Alfvén-Wave Propagation in Solid-State Plasmas. I. Bismuth

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Alfvén-wave propagation velocities in single-crystal bismuth have been studied for the frequency range 13 to 18 kMc/sec using an interference technique. Experimental mass densities obtained with the magnetic field along each principal crystal axis and in the plane of the sample surface are compared with effective-mass components reported in the literature, and a value for n, the number of carriers, is obtained. The existence of quantum oscillations in the mass density of carriers, reflecting oscillations in the position of the Fermi level as the magnetic field is increased, is shown. Numerical results for the band parameters of bismuth obtained from this work are $n=3.10\pm0.10\times10^{17}$ cm^{-3}, $M_1=0.064\pm0.003$, $M_3=0.73\pm0.02$, and $m_4{}^2/m_3=0.37\pm0.04$.

I. INTRODUCTION

It has been pointed out by Buchsbaum and Galt[1] that Alfvén waves can be propagated at microwave frequencies in certain solid-state plasmas. They have reinterpreted the experiments performed by Galt *et al.*[2] on cyclotron resonance in bismuth in the region of high magnetic fields in terms of Alfvén-wave propagation along the direction of magnetic field. Similarly, Smith, Hebel, and Buchsbaum[3] (SHB) interpreted their experiments on hybrid resonance in bismuth in the region of high magnetic fields in terms of Alfvén-wave propagation across the static magnetic field. In both sets of experiments the reflection of electromagnetic waves from an essentially semi-infinite medium was measured and wave velocities were inferred from the impedance mismatch at the sample surface. The purpose of the present paper is to report the results of experiments on transmission of Alfvén waves through thin plasma slabs. We have been able in these experiments to accurately measure the wave velocity by using an interference technique. As will be shown, the microwave "optical thickness" varies with magnetic field. Therefore a Fabry-Perot type of interference fringe is found in the transmitted radiation as the field is varied.

Alfvén waves are magnetohydrodynamic in nature; they propagate in a medium with equal numbers of positive and negative carriers in the presence of a strong magnetic field.[4–6] As will be shown, the phase velocity depends upon the carrier mass density, and therefore the experimental phase velocities can be used to obtain information on the number of carriers and their effective masses. To date we have performed successful experiments on bismuth[7,8] and antimony.[9] Experiments on bismuth have also been reported by Kirsch and Miller,[10] Kirsch,[11] Khaikin *et al.*,[12] Bartelink,[13] and Khaikin, Fal'kovskii, Edel'man, and Mina.[14] Bowers and Steele[15] have published two useful review articles on the general field of plasma effects in solids including the Alfvén wave work. This paper is a report of our work on bismuth for wave propagation transverse to the magnetic field.

In Sec. II we present the theory of Alfvén wave propagation for the simple case of a plasma whose carriers have isotropic masses and infinite relaxation times. In Sec. III the details of the experimental techniques are discussed. Section IV applies the theory to the band structure of bismuth. In Sec. V we present the experimental results and their interpretation.

II. THEORY FOR ISOTROPIC MASSES

We are concerned with the propagation of Alfvén waves through 1 mm or more of material at high magnetic fields. The conditions of the experiments are such that classical skin-effect conditions exist. Therefore, the classical Drude-Zener model is adequate to discuss the results. This model has been used to discuss many of the cyclotron-resonance and other experiments performed on bismuth.[2,16,17] We first discuss the idealized case of

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[1] S. J. Buchsbaum and J. K. Galt, Phys. Fluids 4, 1514 (1961).

[2] J. K. Galt, W. A. Yager, F. R. Merritt, B. B. Cetlin, and A. D. Brailsford, Phys. Rev. 114, 1396 (1959).

[3] G. E. Smith, L. C. Hebel, and S. J. Buchsbaum, Phys. Rev. 129, 154 (1963). Referred to as SHB in the rest of this paper.

[4] H. Alfvén, Nature 150, 405 (1942); Arkiv. Mat. Astr. Fysik 29B, No. 2, 2 (1943).

[5] E. Astrom, Arkiv Fysik 2, 443 (1950).

[6] H. Alfvén, *Cosmical Electrodynamics* (Clarendon Press, Oxford, England, 1950).

[7] G. A. Williams, Bull. Am. Phys. Soc. 7, 409 (1962).

[8] G. A. Williams, Bull. Am. Phys. Soc. 8, 205 (1963).

[9] G. A. Williams, Bull. Am. Phys. Soc. 9, 353 (1963).

[10] J. Kirsch and P. B. Miller, Phys. Rev. Letters 9, 421 (1962).

[11] J. Kirsch, Bull. Am. Phys. Soc. 8, 205 (1963); Phys. Rev. 133, A1390 (1964).

[12] M. S. Khaikin, V. S. Edel'man, and R. T. Mina, Zh. Eksperim. i Teor. Fiz. 44, 2190 (1963) [English transl.: Soviet Phys.—JETP 17, 1470 (1963)].

[13] D. J. Bartelink, Bull. Am. Phys. Soc. 8, 205 (1963).

