angular-momentum components which we have done here, and secondly the derivation of the small-gradient formula given here is more straightforward.

If we had calculated $D(1)$ using the first order approximation to the Fredholm determinant

$$D(1)\approx 1-\int dr\, K(r,r)\,, \tag{50}$$

we would have obtained

$$D(1)\approx 1+\int dr\int dq\,\Phi(qr)\frac{d\chi(q,\omega,r)}{dr}\,. \tag{51}$$

Comparing this with Eq. (47) we see that the screening, expressed by the local dielectric constant $\epsilon(q,\omega,r)$, is included in Eq. (47) but is neglected in Eq. (51). That is, the formula Eq. (47) amounts to a summation of a class of terms in the Fredholm expansion of $D(\lambda)$, which corresponds to the summation of ring diagrams in the electron-gas problem but neglects, e.g., effects due to higher derivatives and powers of the density gradient. We note that

$$d\chi(q,\omega,r)/dr\to 0$$

with the density gradient, i.e., in the limit of zero-gradient density the correct result for a uniform electron gas is guaranteed.

We have begun comprehensive calculations of the statistical spectral functions derived in this paper.

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Mode Competition and Frequency Splitting in Magnetic-Field-Tuned Optical Masers*

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Equations are given describing the beat-frequency variation and mode competition in a gaseous optical maser operated in a magnetic field parallel to the maser axis. The equations include only lowest order nonlinear terms. Important terms in the amplitude- and frequency-determining equations are shown to arise from an induced atomic precession. These terms have a character similar to those describing the effects of selective depletion of the velocity distribution or "hole burning." It is shown that the induced atomic precession causes parametric conversion of an optical field of one circular polarization into one of the other polarization with a frequency shift equal to the rate of precession. This process tends to make the competition between modes of different polarizations important. An additional feature, not found in the scalar theory, is that, for sufficiently large magnetic fields, competition can be important between modes separated in frequency by several Doppler-broadened linewidths.

THIS paper gives equations describing the beat-frequency variation and mode competition in a gaseous optical maser operated in a magnetic field parallel to the maser axis, the equations being obtained from a density-matrix analysis similar to that of W. E. Lamb, Jr.[1,2] Important terms in the amplitude- and frequency-determining equations arise from an induced atomic precession in atoms occupying regions of the velocity distribution analogous to the population depletion holes described by Bennett, Jr.,[3] and by Lamb.[1] Similar calculations by Tang and Statz,[4] and by Culshaw and Kannelaud[5] do not produce the explicit equations given here.

Assuming a $J=1$ to $J=0$ optical transition, circularly polarized components of the optical field of amplitude E_+ and E_-, only one cavity resonance (Ω_+,Ω_-) above oscillation threshold for each polarization (Ω_+ is not necessarily the same resonance as Ω_-), an active medium filling the entire cavity, and otherwise following the assumptions and method of Lamb, one obtains the amplitude- and frequency-determining equations

$$\alpha_+-\beta_+E_+^2-\theta_{+-}E_-^2=0\,, \tag{1a}$$

$$\alpha_--\beta_-E_-^2-\theta_{-+}E_+^2=0\,, \tag{1b}$$

$$\nu_+=\Omega_++\sigma_++\rho_+E_+^2+\tau_{+-}E_-^2\,, \tag{2a}$$

$$\nu_-=\Omega_-+\sigma_-+\rho_-E_-^2+\tau_{-+}E_+^2\,, \tag{2b}$$

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[1] W. E. Lamb, Jr., Phys. Rev. 134, A1429 (1964).

[2] A more complete theory allowing x-y asymmetries in the resonator, transverse magnetic fields, and larger J values is being carried out by the authors in association with W. E. Lamb, Jr.

[3] W. R. Bennett, Jr., Appl. Opt., Suppl. 1, 24 (1962).

[4] C. L. Tang and H. Statz, Phys. Rev. 128, 1013 (1962).

