If the delta function approximation were made, Eqs. (IV.10) and (IV.12) would read, respectively,

$$\dot{E}_1+\tfrac{1}{2}(\omega/Q)E_1=\tfrac{1}{2}(\omega/\epsilon_0)AE_1 \times[Z_i(\xi_1)-(\pi^{1/2})I_1-(\pi^{1/2})I_2\mathfrak{L}(\xi)], \quad \text{(IV.14)}$$

$$(\omega_1+\Phi_1-\Omega_1)=\tfrac{1}{2}(\omega/\epsilon_0)A \times[Z_r(\xi_1)+(\pi^{1/2})I_2\mathfrak{L}(\xi)(\xi/\eta)]. \quad \text{(IV.15)}$$

Now consider one of the neglected contributions to the polarization, having the form of Eq. (IV.4) with $y\rightarrow-y$ in all the exponentials except $\exp-2\eta(x+y)$. Using the integral

$$\int_0^\infty\int_0^\infty dxdy\,\exp[2i(\xi_a x+\xi_b y)-2\eta(x+y)-(x+y)^2] =(8\Delta\xi)^{-1}[Z(\xi_a)-Z(\xi_b)], \quad \text{(IV.16)}$$

where $\Delta\xi=\frac{1}{2}(\xi_b-\xi_a)$, the right-hand sides of Eqs. (IV.10) and (IV.11), respectively, take the form,

$$Z_i(\xi_1)-I_1Z_i(\xi_1)-I_2\mathfrak{L}(\xi)[\bar{Z}_i-(\eta/\xi)\bar{Z}_r] -2\eta I_1[1-\eta Z_i(\xi_1)+\xi_1 Z_r(\xi_1)], \quad \text{(IV.17)}$$

$$Z_r(\xi_1)+(\xi/\eta)I_2\mathfrak{L}(\xi)\{\bar{Z}_i-(\eta/\xi)\bar{Z}_r -\tfrac{1}{2}[1+(\eta/\xi)^2][Z_i(\xi_1)-Z_i(\xi_2)]\} +2\eta I_1[\xi_1 Z_i(\xi_1)+\eta Z_r(\xi_2)], \quad \text{(IV.18)}$$

where use has been made of the identity

$$(\eta/\xi)-(\xi/\eta)\mathfrak{L}(\xi)=(\eta/\xi)\mathfrak{L}(\xi)\,.$$

Thus the effect of not making the delta-function approximation is to introduce additional terms of the order of $\eta\ll1$. Neglecting these terms is equivalent to neglecting the effects of natural broadening on the line-shape for the third-order polarization. Equations (42)–(46) are written in this approximation and for the case of small rotations ($\Delta\xi\ll\eta$).

Note added in proof. It has been brought to my attention that an adaptation of Lamb's model to the traveling-wave case has been presented by C. V. Heer in a report at the 1964 Symposium on Unconventional Inertial Sensors, Polytechnic Institute of Brooklyn Graduate Center, Farmingdale, New York (unpublished).

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Index of Refraction and Sum Rules for Helium*

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Recent measurements by Lowry and by Lukirskii of the atomic photoeffect for helium can be combined with calculations for discrete transitions and for the high-energy photoeffect to give the oscillator density for the entire range. We use dispersion theory to calculate the index of refraction n. We find satisfactory agreement with experiments in the visible and in the near ultraviolet, and we predict the value of n from 2700 to 600 Å. Our predicted values agree satisfactorily with values calculated by Chen and Dalgarno. We use these oscillator densities to calculate the sum rules $S(0)$, $S(1)$, and $S(2)$, which weight with the zeroth, first, and second powers of the photon energy, respectively. Each result agrees with values based on ground-state wave functions to 5% accuracy. We conclude that in general the oscillator density is now known to about 5% accuracy.

I. INTRODUCTION

Lowry *et al.*[1] have recently measured the absorption cross section for helium for photons in the energy range from the photoeffect threshold of 1.807 to 8.82 Ry; while Lukirskii *et al.*[2] have worked up to energies of 20.6 Ry. We are interested in dispersion-theoretic calculations of the index of refraction n, and also in calculations of oscillator-strength sum rules $S(p)$ weighted with the pth power of the photon energy. For our calculations we need both the oscillator strengths for discrete transitions, and we need the oscillator density at all energies. We use Schiff and Pekeris' calculations[3] for discrete transitions, calculations by Stewart and Webb[4] from 20.6 to 130 Ry, and calculations by Salpeter and Zaidi.[5] We have chosen the Stewart-Webb calculation for the velocity matrix element with Hartree-Fock wave functions. We add to this the cross section for double excitation calculated by Salpeter and Zaidi. There is a 10% discrepancy