[14] M. S. Khaikin, La Fal'kovskii, V. S. Edel'man, and R. T. Mina, Zh. Eksperim. i Teor. Fiz. 45, 1839 (1963) [English transl.: Soviet Phys.—JETP 18, 1167 (1964)].

[15] R. Bowers and M. C. Steele, Proc. IEEE 52, 1105 (1964); R. Bowers, Symposium on Plasma Effects in Solids, Paris (1964) (to be published).

[16] W. S. Boyle and G. E. Smith, *Progress in Semiconductors* (Heywood and Company, Ltd., London, 1963), Vol. 7.

[17] J. K. Galt, *Proceedings of the International Conference on High Magnetic Fields, Cambridge, 1961* (John Wiley & Sons, Inc., New York, 1962).

isotropic masses which contains most of the essential physics.

Consider a plasma with n electrons of mass m_e and n holes of mass m_h. Let the relaxation times of each of the carriers be infinite. The Lorentz-force equation for each type of carrier is

$$m(\partial \mathbf{v}/\partial t)=e(\mathbf{E}+\mathbf{v}\times\mathbf{B}/c)\,, \tag{1}$$

where $\mathbf{v}$ is the particle velocity, $\mathbf{E}$ the electric field, and $\mathbf{B}$ the magnetic field. For the experiments under consideration $\mathbf{B}$ is a large static external magnetic field and $\mathbf{E}$ is a microwave electric field parallel to the sample surface. The microwave current can be obtained as $\mathbf{J}=ne(\mathbf{v}_h-\mathbf{v}_e)$, and from this a microwave conductivity $\boldsymbol{\sigma}$. The propagation is most easily described in terms of a complex dielectric constant $\boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon}$ is given by

$$\boldsymbol{\varepsilon}=\epsilon_l\cdot\mathbf{I}-(4\pi i/\omega)\boldsymbol{\sigma}. \tag{2}$$

Here ϵ_l is the frequency-independent or lattice portion of the total dielectric constant, and $\mathbf{I}$ the unit dyadic.

Using the notation of Smith, Hebel, and Buchsbaum,[3] the dielectric tensor for isotropic carriers and a uniform magnetic field along the z axis is given by

$$\boldsymbol{\varepsilon}=\begin{bmatrix}\epsilon_\perp & -\epsilon_\times & 0\\ +\epsilon_\times & \epsilon_\perp & 0\\ 0 & 0 & \epsilon_{||}\end{bmatrix},$$

$$\epsilon_\perp=\epsilon_l-\frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2}-\frac{\omega_{ph}^2}{\omega^2-\omega_{ch}^2}\,,$$

$$\epsilon_\times=j\frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2}\frac{\omega_{ce}}{\omega}-j\frac{\omega_{ph}^2}{\omega^2-\omega_{ch}^2}\frac{\omega_{ch}}{\omega}\,, \tag{3}$$

$$\epsilon_{||}=\epsilon_l-\omega_{pe}^2/\omega^2-\omega_{ph}^2/\omega^2=\epsilon_l-\omega_p^2/\omega^2\,,$$

where $\omega_{ce}=eB/m_ec$ and $\omega_{ch}=eB/m_hc$ are the electron and hole cyclotron frequencies, $\omega_{pe}=(4\pi ne^2/m_e)^{1/2}$ and $\omega_{ph}=(4\pi ne^2/m_h)^{1/2}$ are the electron and hole plasma frequencies, and $\omega_p=[4\pi ne^2(m_e+m_h)/m_em_h]^{1/2}$ is the total plasma frequency. ϵ_l is the lattice dielectric constant.

Maxwell's equations yield the wave equation which must be satisfied by the electromagnetic wave:

$$\nabla\times\nabla\times\mathbf{E}-(\omega^2/c^2)\boldsymbol{\varepsilon}\cdot\mathbf{E}=0. \tag{4}$$

In this paper we will discuss solutions corresponding to a plane wave propagating perpendicular to the magnetic field which lies in the plane of the sample surface. A subsequent paper will discuss propagation along the magnetic field. The dispersion equation obtained from Eq. (4) for the present case is identical to that discussed by SHB.[3]

$$[k^2-\epsilon_{||}\omega^2/c^2][k^2\epsilon_\perp-(\epsilon_\perp^2+\epsilon_\times^2)\omega^2/c^2]=0. \tag{5}$$

The wave corresponding to the first root $k_0^2=(\omega^2/c^2)\epsilon_{||}$ does not propagate under the conditions of our experiment. Alfvén wave propagation corresponds to the high-field limit of the root

$$k_e^2=(\omega/c)^2(\epsilon_\perp^2+\epsilon_\times^2)/\epsilon_\perp. \tag{6}$$

In this limit $\epsilon_\times/\epsilon_\perp\cong\omega^2/\omega_c^2\ll 1$, and

$$k_e^2\cong(\omega^2/c^2)\epsilon_\perp\,, \tag{7a}$$

$$=(\omega^2/c^2)\{\epsilon_l+\omega_{pe}^2/\omega_{ce}^2+\omega_{ph}^2/\omega_{ch}^2\} \tag{7b}$$

$$=(\omega^2/c^2)\{\epsilon_l+(4\pi c^2/B^2)n(m_e+m_h)\} \tag{7c}$$

$$=(\omega^2/c^2)\{\epsilon_l+c^2/v_a^2\}\,, \tag{7d}$$

where $v_a=\{(B^2/4\pi)[n(m_e+m_h)]^{-1}\}^{1/2}$ is the Alfvén velocity. The propagation constant k_e is that of wave with $\mathbf{E}$ vector perpendicular to the magnetic field.

