiffness in natural units is small; but it is so central to the problem that we have been unable to obtain a consistent and manageable perturbative expansion in A for any case of interest. Only when $A=0$ or $\tilde{\sigma}(q)=\sigma_0$ does there appear to be a simple expression for the surface impedance. These two cases are more easily handled without the full apparatus of our method.

The complexity of our solution is unfortunate from the point of view of general insight, but direct solution of the above equations for specific cases is a fairly straightforward computer problem. A FORTRAN program has been written which requires about 10 sec of computing time per surface impedance value. Computations for various cases of interest have been carried out on the IBM-7094 at the University of Maryland Computer Science Center.

III. PHYSICAL ASSUMPTIONS

The solution just obtained is mathematically exact, but of course is no better than the model used. Our primary assumption is the separability of the magnetic properties from the conduction properties of the electrons, i.e., the use of independent permeability and conductivity functions, in spite of the fact that the conduction-band electrons have some magnetization and the d-band electrons have some mobility. In so doing we have followed the phenomenological philosophy of the Landau-Lifshitz equation which has proved successful in other cases.

The second important set of assumptions concerns the simplification of the conductivity function. In gen-

eral, we cannot expect the conductivity to be a diagonal tensor. However, if the electric field is in a principle direction a single component suffices. Note that complications in the effects of magnetic anisotropy arise if this direction does not coincide with a principle direction of magnetization. We thus confine ourselves to this case.

Next, we have neglected the effects of the magnetic field on the orbital motions of the electrons. We shall, by sticking to a range of parameters such that $\omega_c\tau \lesssim \frac{1}{2}$, where ω_c is the cyclotron frequency, and τ the transport mean free time, reduce these effects a great deal. In this case, the only effect of the magnetic field is to curve the orbits slightly as they pass within the skin depth. The result of this curvature is to give a small negative dynamical magnetoresistance for nonmagnetic materials (at moderate field strengths). There is no sharp resonant structure. The curvature effect is further minimized by the fact that the most important electrons have velocities nearly parallel to the magnetic field. Although the question is not settled, we believe that the neglect of orbital curvature within the skin depth is not a seriously misleading approximation, especially at a resonance where the skin depth is small. In the same way we expect the dynamic Hall effect to make but a small contribution. In fact, for the geometry we have assumed the Hall currents make no contribution whatsoever.

For the conductivity we thus use the form familiar from anomalous skin-effect calculations:

$$\tilde{\sigma}(q)=\sum_s \frac{C_F{}^{(s)}}{i\pi q}\left[\left(1-\frac{x^{(s)}}{q}\right)^2 \ln\frac{x^{(s)}-q}{x^{(s)}+q}-\frac{2x^{(s)}}{q}\right], \quad (17)$$

where

$$x^{(s)}=(\omega+i/\tau_{\text{elec}}{}^{(s)})/v_F$$
$$\tau_{\text{elec}}{}^{(s)}=3\pi\sigma_0(s)/4v_F C_F{}^{(s)}.$$

The index s gives a summation over sheets of the Fermi surface; $\sigma_0{}^{(s)}$ is the dc conductivity due to a given sheet and $C_F{}^{(s)}/q$ is the extreme anomalous conductivity, where[11]

$$C_F{}^{(s)}=(e^2/4\pi^2\hbar)\oint_s \rho_y dq_y. \quad (18)$$

Since the formula is insensitive to the Fermi velocity $v_F \approx 10^8$ cm/sec, this conductivity function is essentially determined by the $C_F{}^{(s)}$ and $\sigma_0{}^{(s)}$ values. In the absence of an applied field it can be derived from the Boltzmann equation in the case of a spherical Fermi surface. Our Fermi surfaces are not spherical, but in any case (17) is a reasonable interpolation between the static and extreme anomalous limits.