[5] W. Culshaw and J. Kannelaud, Phys. Rev. 136, A1209 (1964).

where

$$\alpha_+ = -\tfrac{1}{2}(\nu/Q_+) + \Gamma_+ Z_i(\nu_+ - \omega_+),$$
$$\sigma_+ = \Gamma_+ Z_r(\nu_+ - \omega_+).$$

In the following α_-, σ_-, β_-, ρ_-, θ_{-+}, and τ_{-+} are obtained by making the subscript exchange $(+\leftrightarrow-)$ in the appropriate given function. Here ν is the optical frequency; Q_+ is the quality factor for the $+$ polarization (we assume axial symmetry and that $Q_+=Q_-$); Z_r and Z_i are the real and imaginary parts of the plasma dispersion function[1]; and $\Gamma_+ = \frac{1}{2}\nu\bar{N}_+[p^2/(\epsilon_0\hbar Ku)]$. Here $pE_{+(-)}$ is the perturbation producing the transition $|J=1, m=+(-)1\rangle \rightarrow |J=0, m=0\rangle$, (that is, p is the reduced matrix element of the dipole operator taken between electronic states); $\bar{N}_{+(-)}$ is the excitation density[1] for that transition (we assume $N_+=N_-$ and set $\Gamma_+=\Gamma_-\equiv\Gamma$); $\omega_{+(-)}$ is the optical frequency for that transition; and $Ku(\ln 2)^{1/2}$ is the Doppler half-width. The other coefficients are

$$\beta_+ = (\Gamma'/\gamma_a\gamma_b)[1+\gamma_{ab}{}^2\mathcal{L}_{ab}(\omega_+ - \nu_+)], \tag{3}$$

$$\begin{aligned}\theta_{+-} = {} & (\Gamma'/2\gamma_b\gamma_{ab})[\gamma_{ab}{}^2\mathcal{L}_{ab}(\delta)+\gamma_{ab}{}^2\mathcal{L}_{ab}(\omega_0-\nu_0)] \\ & +(\Gamma'/2)\mathcal{L}_a(2\delta)\{(\gamma_a\gamma_{ab}-2\delta^2)\mathcal{L}_{ab}(\delta) \\ & +[\gamma_a\gamma_{ab}-2(\omega_+-\nu_+)\delta]\mathcal{L}_{ab}(\omega_+-\nu_+)\},\end{aligned} \tag{4}$$

$$\rho_+ = -(\Gamma'/\gamma_a\gamma_b)\gamma_{ab}(\omega_+-\nu_+)\mathcal{L}_{ab}(\omega_+-\nu_+), \tag{5}$$

$$\begin{aligned}\tau_{+-} = {} & -(\Gamma'/2\gamma_b\gamma_{ab})[\gamma_{ab}\delta\mathcal{L}_{ab}(\delta) \\ & +\gamma_{ab}(\omega_0-\nu_0)\mathcal{L}_{ab}(\omega_0-\nu_0)] \\ & -(\Gamma'/2)\mathcal{L}_a(2\delta)\{(2\gamma_{ab}+\gamma_a)\delta\mathcal{L}_{ab}(\delta) \\ & +[2\gamma_{ab}\delta+\gamma_a(\omega_+-\nu_+)]\mathcal{L}_{ab}(\omega_+-\nu_+)\}.\end{aligned} \tag{6}$$

Here $\mathcal{L}_n(\omega) = [\gamma_n{}^2+\omega^2]^{-1}$; γ_a and γ_b are the decay rates for the $J=1$ and $J=0$ level, respectively; $\gamma_{ab}=(\gamma_a+\gamma_b)/2$; $\nu_0=(\nu_++\nu_-)/2$; $\omega_0=(\omega_++\omega_-)/2$; $\delta=[\gamma H-(\nu_+-\nu_-)/2]$; $\gamma=\mu_B g/\hbar$; g is the g factor for level a; μ_B is the Bohr magneton; H is the magnetic field; and $\Gamma'=\Gamma\pi^{1/2}(p^2/8\hbar^2)$.