A similar result holds for the case of the magnetic field perpendicular to the sample surface, that is, the propagation is along the magnetic field, except that two waves of orthogonal polarization can propagate. For isotropic masses, these can be right and left circularly polarized. For anisotropic masses, they are linearly polarized at right angles to one another along principal crystal axes. Alfvén waves in bismuth and antimony in this orientation will be discussed in a later publication.

For magnetic fields which are not extremely large, that is, $\epsilon_l\ll c^2/v_a^2$, we have

$$k_e^2=\omega^2/v_a^2. \tag{8}$$

Thus there exists a wave whose phase velocity is proportional to the magnetic field, inversely proportional to the mass density of carriers, and independent of frequency. The frequency independence distinguishes the Alfvén wave from the helicon or "whistler" mode of propagation, which is found when carriers of one sign predominate. Helicon waves with a frequency-dependent dispersion relationship have been discussed by several authors.[15,18–20] Equation (8) is nearly correct for bismuth for magnetic fields less than about 100 kG provided, of course, that $\omega\ll\omega_c$.

III. EXPERIMENTAL DETAILS

The experimental measurement of the propagation velocity has been achieved by using an interference technique. Samples of bismuth 1 to 3 mm thick were prepared with flat, parallel surfaces. The transmission of linearly polarized microwave radiation, 13 to 18 kMc/sec, was observed as a function of magnetic field. Above a few thousand gauss, the attenuation is sufficiently weak that a Fabry-Perot-type interference pattern or geometric resonance is observed in most samples. The intensity maxima occur when

$$N\lambda_0=2d(c/\omega)k\,, \tag{9}$$

[18] P. Aigrain, *Proceedings of the International Conference on Semiconductor Physics*, Prague (1960) (Czechoslovakian Academy of Sciences, Prague, 1961).
[19] A. Libchaber and R. Veilex, Phys. Rev. **127**, 774 (1962).
[20] R. Bowers, C. Legéndy, and F. Rose, Phys. Rev. Letters **7**, 339 (1961).

where N is the fringe index, d the sample thickness and λ_0 the free space wavelength. The term $\cos\theta$ in the usual interference formula is set equal to 1 because the refractive index is sufficiently high in the medium that propagation is always normal to the surface for all incident waves. Combining Eqs. (8) and (9) we find

$$N=(4d\omega\pi^{\frac{1}{2}}/B)\{n(m_e+m_h)\}^{1/2}. \tag{10}$$

Therefore a plot of fringe number versus $4d\omega\pi^{\frac{1}{2}}/B$ should yield a straight line with a slope equal to the square root of the mass density. One should note that this is independent of any knowledge of the absolute value of N.

The experimental arrangement used in these experiments is shown in Fig. 1. The sample was held between two irises so as to completely block the cylindrical waveguide. Microwaves with frequencies between 13 and 18 kMc/sec were incident from above. If the sample had been made part of a resonant cavity, signal strengths would have been improved, but varying the frequency would have been less convenient. Since sufficient signal to noise ratio was available, a resonant cavity was not used. The apparatus was immersed in liquid helium at 1.2°K, in order that values of $\omega\tau>1$ might be attained, where τ is the scattering time. If $\omega\tau<1$ the wave is strongly damped. The low temperature also facilitated the use of a carbon bolometer detector.

Two methods of mounting samples were used. In one case the sample was simply pressed between the two irises mechanically. This led to a certain amount of microwave leakage around the sample. The leakage signal mixed with the transmitted signal and lead to an interference pattern with a period twice that of the internal "Fabry-Perot" pattern. At higher magnetic fields the internal pattern became evident. Since the period of the two patterns is different they can be distinguished from one another. In other instances the sample was soldered to the upper iris, and no leakage interference pattern occurred. This had the disadvantage that the sample quality deteriorated much more rapidly with cycling between room and liquid-helium temperatures, and once it was found that the leakage and interval patterns could be easily distinguished, the simpler, unsoldered arrangement was used.

A bolometer cut from a 10-Ω, 2-W, carbon radio resistor, similar to those first used by Boyle and Rodgers[21] in the infrared was mounted below the sample. The lower end of the bolometer cavity consisted of a tuning piston. Copper was plated on the ends of the bolometer and leads soldered to the copper. An exchange gas pressure of approximately 20μ of helium gas was maintained to obtain thermal contact between the bolometer and the bath. A bias current of 45 μA was used, and the voltage across the bolometer amplified by a Baird-Atomic coherent detector tuned to 27 cps. The incident microwave signal was also sine-wave amplitude modulated at 27 cps. Minimum signal sensitivity has been measured as better than 5×10^{-11} W with this detector system as used. The experimental plots therefore show the amplitude of the microwave signal at the lower side of the sample as a function of magnetic field.