Effective mass ratios determined from specific-heat data suggest that $\sigma_0{}^{(d)}$ will be smaller than $\sigma_0{}^{\text{conduction band}}$ by a factor of 10 to 30. This means that the d-band conductivity will not come into play until fairly low temperatures. Eventually, however, (17) will go over to the simpler extreme anomalous form C_F/q, where $C_F=\sum_s C_F{}^{(s)}$. This typically requires a resistivity ratio of 1000. C_F is determined according to (18) by the Fermi-surface geometry and is expected to vary appreciably with crystal orientation; it is not well known for any ferromagnetic metal, and is one of the quantities about which one would like to gain information.

Crude but useful estimates of C_F can be made by using the formula for its average over angles,

$$\langle C_F\rangle = e^2S/16\pi^2\hbar, \quad (19)$$

where S is the total Fermi-surface area. Taking Phillips'[12] values for the number of electrons in the various bands of nickel and using (19) with an asymmetry correction suggested by the neighboring element copper, we estimate, for nickel cut with a (100) surface, $C_F=0.7\times10^{24}$ cm^{-1}sec^{-1}. This value will be used in the numerical calculations to be discussed in Sec. IV. It does not differ greatly from that found by Pippard[13] for copper, where all the electrons are in the s-p conduction band.

The Landau-Lifshitz relaxation damping term (as modified by Gilbert)[9,10] is introduced into Eq. (4) in a purely phenomenological way, and in the absence of a satisfactory microscopic model there are no theoretical grounds for supposing the coefficient λ to be independent of temperature or frequency. Nevertheless, measurements by Rodbell[14] of linewidths in nickel whiskers indicate that λ remains constant for temperatures from -100°C to near the Curie temperature, and over a change in frequency from 9.2 to 35 kMc/sec, having a value of about 2.5×10^8 sec^{-1} in his best samples. In this laboratory linewidths have been measured in bulk single crystals of nickel corresponding to about this same value.[15] The linewidths measured earlier by Bloembergen[16] in polycrystalline samples of nickel and Supermalloy correspond to λ values which remain roughly constant from room temperature to near the Curie temperature, the values being about 3×10^8 sec^{-1} and 1.5×10^8 sec^{-1} for nickel and Supermalloy, respectively.

The free-spin magnetic boundary condition which we have used,

$$\mathbf{m}'(0)=d\mathbf{m}/dy|_{y=0}=0$$

is re-derived in Ref. 2 as a special case of a more general boundary condition including a surface anisotropy constant K_{surf} to describe the tendency of the spins to be "pinned" at the surface. If the direction of the surface anisotropy is such as to tend to pin the spins parallel to z, the magnetic boundary condition becomes

$$2A\mathbf{m}'(0)-K_{\text{surf}}\mathbf{m}(0)=0. \quad (20)$$

[11] A. B. Pippard, Rept. Progr. Phys. **23**, 176 (1960).

[12] J. C. Phillips, Phys. Rev. **133**, A1020 (1964).

[13] A. B. Pippard, Phil. Trans. Roy. Soc. **A250**, 325 (1957).

[14] D. S. Rodbell, Phys. Rev. Letters **13**, 471 (1964).

[15] S. M. Bhagat and L. L. Hirst (to be published).

[16] N. Bloembergen, Phys. Rev. **78**, 572 (1950).

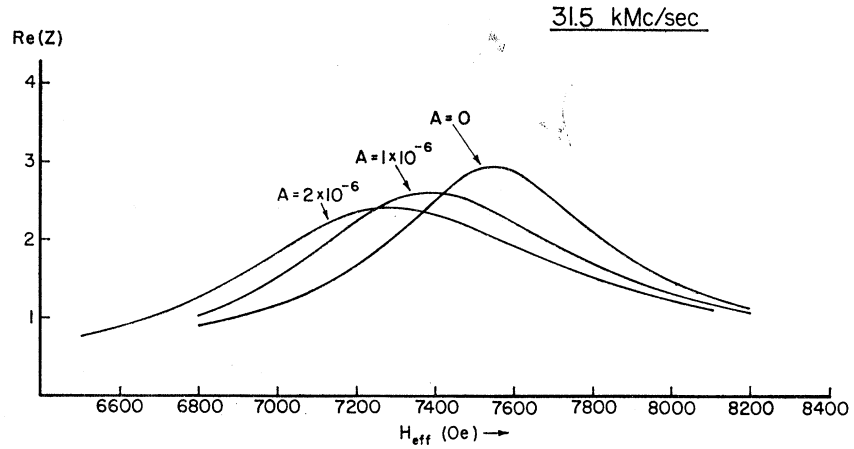

FIG. 1. Real part of surface impedance versus H_{eff}, model calculation for nickel, 31.5 kMc/sec.