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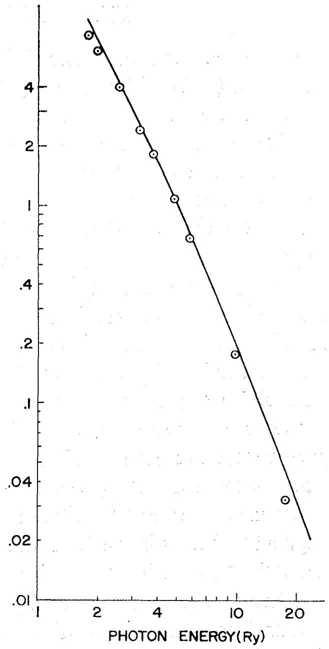

FIG. 1. Log-log plot of cross section for helium photoeffect in Mb *versus* photon energy W in Ry. The solid curve shows Lowry's and Lukirskii's data, Refs. 1 and 2; the points show calculations of Stewart and Webb, Ref. 4, using the velocity matrix element for Hartree-Fock wave functions.

between theory[4,5] and experiment[2,6] near 20 Ry as shown in Fig. 1. Above 130 Ry we use Salpeter-Zaidi's calculation for single and double excitation.

Lowry *et al.*[1] used their measurements to evaluate the oscillator sum rules $S(-2)$, $S(-1)$, $S(0)$, and $S(1)$. Here

$$S(p)=\sum_n f_{0n}W^p \tag{1}$$

where f_{0n} is the oscillator strength (or density) and W is the photon energy in Ry. We follow Lowry in the treatment of oscillator strength for discrete transitions[3] and oscillator density up to 8.82 Ry. We differ from Lowry in the treatment of the oscillator density above 8.82 Ry, since we now have Lukirskii's measurements to rely on. We also evaluate $S(2)$, which is sensitive to the high-energy photoeffect. Lowry's results and ours both give good agreement with values for the sum rules based on the helium ground-state wave function.

In the next section we calculate the index of refraction n for helium gas, using Fig. 1 for the oscillator densities. Of course, the value of n at very low frequencies corresponds to the dc polarizability, or $S(-2)$; and in fact n at other frequencies has been treated as an expansion in $S(p)$ for p a negative even integer. The main purpose of our calculation of n is to provide predictions of the index of refraction in the region around 1000 Å. Measurements in this region would provide a sensitive test of the Pekeris-Schiff calculations.

After we had completed our calculations of the index of refraction, we learned of the very recent Chan-Dalgarno[7] variational calculations of n. The comparison below with their calculations gives satisfactory agreement, thus providing a check on each calculation.

II. THE INDEX OF REFRACTION

From the usual dispersion relation,

$$n(\omega)-1=(Nc/\pi)\int\sigma(\omega')d\omega'/(\omega'^2-\omega^2). \tag{2}$$

Here N is the number of helium atoms per cm^3 at standard conditions, and $\sigma(\omega')$ is the photoeffect cross section which we take from Fig. 1, supplemented by calculations[3,4,5] at lower and higher photon energies.

Our results for the index of refraction are given in Table I and are illustrated in Fig. 2. We see that there is satisfactory agreement between our calculations from Eq. (2) and the measurements[8] of n in the visible and near ultraviolet. The discrepancy shown of only 2% in $(n-1)$ is not serious, for two reasons. First, the microwave polarizability of helium[9] is 1% above that found by extrapolation from Cuthbertson,[8] thus reducing the discrepancy. Second, the experimental values of the cross sections have errors of 5% or more, so a 2% discrepancy in our calculated $(n-1)$ is to be anticipated.

TABLE I. Calculated index of refraction of helium. The index of refraction at 1 atm and 0°C calculated from Eq. (2), using absorption cross section from Refs. 1–5.

λ (Å)	$n-1$	λ	$n-1$
600	215.5×10^{-6}	3700	35.9×10^{-6}
700	75.2×10^{-6}	4000	35.8×10^{-6}
800	57.6×10^{-6}	4300	35.7×10^{-6}
900	50.3×10^{-6}	4600	35.7×10^{-6}
1000	46.3×10^{-6}	4900	35.6×10^{-6}
1300	40.9×10^{-6}	5200	35.6×10^{-6}
1600	38.8×10^{-6}	5500	35.5×10^{-6}
1900	37.7×10^{-6}	5800	35.5×10^{-6}
2200	37.0×10^{-6}	6100	35.5×10^{-6}
2500	36.6×10^{-6}	6400	35.5×10^{-6}
2800	36.3×10^{-6}	6700	35.4×10^{-6}
3100	36.1×10^{-6}	7000	35.4×10^{-6}
3400	36.0×10^{-6}		

We note that earlier workers[10] have also calculated values of the index of refraction that were in satisfactory agreement with experiments on the index of refraction and on the Verdet constant.