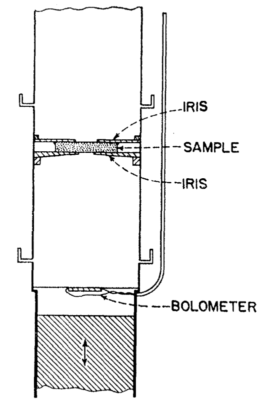

FIG. 1. Diagram of the sample holder and detector arrangement. The entire region was operated with an exchange-gas pressure of He of about 20-μ Hg. The inside diameter of the waveguide was 0.82 in.

Samples 0.5 to 3.0 mm thick were prepared from boules grown by the Bridgeman technique.[22] The starting material was 99.9999%-pure zone-refined bismuth.[23] Preformed, pointed, polycrystalline slugs were embedded in pure powdered graphite in a quartz tube, the tube was evacuated, and then lowered through a furnace at a rate of 0.13 in./h. These boules were oriented with x rays, and slabs cut using an acid string saw with a cutting acid of 50% nitric acid to which a few drops of hydrochloric acid were added. The samples were carefully oriented, then mechanically lapped. The final 0.020 in. was removed from the surface by lapping on a surface of "twill jean" cloth with an acid etch of 3 parts fuming nitric to 1 part glacial acetic acid. Before using the sample, a back-reflection Laue photograph was taken to check for residual strain and orientation. All samples used had one principal crystal axis perpendicular to the sample surface, and therefore the other two in the plane of the surface. Orientation of the sample surface with respect to the crystal axis perpendicular to the surface was better than 1° in all cases.

The orientation of the magnetic field with respect to the crystal axes in the sample surface was performed in two steps. First the sample was mechanically mounted with one axis parallel to a mark on the apparatus. Then data was taken for various orientations of the magnetic field near this line, until an extremum in the data was found. This was taken as the desired orientation. A check was obtained by ensuring that a similar extremum existed 90° from the first.

Data at several frequencies between 13 and 18 kMc/

[21] W. S. Boyle and K. F. Rodgers, J. Opt. Soc. Am. **49**, 66 (1959).

[22] H. E. Buckley, *Crystal Growth* (John Wiley & Sons, Inc., New York, 1951).

[23] Obtained from Cominco Products, Inc., Spokane 4, Washington.

sec were obtained for all samples in order to verify the frequency independence of the wave velocity predicted by Eq. (7).

IV. THEORY FOR ANISOTROPIC MASSES

Since the effective masses in bismuth are anisotropic, the dielectric constant used in Sec. II is inadequate. The model of the band structure of bismuth most frequently used to discuss experiments on bismuth is the ellipsoidal model recently reviewed by Boyle and Smith.[16] In this model the electron band is made up of three ellipsoids slightly tilted out of the plane perpendicular to the threefold axis. For a typical ellipsoid, the electron energy is given by

$$E=(\hbar^2/2m_0)(\alpha_1k_1^2+\alpha_2k_2^2+\alpha_3k_3^2+2\alpha_4k_2k_3), \quad (11)$$

where k_1, k_2, and k_3 are the wave-vector components parallel to the binary, bisectrix and trigonal axes, and the α's are components of the reciprocal mass tensor. For such an ellipsoid, the effective mass tensor has the form

$$m_a=m_0\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & m_4 \\ 0 & m_4 & m_3 \end{bmatrix}, \quad (12)$$

where m_1, m_2, m_3, and m_4 are the effective mass components in units of m_0. The two remaining ellipsoids are obtained by 120° rotations about the threefold axis.

The hole band is made up of a single ellipsoid of revolution about the trigonal axis. For this band

$$E-E_b=(\hbar^2/2m_0)(k_1^2/M_1+k_2^2/M_1+k_3^3/M_3), \quad (13)$$

where E_b is the band overlap energy, and M_1 and M_3 the effective mass components in units of m_0.

The magnetoconductivity tensor for this model has been calculated,[24] and the high-field values of the conductivity can be computed directly from it. The Alfvén velocity becomes

$$v_a^2=B^2/4\pi nf(m), \quad (14)$$

where $nf(m)$, an effective mass density, is a function of the mass components obtained from the high-field conductivity. Values of $nf(m)$ for the magnetic field along all principal crystal directions are given in Table I for comparison with experiment. It should be noted that for a given orientation of magnetic field, the function $nf(m)$ depends on the direction of polarization of the E field in the plane at right angles to B. The expressions in Table I are valid only for E parallel to one of the principal crystal axes.

When the magnetic field is along the binary or trigonal axes, the high-field limit of the dielectric tensor is nearly diagonal. For the magnetic field along the bisectrix direction, Eq. (5) is more complex and does not separate. In the high-field limit, however, it separates approximately.

In contrast to most of the other experiments which

TABLE I. Experimental mass densities, expressions for $nf(m^*)$ from the conductivity tensor,[a] and values of n calculated from this data for several sets of masses. Experimental errors quoted in this table are experimental standard deviations of the averages, with no estimate of systematic error.