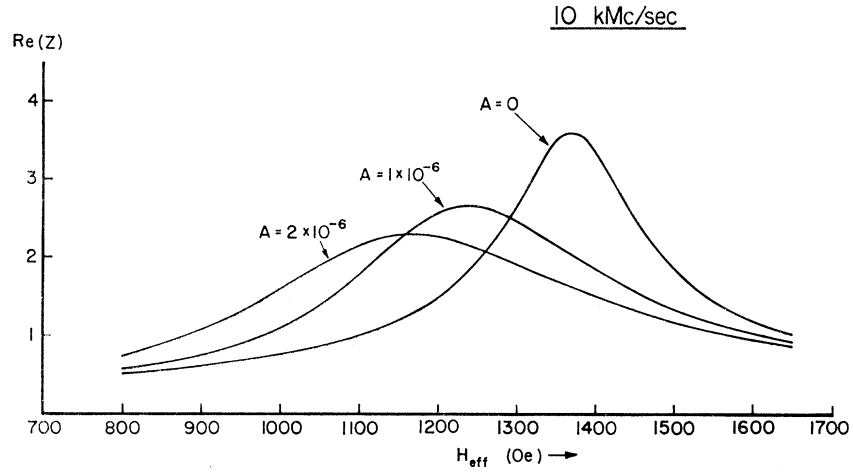

FIG. 2. Real part of surface impedance versus H_{eff}, model calculation for nickel, 10 kMc/sec.

If this more general boundary condition is used in our derivation, the result is that the terms in $m_x(0)$ and $m_y(0)$ in Eq. (13) are multiplied by $(1+K_{\text{surf}}/2Aiq)$ while Eq. (14) is unchanged. Such surface anisotropy causes an increase in exchange effects. The free-spin boundary condition is generally thought to be a good approximation under typical ferromagnetic resonance conditions; but see also Ref. 17.

Note added in proof. More recent experimental data calls for surface anisotropy values which are not negligible, although nearer to the "free" than to the "pinned" limit [Z. Frait and H. MacFaden (to be published); J. R. Anderson, S. M. Bhagat, and L. L. Hirst (to be published)]. When exchange dominates, the linewidth in the "pinned" limit is about three times that of the "free" case. Details will be reported in a future publication.

Our method of solution can also be applied to the geometry with the applied field perpendicular to the surface. This geometry has been dealt with elsewhere by another method.[18]

[17] Z. Frait and B. Heinrich, J. Appl. Phys. **35**, 904 (1964).
[18] V. L. Gurevich, Zh. Eksperim. i Teor. Fiz. **33**, 1497 (1957) [English transl.: Soviet Phys.—JETP **6**, 1155 (1958)].

IV. NUMERICAL RESULTS

In Figs. 1, 2, and 3 we show computed curves of the real part of the surface impedance versus applied field for nickel at 5, 10, and 31.5 kMc/sec. We have used extreme anomalous conductivity with $C_F=0.7\times10^{24}$ $\text{cm}^{-1}\text{sec}^{-1}$ as discussed above. The damping constant is taken as $\lambda=2.5\times10^8$ sec^{-1}. We show curves for $A=0$, 1, and 2×10^{-6} erg/cm. The linewidths of these curves

TABLE I. Results from model calculation for nickel.