Recently, Chan and Dalgarno[7] have performed a variational calculation of the index of refraction for wavelengths from 600 to 2500 Å. Their variational calculation employs four adjustable parameters, and appears to converge very rapidly when the wavelength

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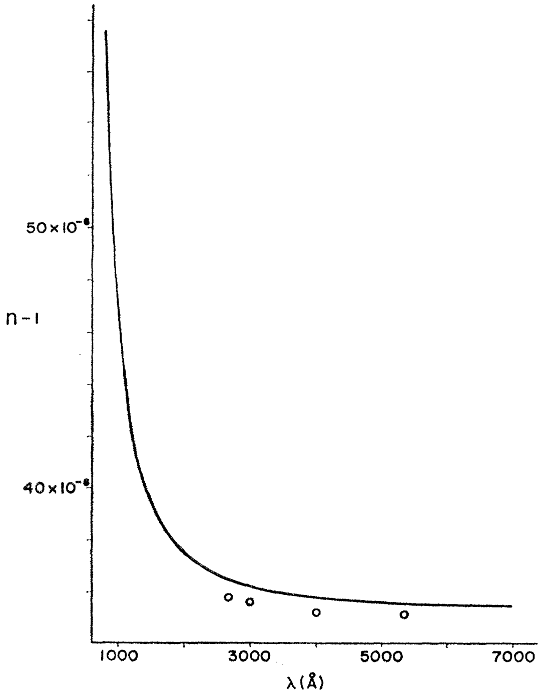

FIG. 2. Index of refraction n of helium *versus* wavelength λ. The curve shows our calculations: see Eq. (2) and Table I. The points show measurements, Ref. 7.

is not close to a resonant line. Our values of $(n-1)$ given in Table I vary from 0.3 to 2.8% higher than those taken from Table II of Chan and Dalgarno. (At 2000 Å they find $n-1=36.4\times10^{-6}$ while we find 37.4×10^{-6}). As noted above, a discrepancy of this magnitude in our calculations in not unexpected; so this comparison provides a satisfactory check of each calculation of n. Clearly, an experimental confirmation would be desirable.

TABLE II. Calculations of summed oscillator strength. The quantity $S(p)$ is defined in Eq. (1) and is given in units of Ry^p. The input data, *b*, *c*, and *d*, are shown in Fig. 1.

Energy region	Ref.	$S(-2)$	$S(-1)$	$S(0)$	$S(1)$	$S(2)$
Discrete, to 1.807 Ry	a	0.157	0.247	0.40	0.65	1
Threshold to 8.82 Ry	b	0.190	0.498	1.45	5.20	22
8.82 to 20.6	c	0.001	0.011	0.12	1.52	21
20.6 to ∞	d	0.000	0.001	0.02	0.99	83
Total		0.348	0.757	1.99	8.36	127
Other calculation		0.349[e]	0.752[f]	2.00[f]	8.17[f]	121[f]

[a] Schiff and Perkeris, Ref. 3.
[b] Lowry, Ederer, and Tomboulian, Ref. 1.
[c] Lukirskii, Brytov, and Zimkina, Ref. 2.
[d] Stewart and Webb, Ref. 4; and Salpeter and Zaidi, Ref. 5.
[e] Essen, Ref. 9 and Johnson, *et al.*, Ref. 9.
[f] Pekeris, Ref. 12.

III. SUM RULES

As noted above, Lowry[1] has evaluated sum rules $S(-2)$, $S(-1)$, $S(0)$, and $S(1)$ using Eq. (1) with input data for the oscillator strength (or density) similar but not identical to those used here. Our calculation above for the index of refraction essentially repeats Lowry's $S(-2)$; and we have also found nothing new in the sums $S(-1)$ and $S(0)$, since they are insensitive to the region above 8.82 Ry where our input differs from Lowry's. However, $S(1)$ and $S(2)$ are sensitive to the photoeffect cross section above 8.82 Ry, and thus provide a good check on Lukirskii's measurements,[2] and on calculations by Stewart and Webb[4] and by Salpeter and Zaidi.[5] (We note that Lukirskii provides both actual measurements, and gives a 10% correction to give better agreement with Tomboulian's earlier data.[11] This correction also gives excellent agreement with the early measurement by Dershem and Schein.[6] Lukirskii suggests that his helium may have impurities, thus necessitating this correction. We use the Lukirskii measurement, as corrected by him.)

Our five sum-rule results are given in Table II. We show the contributions from various energy regions, so that one can judge which energy region is being tested most severely by which sum rule. We also show in the table the value $S(-2)$ from Essen's polarizability measurements, and values of $S(-1)$, $S(0)$, $S(1)$, and $S(2)$ taken from Pekeris's calculation[12] of the helium ground-state wave function.

We see from Table II that our input data[1–5] for Eq. (1) give remarkably good agreement with other calculations of $S(p)$ based on polarizability or on Pekeris' wave function for the ground state. The discrepancies are never as large as 5%, and since the sum rules for different p test different energy regions, we conclude that in general the photoeffect cross section is now known to about 5% accuracy at all energies. Of course, we still have the problem of the 10% jump in joining experiment and calculation at 20.6 Ry.

Helium is the second atom for which all five sum rules have been verified[13] by recent experiments.

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