$B_{\parallel}$ Axis	$E_{rf}\parallel$ Axis	$[nf(m^*)]^{\frac{1}{2}}$ (observed) $\times10^{-9}$	$nf(m^*)$	n (calculated) ($\text{cm}^{-3}\times10^{-17}$) Galt[b]	SHB[c]	Kao[d]	SBR[e]
1	2	0.484±0.002	$n\left\{m_3-\frac{2m_4^2}{m_1+3m_2}+M_3\right\}$	2.5^0	2.9^4	3.0^0	3.2^8
1	3	0.318±0.002	$n\left\{\frac{m_2}{3}+\frac{8}{3}\frac{m_1m_2}{m_1+3m_2}+M_2\right\}$	1.4^6	2.0^5	1.5^5	2.1^0
2	1	0.470±0.002	$n\left\{m_3-\frac{m_4^2}{3m_2}-\frac{2}{3}\frac{m_4^2}{3m_1+m_2}+M_3\right\}$	2.3^7	2.8^5	2.8^6	3.1^5
2	3	0.162±0.002	$n\left\{\frac{m_1}{3}+\frac{8}{3}\frac{m_1m_2}{3m_1+m_2}+M_1\right\}$	2.7^5	3.4^5	2.9^0	3.7^3
3	1	0.393±0.002	$\frac{n}{2}\left\{(m_1+m_2)-\frac{m_4^2}{m_3}+2M_2\right\}$	3.5^7	3.0^6	3.8^4	3.3^5
3	2	0.460±0.002	$\frac{n}{2}\left\{(m_1+m_2)-\frac{m_4^2}{m_3}+2M_1\right\}$	4.9^0	4.2^1	5.2^8	4.6^1

[a] Reference 24. [b] Reference 2. [c] Reference 3.
[d] Reference 25. [e] Reference 26.

[24] B. Lax, K. J. Button, H. J. Zeiger, and L. Roth, Phys. Rev. 102, 715 (1956).

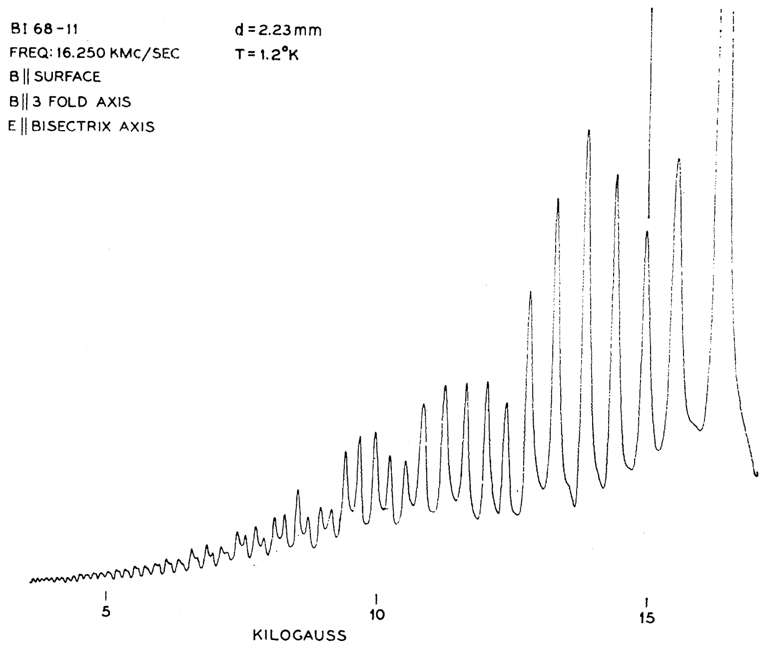

FIG. 2. Interference fringes in a sample of bismuth 2.23 mm thick. The magnetic field was parallel to the threefold axis and in the plane of the sample surface. The microwave **E** field was parallel to a bisectrix axis.

measure effective mass components, the Alfvén wave experiments are most sensitive to the heavier mass components. In this way they complement the other experiments.

One can obtain a feeling for the quantities measured by the Alfvén-wave velocity by noting the following. For a band consisting of a single ellipsoid, such as the holes in bismuth, if the magnetic field is along axis 1, and the microwave electric field along axis 2, the mass component contributing to the mass density is M_3. It is the component corresponding to the axis orthogonal to both the magnetic and electric fields.

V. RESULTS AND ANALYSIS

Figure 2 shows a typical interference pattern as obtained in these experiments. For this sample the magnetic field is along the threefold axis, and the microwave field parallel to the bisectrix axis. For the frequency used, 16.25 kMc/sec, the highest cyclotron resonance occurs near 400 G as can be seen in the data of SHB,[3] or calculated from the effective mass components. The thickness of this sample was 2.23 mm. In Fig. 2 the intensity of signal received at the bolometer below the sample is plotted as a function of magnetic field. Below about 6000 G, a "leakage" pattern, as described in Sec. III, is observed. As the attenuation decreases with increasing magnetic field above this, the Fabry-Perot, or geometric resonances appear. This has half the period of the leakage pattern as previously discussed.