Frequency (kMc)	Conductivity	A (erg/cm)	Linewidth (Oe)	$H_{\text{eff}}^{\text{res}}$ (Oe)
5	normal	...	64	...
5	extreme anomalous	0	52.5	390
5	extreme anomalous	1×10^{-6}	150	285
5	extreme anomalous	2×10^{-6}	215	225
10	normal	...	145	...
10	extreme anomalous	0	110	1370
10	extreme anomalous	1×10^{-6}	228	1235
10	extreme anomalous	2×10^{-6}	300	1170
31.5	normal	...	400	...
31.5	extreme anomalous	0	360	7550
31.5	extreme anomalous	1×10^{-6}	500	7390
31.5	extreme anomalous	2×10^{-6}	575	7280

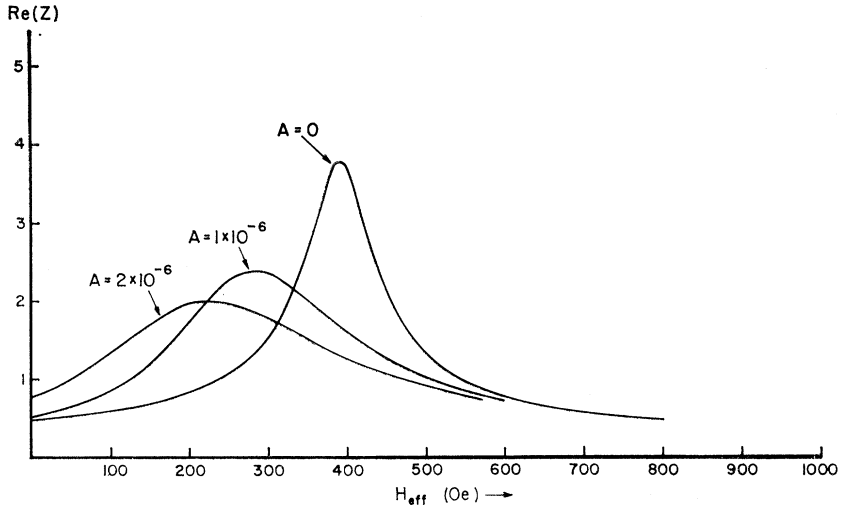

FIG. 3. Real part of surface impedance versus H_{eff}, model calculation for nickel, 5 kMc/sec.

are tabulated in Table I, with the room-temperature (normal-conductivity) widths corresponding to the same assumed value of λ shown for comparison. The linewidth given is the peak-to-peak width of the derivative of the real part of the surface impedance with respect to H_{eff}.

As expected, these curves show a broadening of the resonance and a shift of the peak toward lower fields. These effects increase with frequency in absolute terms but are relatively larger at low frequencies, and it follows, as first realized by Rado and Weertman,[1] that one should work at as low a frequency as possible to detect exchange effects.

To detect a shift in resonance upon lowering the temperature is experimentally simple, but interpretation is upset by the fact that the effective field we have used includes the anisotropy field, which, in pure metals, takes on values of several kilogauss in going to low temperatures. The anisotropy field would have to be known very accurately to enable one to isolate the exchange shift, which means that except in special alloys of low magnetic anisotropy one must proceed by observing linewidths. Careful work is required, since such difficulties as poor sample surfaces or loose mounting will introduce spurious broadening.

For the case of extreme anomalous conductivity the natural dimensionless measure of exchange is $A(4\pi\omega C_F/c^2)^{2/3}$, so that observation of exchange broadening or exchange shift in the resonance essentially measures $AC_F^{2/3}$. When exchange broadening dominates the magnetic damping, the linewidth and shift should be approximately proportional to $(AC_F^{2/3})^{1/2}$.

In addition to broadening the line, the exchange-conductivity mechanism gives a qualitative change in lineshape, causing the peak to be more rounded. This is brought out in the derivative curves for the 10 kMc case, Fig. 4. There is also a skewing of the resonance shape, since the center of the line is shifted more strongly than the fringes.