The mode intensities and frequencies can be calculated directly from Eqs. (1) through (6). By analogy with Lamb's analysis of two-mode operation, the general features of these equations are: (1) Only one polarization oscillates if $\beta_+\beta_- < \theta_{+-}\theta_{-+}$ (strong coupling), or if $\beta_+\beta_- > \theta_{+-}\theta_{-+}$ (weak coupling), and either $\alpha_+-\theta_{+-}E_-{}^2$ or $\alpha_- - \theta_{-+}E_+{}^2$ are negative[6]; (2) both polarizations oscillate if $\beta_+\beta_- > \theta_{+-}\theta_{-+}$ and neither $\alpha_+-\theta_{+-}E_-{}^2$ nor $\alpha_- - \theta_{-+}E_+{}^2$ are negative[7]; (3) if both fields oscillate ν_+ and ν_- will generally differ even for $\Omega_+=\Omega_-$, the beat in the latter case being[8]

$$\nu_+-\nu_- \approx 2\gamma H\{-\sigma_0+\rho_0(E_+{}^2+E_-{}^2)\gamma_{ab}{}^2\mathcal{L}_{ab}(\gamma H)\}.$$

Typical values of σ_0, the slope of the dispersion curve near line center, and $\rho_0(E_+{}^2+E_-{}^2)$, the pushing coefficient [e.g., $\sigma_0 \cong 3\times10^{-3}$ and $\rho_0(E_+{}^2+E_-{}^2)\cong 5\times10^{-3}$ from data taken in a study of two-mode operation], show that except for operation very near threshold, the difference $\nu_+-\nu_-$ passes through zero and changes sign as the magnetic-field splitting is increased. That is, initially the pushing dominates only to give way to pulling as $2\gamma H$ exceeds γ_{ab}. Such a variation has been observed by Culshaw and Kannelaud[5] without, however, noting the change in sign of $\nu_+-\nu_-$.

A portion of the mode competition [the part of the θ term containing the factor $\mathcal{L}_a(2\delta)$] arises from a coupling of the two fields (E_+, E_-) by means of an induced atomic precession. The interaction of $\bar{E}_-(\nu_-)$ with this precessing moment produces a field $\bar{E}_+(\nu_+)$ and vice versa. Since polarization conversion as well as frequency conversion occurs, the orthogonality of the polarizations does not preclude a strong competition. Mathematically, this precessing moment appears as a coherent superposition of the three states $|J=1, m=1\rangle$, $|J=1, m=-1\rangle$, and $|J=0, m=0\rangle$ produced by interactions of the same atom with first one polarization component and then the other. An additional consequence of the coherence of the magnetic sublevels of the $J=1$ state is the existence of magnetic multipole radiation (also noted by Javan[9]).

The induced precessing moment, and hence coupling, can remain significant for $\gamma H \gg Ku$ provided only that $\nu_+-\nu_- \approx 2\gamma H$. It seems highly probable that this coupling played an important role in the strong competition reported earlier between orthogonally polarized maser modes oscillating on well-resolved Zeeman components.[10]

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[6] One of the authors (RLF) has observed such a condition in an optical maser operating on the xenon $5d[3/2]_1{}^0-6p[1/2]_0$ (2.65 μ) transition.

[7] The equations indicate that weak coupling is the most typical case; however, neutral coupling is predicated in this approximation for $\delta=0$.

[8] The expression was specialized to the case $\Omega_+=\Omega_-=\omega_0$ and the τ terms were neglected. The general case can be easily obtained from the given equations.

[9] A. Javan and A. Szöke, Phys. Rev. **137**, A536 (1965).

[10] R. L. Fork and C. K. N. Patel, Appl. Phys. Letters **2**, 180 (1963).