In Fig. 3 the fringe index of the first 35 peaks is plotted versus $1/B$. The index of the fringe at the highest magnetic field is arbitrarily set equal to one. We obtain an excellent straight line as predicted in Eq. (10). From Eq. (10) the slope of this line is $4d\omega(\pi m_0)^{\frac{1}{2}}[nf(m^*)]^{1/2}$. Here $f(m^*)$ is in units of the free-electron mass m_0. From the measured slope we can find the experimental mass density $nf(m^*)$. The excellent straight line obtained gives confidence in the form of Eq. (10). The $1/B$ dependence shown is partial proof that we are observing a true Alfvén wave. True Alfvén-wave behavior as described by Eq. (7) also requires that the wave velocity be independent of frequency. This was checked by studying each sample at several fre-

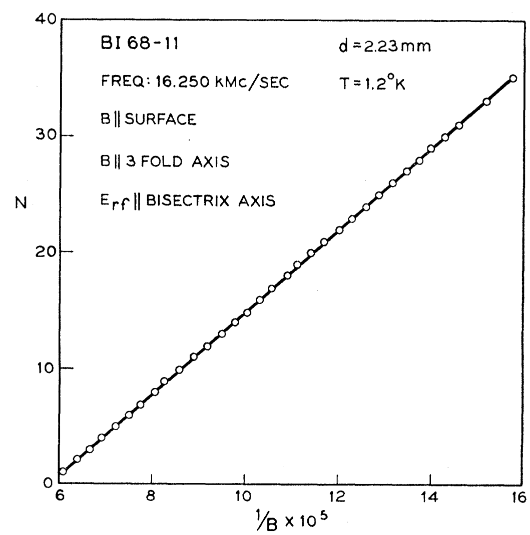

FIG. 3. A plot of fringe index versus position in $1/B$ for the data of Fig. 2. The index of the fringe at highest magnetic field is arbitrarily set equal to 1.

quencies between 13 and 18 kMc/sec. The wave velocity was found to be independent of frequency to approximately 1%, thus confirming Alfvén-wave behavior.

The wave velocity will only be independent of frequency if the damping is sufficiently weak. The extinction coefficient $k=\eta/2\omega\tau$, so that

$$\epsilon_{\text{real}}=\eta^2-k^2 \tag{15}$$

yields

$$\epsilon_{\text{real}}=\eta^2(1-1/4\omega^2\tau^2). \tag{16}$$

The experiment measures η, the effective refractive index, so that for $\omega\tau<1$

$$\eta\cong\epsilon_{\text{real}}^{1/2}(1+1/8\omega^2\tau^2+\cdots). \tag{17}$$

Therefore $\omega\tau>3.5$ yields a frequency-dependent correction of 1% or less to the measured refractive index. If the correction is 2% at 13 kMc/sec it will be approximately 1% at the highest frequencies used. Since our estimated frequency independence is 1%, there may be a correction of the order of 1–2% to the measured refractive index. This error is in the sense that the experimental refractive index will be high. This would lead to high values for the experimental mass densities.

There are six possible orientations of the steady magnetic field and microwave electric field with respect to principal crystal axes. Two or more samples were studied for each possible orientation and data was taken at several frequencies for each sample. In each case preliminary orientation of the crystal axes in the plane with respect to magnetic field was accomplished using the x-ray orientation data. Final orientation was achieved by finding extrema in the data as a function of angle (B still parallel to the plane of the sample). In only one case did this procedure lead to difficulties as discussed below.

The mass densities obtained from the experimental data are collected in Table I. Also in Table I are the mass-density expressions as obtained from the conductivity tensor.[24] The off-diagonal contributions to these terms have been shown to be small by direct calculation. In order to compare the experimental mass densities with those calculated from the various sets of effective masses in the literature, a value for the number of carriers n must be chosen. Alternatively, the various sets of mass components in the literature can be used to calculate $f(m^*)$; this combined with the experimental mass densities yields a value for n, the number of carriers. This has been done in Table I for the mass parameters of Galt *et al.*,[2] Smith, Hebel, and Buchsbaum,[3] Kao,[25] and Smith, Baraff, and Rowell,[26] (SBR) as representative sets. The experimental agreement between the data of this work and that of Kirsch,[11] and Khaikin, *et al.*,[14] is not good. The reasons for this are not clear. Most of the data of Kirsch were taken with the magnetic field normal to the surface, so a direct comparison is difficult. His data were taken at a lower frequency, and therefore damping effects, as discussed previously, may play a role in the differences. The data of Khaikin *et al.*, were taken at two frequencies, both above, and below ours. Their data for S waves, which correspond to these experiments, do not agree well with ours. In particular the high velocity reported for $H\|C_1$, $k\|C_2$, is hard to understand. One possible source of error in interpreting results of these experiments is the "leakage" versus Fabry-Perot type of interference pattern.

Several things are immediately apparent from Table I. The values for n are reasonably consistent except for lines 2 and 6. Line 6 is in error, and will be discussed later. The over-all average of all 20 determinations of n in lines 1–5 is 2.84×10^{17}/cc. A tabulation of reported values of n is included as Table II.