V. THE EXPERIMENTS OF RADO AND WEERTMAN

The most careful attempt to experimentally observe exchange effects in ferromagnetic resonance has been made by Rado and Weertman,[2] who measured both the real and the imaginary part of the surface impedance in Permalloy at 3 and 4 kMc. They discuss their results in terms of the equivalent permeability,

$$\mu_{equ}=(ic^2\sigma_0/4\pi\omega)Z^2=(4\pi i\sigma_0/\omega)[e_z(0)/h_x(0)]^2$$

which is that wave-number-independent permeability which would produce a given surface impedance Z if the conductivity were normal and exchange effects were absent. The normal-conductivity theory of Ament and Rado[3] shows that if the resonance shape is determined entirely by magnetic damping then μ_{equ}, plotted in the complex plane, traces out a circle tangent to the real axis at the origin as the field is varied, whereas a

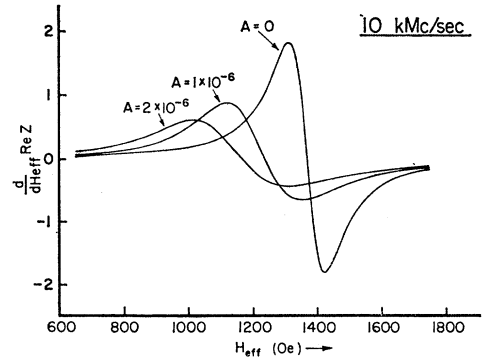

FIG. 4. Derivative of real part of surface impedance versus H_{eff}, model calculation for nickel, 10 kMc/sec.

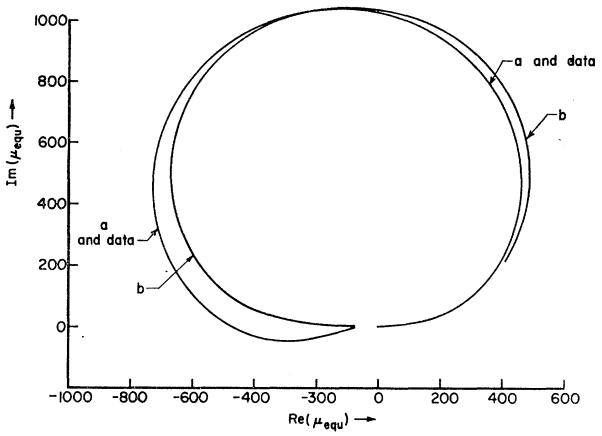

FIG. 5. Plot in complex plane of equivalent permeability of Permalloy; room temperature.

characteristic asymmetric "egg-shaped" curve results if the line shape is completely dominated by exchange. Rado and Weertman obtained a good fit to their experimental data at room temperature with the "egg-shaped" curve, but their results were somewhat puzzling in that the values they obtained for A were 2.88 and 3.80×10^{-6} erg/cm at 3 and 4 kMc, respectively, showing an apparent dependence of A on frequency and disagreeing with values of approximately 1.0×10^{-6} obtained by other methods. Discrepancies with further measurements taken at liquid-nitrogen temperatures were attributed to effects of nonlocal conductivity.

The dc resistivity of the Permalloy sample was 18.25 and 6.47 Ω-cm at room temperature and liquid-nitrogen temperature, respectively. We note that this is a resistivity ratio of only about 3 and that the nitrogen resistivity is about that of a pure metal at room temperature. For this reason the electron relaxation time is short and the d-band contribution to the conductivity remains negligible. From the known saturation magnetization and g value of the Permalloy we find it has 1.29 aligned electrons per atom, and from its 66:34 composition it follows that there are 0.61 conduction electrons per atom, assuming the conduction electrons to be unaligned. This gives $\langle C_F\rangle=0.23\times10^{24}$ $\text{cm}^{-1}\text{sec}^{-1}$, and we take $C_F=\langle C_F\rangle$, assuming the sample to be a random polycrystal. Effects of nonlocal conductivity turn out to be small despite the $\mu^{1/2}$ factor in skin depth; they cause the curve at nitrogen temperature to expand by about 10% as compared with the normal-conductivity curve. This agrees with an approximate calculation by Rado,[7] and indeed the two calculations should be equivalent when, as in the present case, Rado's approximations are valid.

The situation for the 4 kMc/sec data is shown in Figs. 5 and 6. In Fig. 5, curve a, a good fit is obtained by assuming magnetic exchange constant $A=4.00\times10^{-6}$ erg/cm (increased by 5% over the value found by Rado and Weertman due to the slight effects of nonlocal conductivity). But when one calculates the nitrogen-temperature curve, using the same value of A, the calculated μ_{equ} is smaller than the experimental values (Fig. 6, curve a). (The experimental curve cannot be fit even if one ignores the above estimate of the conductivity and regards C_F as an adjustable parameter.)

Over-all agreement at the two temperatures can be improved somewhat by introducing magnetic damping and decreasing the exchange constant correspondingly, the damping constant and exchange constant being required to be independent of temperature and frequency. The best results are obtained for values in the neighborhood of $A=1\times10^{-6}$, $\lambda=1.25\times10^{8}$, but the fit still cannot be called good, as can be seen from curves b, Figs. 5 and 6. These values, however, are in good agreement with the evidence of other experiments. Negligible damping, on the other hand, would mean $\lambda\lesssim10^{7}$ sec^{-1} (about ten times too small on the basis of the measurements discussed in Sec. III) and gives a value of A three or four times larger than the values obtained from thin-film spin-wave experiments. It is also interesting to note[19] that if the line shapes are ignored and attention is centered on the shift in resonant field due to exchange effects, then one deduces from these experiments $A\approx1\times10^{-6}$.

We are unable to suggest any satisfactory explanation for the remaining discrepancies. We emphasize that for the case of normal conductivity our results are identical with those of Ament and Rado, and thus our calculation is important only in the negative sense of showing that the discrepancies encountered by Rado and Weertman in trying to fit their nitrogen-temperature data cannot be attributed to nonlocal conductivity. If the values $A=1\times10^{-6}$ erg/cm and $\lambda=1.25\times10^{8}$ sec^{-1} are approximately correct, the linewidth at room temperature is about half due to magnetic damping and half to exchange broadening. Our conclusion is that exchange effects have been observed but are not sufficiently

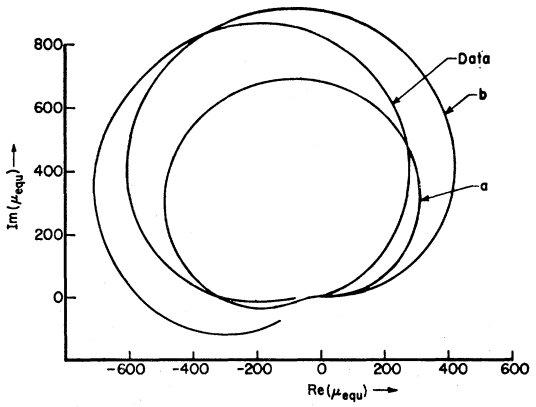

FIG. 6. Plot in complex plane of equivalent permeability of Permalloy; liquid-nitrogen temperature.

[19] G. T. Rado (private communication).

dominant to permit a reliable value of the exchange constant to be extracted.

VI. CONCLUSIONS

Using a physically reasonable model, we have solved the mathematical problem of relating the surface impedance of a ferromagnetic metal to the exchange stiffness, conductivity function, and other constants of the problem. The principal difficulty remaining is the uncertainty in the conductivity function, exchange constant, and surface anisotropy. Although it would be unrealistic to expect low-temperature ferromagnetic resonance experiments to determine these unknowns completely, such experiments should nevertheless yield some useful information.