In order to obtain the most reliable value for n from the data, we must investigate whether any of the values of $f(m^*)$ calculated above can be considered more reliable than others. For this purpose we include Table III, which is a summary of various sets of mass parameters in the literature. We see immediately that the values for the hole masses, and in particular, the more recent determinations, vary less from one to the other than the electron-mass components, and therefore can probably be considered more reliable. The mass densities in lines 1 and 3 are the only ones that depend almost entirely on hole mass components. The average of the four most recent values of M_3 [SHB,[3] Kao,[25] SBR,[26] Grenier, Reynolds, and Sybert (GRS)[27]] yields $n=2.94\times10^{17}$/cc from the experimental mass densities in lines 1 and 3.

A more appropriate method of handling the data is to note that all of the cyclotron resonance experiments

TABLE II. Values reported for the number of carriers in bismuth.

	n_e/cm^3 ($\times10^{-17}$)	n_h/cm^3 ($\times10^{-17}$)
Schoenberg[a]	4.2	
Boyle and Brailsford[b]	4.4	
	3.9 (corrected[d])	
Brandt[c]		3.4
Jain and Koenig[d]	4.2	3.9
Mase, von Molnar, and Lawson[e]	3.75	
Weiner[f]	3.6	3.4
Zitter[g]	2.5	
Kao[h]	4.05	
	3.87	3.5
SBR[i]	2.75	2.75
This work	3.10±0.10	3.10±0.10

[a] D. Schoenberg, Phil. Trans. Roy. Soc. (London) **A245**, 1 (1952).
[b] Reference 28.
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TABLE III. Electron and hole mass components reported in the literature and the values from this work.

	Electrons					Holes	
	m_1	m_2	m_3	m_4	m_4^2/m_3	$M_1=M_2$	M_3
Galt *et al.*[a]	0.0088	1.80	0.023	$\pm$0.16	1.11	0.068	0.92
Boyle and Brailsford[b]	0.0088	1.27	0.0163	0.085	0.44		
Jain and Koenig[c]	0.0084	1.71	0.0219	0.144	0.95		
SHB[d]	0.0062	1.30	0.017	−0.085	0.42	0.057	0.77
Kao[e]	0.0071	1.70	0.0301	0.177	1.04	0.0675	0.76
Weiner[f]	0.00495	1.03	0.0245	−0.103	0.43		
SBR[g]	0.00521	1.20	0.0204	−0.090	0.40	0.064	0.69
Greiner *et al.*[h]						0.065	0.84
This work					0.37±0.04	0.064 ±0.003	0.73±0.02

[a] Reference 2.
[b] Reference 28.
[c] A. L. Jain and S. H. Koenig, Phys. Rev. **127**, 442 (1962).
[d] Reference 3.
[e] Reference 25.
[f] D. Weiner, Phys. Rev. **125**, 1226 (1962).
[g] Reference 26.
[h] Reference 27.

measure the quantity $(M_1M_3)^{1/2}$ directly. We can combine this with the experimental results for nM_3 from lines 1 and 3 and $n(M_1+3m_1)$ from line 4 of Table I and solve simultaneously for n, M_1, and M_3. The results are tabulated in Table IV. The errors quoted in Table IV are the spread which results from the various possible choices for m_1. Also included for completeness are the data of GRS[27], from de Haas–Shubnikov measurements.

The best value for n from this work is almost certainly that obtained from this last procedure, since it relies most directly on experimentally determined quantities, and avoids some of the obvious errors included by a blind averaging of the results in Table I. If we take the more recent cyclotron resonance data of SHB,[3] Kao,[25] and SBR[26] as being the more reliable, the numerical values obtained are $n=3.10\pm0.10\times10^{17}$/cc, $M_3=0.73\pm0.02$, and $M_1=0.064\pm0.003$, where the errors quoted here are the spread in values obtained from the two sets of data. It is believed that this error is larger than any systematic error in our data. The data of GRS[27] do not yield directly M_1M_3 and therefore have not been included. If one tries to solve for n, M_1 and M_3 using their value of 12.9 for M_3/M_1, a value for m_1 of 0.0094 is found necessary. This is larger than any of the reported values, and indicates that their ratio M_3/M_1 is probably too high.

The predicted mass densities in lines 5 and 6 are identical. The experimental values are not. If one studies the theoretical angular dependence of the mass densities, one finds that in the 2-3 plane the result is not symmetric about the principal axis when the magnetic field is near the threefold axis. In fact, on one side of the principal axis, with the magnetic field between 6 and 9° (depending on the numerical values of the mass components) from the threefold axis, the term $-m_4^2/m_3$ in the mass density is canceled. Since the last 5–10° of sample orientation was achieved by searching for an extremum in the data, the measured $f(m^*)$ for line 6 is high by m_4^2/m_3. No check was available in this plane with the magnetic field at right angles to axis 3, since the mass density there goes through a broad flat minimum, and the data cannot be used for orientation purposes. The value of m_4^2/m_3 obtained by comparing the data in lines 5 and 6 in this manner is 0.37±0.04. The calculated value of m_4^2/m_3 for the various sets of masses is included for comparison in Table III. The values of Boyle and Brailsford,[28] SHB,[3] and SBR[26] fit the best.

The presence or absence of a band of "heavy" holes in bismuth has been the subject of controversy for some time.[29–33] Lerner[32,33] has interpreted his data in terms of a band of heavy holes with nearly isotropic masses. Since the Alfvén-wave velocity is directly dependent on the mass density, the experimental values can be used to place an upper limit on the number of holes that might exist in such an isotropic band with an effective mass of the order of the free-electron mass. The smallest mass density observed is that in Table I, line 4. 0.02×10^{17}/cc carriers of effective mass of the order of m_0 in addition to the "light"-hole band would approximately double the mass density observed in this orientation. Therefore, the number of isotropic holes with

TABLE IV. Values of n, M_1, and M_3 calculated using the mass densities from this work, and the product M_1M_3 from other work. The errors shown are the spread arising from varying m_1 from 0.0052[a] to 0.0088.[b]

	Obs.	Calculated		
Experiment	(M_1M_3)	$n\times10^{-17}$ cm^{-3}	M_3	M_1
Galt *et al.*[b]	0.0625	2.75±0.09	0.83±0.03	0.074±0.003
SHB[c]	0.044	3.20±0.12	0.71±0.03	0.061±0.003
SBR[a]				
Kao[d]	0.0513	2.99±0.11	0.76±0.03	0.066±0.003
Grenier *et al.*[e]	0.0556	2.87±0.09	0.79±0.03	0.070±0.002

[a] Reference 26.
[b] Reference 2.
[c] Reference 3, and G. E. Smith (private communication).
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[31] D. Weiner, Phys. Rev. **125**, 1226 (1962).
[32] L. S. Lerner, Phys. Rev. **127**, 1480 (1962).
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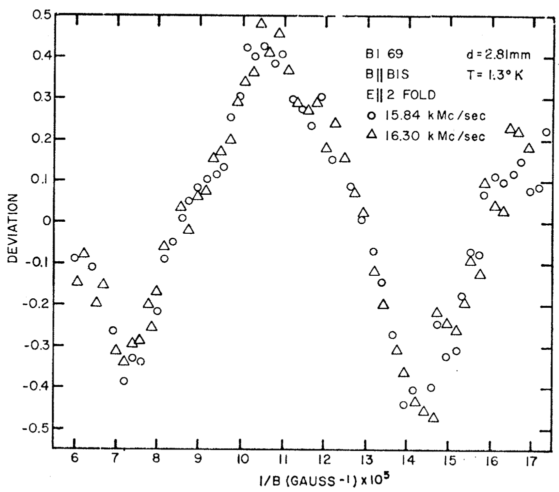

FIG. 4. A plot of the deviation from straight-line behavior of several data plots similar to Fig. 3. The deviations are in fractions of a whole fringe, and are measured from least-squares straight lines through the data points.

this mass must be less than 1% of the total number of carriers. The "heavy"-hole band would be more difficult to eliminate using the Alfvén-wave data if its shape were similar to that of the "light" holes, where $M_1=M_2$, and $M_3/M_1 \cong 10$–12; because it would then simply increase the experimental mass densities, so that our experiment would appear to yield unduly high values for the parameters of the "light" band, or high values for n, if the mass parameters from other measurements are used. However any hole band with $M_1 \approx m_0$ can be stated to have $n<0.01\times10^{17}$/cc.

The low values for n appearing in line 2 indicate that too large a value for m_2 is being used, or possibly that in this orientation the effects of nonparabolic bands[34] are seen. This possibility is currently being studied.

We have observed another phenomenon in these experiments which is most striking when the magnetic field is along a bisectrix axis. We find for this orientation of the magnetic field that when the peak positions of interference fringes are plotted as in Fig. 3 there are small periodic deviations from straight line behavior. We interpret this as a direct measurement of oscillations in the mass density of carriers which results from a shift in the Fermi energy as that energy is crossed by various Landau levels. This phenomenon occurs as an accompaniment to the de Haas–van Alphen fluctuations in the density of states at the Fermi surface. We believe that, so far, it is only direct measurement of the oscillatory effects in the mass density of carriers, and thus indirectly on fluctuations in the value of the Fermi energy. In Fig. 4 the deviations from the least-squares line of several plots similar to Fig. 3, except that the magnetic field is along the bisectrix direction, are plotted versus $1/B$. The period observed agrees with the de Haas–van Alphen period for this orientation. Further details of this work can be found elsewhere.[35]

In summary, we have measured Alfvén wave velocities for all orientations of magnetic field and rf electric field in the plane of the sample. We find excellent agreement with the predicted $1/B$ dependence, and frequency independence of wave velocity. The data yield the most direct available measurement of the number of carriers. Our best value is $n=3.10\pm0.10\times10^{17}$/cc. We also obtain values of $M_1=0.064\pm0.003$, $M_3=0.73\pm0.02$ and $m_4^2/m_3=0.37\pm0.04$. We find a large discrepancy with previous values for m_2, the heaviest electron component. We have made a direct measurement of oscillations in the mass density of carriers as a function of magnetic field.

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