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APPENDIX

The solution of Eqs. (4)–(6) of the text is obtained by the conversion of the equations to a Hilbert problem.[20] In this technique, one recognizes that the functions $\tilde{e}_z(\zeta)$, etc. defined as

$$\tilde{e}_z(\zeta)=\int_0^\infty dy e^{-i\zeta y}e_z(y) \tag{A1}$$

are analytic in the lower half-complex-ζ-plane, and that the physical function $\tilde{e}_z(q)$ is

$$\tilde{e}_z(q)=\lim_{\eta\to 0+}\tilde{e}_z(q-i\eta). \tag{A2}$$

We introduce $\tilde{N}(\zeta)$, analytic in the upper half-plane, defined by

$$\tilde{N}(\zeta)=\int_{-\infty}^0 dy e^{-i\zeta y}(4\pi/c)\int_0^\infty \sigma(y-y')e_z(y')dy'. \tag{A3}$$

[We shall avoid explicit evaluation of $\tilde{N}(\zeta)$.]

Then Eqs. (5) and (6) of the text may be Fourier transformed to yield

$$h_x(0)-iq\tilde{h}_x(q)+\tilde{N}(q)=(4\pi/c)\tilde{\sigma}(q)\tilde{e}_z(q) \tag{A4}$$

$$-e_z(0)+iq\tilde{e}_z(q)=(i\omega/c)[\tilde{h}_x(q)+4\pi\tilde{m}_x(q)]. \tag{A5}$$

By introducing the dynamic permeability according to (10), we obtain

$$q^2g(q)\tilde{e}_z(q)=-iqe_z(0)-(i\omega/c)\tilde{\mu}(q)\tilde{N}(q)-(i\omega/c)\tilde{\mu}(q)h_x(0)$$
$$-\frac{iM_s\omega q^2}{2A\gamma c}\frac{(H_{\rm eff}+4\pi M_s+2Aq^2/M_s-i\lambda\omega/\gamma^2M_s)4\pi\gamma m_x(0)+4\pi i\omega m_y(0)}{(q^2-q_1^2)(q^2-q_2^2)}, \tag{A6}$$

where the symbols $g(q)$, q_1, and q_2 have been introduced in the text [Eqs. (11)–(12)].

Let us introduce the function

$$X(\zeta)=\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^\infty dq\frac{\ln g(q)}{q-\zeta}\right\}, \tag{A7}$$

with

$$X^\pm(q)=\lim_{\eta\to 0+}X(q\pm i\eta). \tag{A8}$$

Note that $X(\zeta)$ is analytic, except possibly on the real ζ axis, and further, it does not vanish. What is more,

$$X^+(q)/X^-(q)=g(q). \tag{A9}$$

Multiplication of Eq. (A6) by $(q-q_1)(q-q_2)/X^+(q)$ yields

$$(q-q_1)(q-q_2)q^2\tilde{e}_z(q)/X^-(q)=\frac{(q-q_1)(q-q_2)}{X^+(q)}\left\{-\frac{i\omega}{c}\tilde{\mu}(q)[\tilde{N}(q)+h_x(0)]-iqe_z(0)\right.$$
$$\left.-\frac{M_si\omega q^2}{2A\gamma c}\frac{\gamma(H_{\rm eff}+4\pi M_s+2Aq^2/M_s-i\lambda\omega/\gamma^2M_s)4\pi m_x(0)+i\omega 4\pi m_y(0)}{(q^2-q_1^2)(q^2-q_2^2)}\right\}. \tag{A10}$$

[20] N. I. Mushkelishvili, *Singular Integral Equations* (P. Noordhoff Ltd., Groningen, The Netherlands, 1963).

The left-hand side is the boundary value of a function analytic in the lower half-plane whereas the right-hand side is analytic in the upper half-plane. It is a fundamental theorem of analysis[20] that the two functions can be regarded as being restrictions of a single entire function which in this case is simply a polynomial $P(q)$. Thus,

$$\tilde{e}_z(q)=P(q)X^-(q)/(q-q_1)(q-q_2)q^2. \quad \text{(A11)}$$

Some of the coefficients of $P(q)$ are determined from the conditions at infinity and zero. For small q, $X^-(q)\propto q$ and $\tilde{e}_z(q)$ is finite, thus implying that $P(0)=0$. For large q, integrations by parts show

$$\tilde{e}_z(q)=[e_z(0)/iq]+(i\omega/c)[h_x(0)+4\pi m_x(0)]/(iq)^2 +O(1/q^3) \quad \text{(A12)}$$

and

$$X^-(q)=1+(iI/\pi q)+O(1/q^2). \quad \text{(A13)}$$

We find, consequently,

$$\tilde{e}_z(q)=\frac{-ie_z(0)[q^3+Bq^2+Cq]X^-(q)}{q^2(q-q_1)(q-q_2)} \quad \text{(A14)}$$

and

$$\frac{i\omega}{c}[h_x(0)+4\pi m_x(0)] =ie_z(0)[(iI/\pi)+B+q_1+q_2], \quad \text{(A15)}$$

where B and C are still to be determined.

A straightforward way of obtaining the unknown coefficients B and C, as well as h_x, etc. is to perform the inverse Fourier transform on $\tilde{e}_z$ to obtain the functional form of $e_z(y)$ in terms of integration constants; thus,

$$e_z(y)=\sum_{i=1}^{3} A_i f_i(y) \quad [\textstyle\sum A_i f_i(0)=e_z(0)].$$

The microwave magnetic field may be determined from this expression by quadrature, and $m_x(y)$ in turn from $h_x(y)$ and $e_z'(y)$. Finally, $m_y(y)$ is found from the Landau-Lifshitz equation. Having found the functional forms of the fields in terms of the three unknown constants A_i, it is simple, in principle, to impose the boundary conditions. In the case of normal conductivity, the functions $f_i(y)$ may be taken to be simple exponentials $\exp(-\alpha_i y)$, where α_i are roots of the equation discussed by Ament and Rado. This method is very cumbersome, however, and we shall employ a trick based on analytic continuation to obtain the surface impedance.

The technique we use is valid if $\tilde{\sigma}(q)$ is not too pathological. We have chosen the standard formula familiar from anomalous skin-effect calculations for $\tilde{\sigma}(q)$, namely (17), with $\tilde{\sigma}(q)=\tilde{\sigma}(-q)$. We take $\tilde{\sigma}(\zeta)$ to be analytic except for branch cuts beginning at $\pm x^{(s)}$, and along the imaginary axis.

We now consider Eq. (A10) evaluated at $q=\pm q_3$, $\pm q_4$, i.e., the zeros of $\tilde{\mu}(q)$. In order to do this, we have to continue $\tilde{e}_z(q)$ and $\tilde{N}(q)$ into the upper and lower half-planes, respectively. In view of the analytic properties of $g(q)$, we can continue $X^-(q)$ and thus $\tilde{e}_z(q)$ by maintaining the relation $X^+(q)/X^-(q)=g(q)$. The function $\tilde{N}(q)$ can be continued into the lower half-plane by indenting the contour of

$$\tilde{N}(q)=\frac{4\pi}{c}\int_{-\infty}^{\infty}\frac{dq'}{2\pi i}\frac{\tilde{\sigma}(q')\tilde{e}_z(q')}{q'-q}$$

so that, for ζ in the lower half-plane

$$\tilde{N}(\zeta)=\frac{4\pi}{c}\int_{-\infty}^{\infty}\frac{dq'}{2\pi i}\frac{\tilde{\sigma}(q')\tilde{e}_z(q')}{q'-\zeta}+\frac{4\pi}{c}\tilde{\sigma}(\zeta)\tilde{e}_z(\zeta).$$

In particular, $\tilde{N}(-q_3)$ and $\tilde{N}(-q_4)$ are finite.

We have thus extended (A10) to complex q, and we can now substitute the particular values $q=\pm q_3$, $\pm q_4$ for which $\tilde{\mu}(q)=0$. It is further convenient to replace $B=B'-q_1-q_2$, $C=C'+q_1q_2$ since B' and C' remain finite for $A\rightarrow 0$ while B and C do not. We thus obtain Eqs. (13) of the text.