Transport Equations for Superconductors

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Using the microscopic theory of superconductivity due to Bardeen, Cooper, and Schrieffer, transport equations, analogous to the Boltzmann transport equation for normal metals, are developed for superconductors. The damping is assumed to arise from collisions of the electrons with randomly distributed impurities. The collisions are treated in the Born approximation. By making suitable assumptions about the solutions to the transport equation, the two-fluid model of a superconductor is derived. The response of a superconductor to various driving forces is investigated, giving expressions for the phenomenological relations and thermal conductivity of a superconductor.

1. INTRODUCTION

THE Boltzmann transport equation as applied to normal metals is very well known and has been used extensively to describe all types of transport phenomena.[1] The object of the present paper is to develop an analogous transport theory for superconductors. This transport theory is based on the microscopic theory of superconductivity due to BCS[2] in the Gorkov formulation.[3] For a description of transport phenomena in superconductors it is not sufficient to study only the excitations. Thus, for a superconductor two distribution functions are required (or four if spin is considered), which can be regarded as describing the superfluid and normal components. There are also two (or four) transport equations in general coupled together, giving the evolution of these distribution functions. The scattering of the electrons is assumed to arise from elastic scattering by randomly distributed impurities. The collision term in the transport equation is treated in much the same spirit as the collision term in the Boltzmann equation for normal metals. The transport equation obtained here is a generalization of the Boltzmann equation found by Bardeen, Rickayzen, and Tewordt[4] for the excitations in a superconductor.

The two-fluid model of a superconductor is derived by making suitable assumptions about the distribution functions which are solutions of the transport equation. It is assumed that the system is in local thermodynamic equilibrium, i.e., equilibrium in the superconductor is attained much more rapidly than any other process that we consider. The actual interactions that lead to this equilibrium are not contained in the transport equation given here. These assumptions concerning the distribution functions lead to the two-fluid equations proposed by Landau[5] for He II extended to the case where the particles are charged. The two-fluid equations of a superconductor are not very realistic because lattice scattering is generally more important than the electron-electron scattering required to maintain local equilibrium.

In Sec. 4 the linear response of a superconductor to various driving forces is found. Only the case where the driving forces are very slowly varying (with respect to the coherence distance) is considered, i.e., London-type superconductor. These considerations lead to expressions for the phenomenological relations and thermal conductivity of a London-type superconductor. These relations agree with those suggested recently by Luttinger.[6] Not unexpectedly, it is found that the excitations in a superconductor behave very similarly to electrons in a normal metal as far as their response to driving forces is concerned.

Applications of the transport equations are only made to London-type superconductors. But the equations may equally well be used to describe transport properties of Pippard-type superconductors, e.g., for the electromagnetic properties results are obtained in agreement with those of other authors[7] using the Kubo approach.

2. THEORY

In order to develop the transport equations we use the simple phenomenological Hamiltonian for a superconductor proposed by Gorkov[3]

$$\mathcal{H}=\sum_s \int d\mathbf{r}\psi_s^\dagger(\mathbf{r}) \times\left\{\frac{1}{2m}\left[\mathbf{p}+\frac{e}{c}\mathbf{A}(\mathbf{r},t)\right]^2+V_p(\mathbf{r})-e\phi(\mathbf{r},t)\right\}\psi_s(\mathbf{r}) -\frac{g}{2}\sum_{s_1s_2}\int d\mathbf{r}\psi_{s_1}^\dagger(\mathbf{r})\psi_{s_2}^\dagger(\mathbf{r})\psi_{s_2}(\mathbf{r})\psi_{s_1}(\mathbf{r}), \quad (2.1)$$

[1] A. H. Wilson, *The Theory of Metals* (Cambridge University Press, New York, 1958).
[2] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).
[3] L. P. Gorkov, Zh. Eksperim. i Teor. Fiz. **34**, 735 (1958) [English transl.: Soviet Phys.—JETP **7**, 505 (1958)].
[4] J. Bardeen, G. Rickayzen, and L. Tewordt, Phys. Rev. **113**, 982 (1959).
[5] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, Inc., New York, 1959). C. J. Gorter, *Progress in Low-Temperature Physics*, edited by C. J. Gorter (John Wiley & Sons, Inc., New York, 1961), Vol. 3.
[6] J. M. Luttinger, Phys. Rev. **136**, A1481 (1964).
[7] D. C. Mattis and J. Bardeen, Phys. Rev. **111**, 412 (1958); P. B. Miller, *ibid.* **113**, 1209 (1959); A. A. Abrikosov and L. P. Gorkov, Zh. Eksperim. i Teor. Fiz. **35**, 1558 (1958) [English transl.: Soviet Phys.—JETP **8**, 1090 (1959)].

where $V_p(\mathbf{r})=\sum_i V(\mathbf{r}-\mathbf{R}_i)$ is the potential due to a set of impurities at the positions $\mathbf{R}_i$ and $\mathbf{A}(\mathbf{r},t)$ and $\phi(\mathbf{r},t)$ are potentials depending, in general, on space $\mathbf{r}$ and time t. The Green's functions are defined in the usual way by

$$\begin{aligned} G_+(1,1')&=-i\langle T\psi_+(1)\psi_+^\dagger(1')\rangle,\\ F^\dagger(1,1')&=-i\langle T\psi_-^\dagger(1)\psi_+^\dagger(1')\rangle,\\ F(1',1)&=-i\langle T\psi_+(1)\psi_-(1')\rangle,\\ G_-(1',1)&=-i\langle T\psi_-(1')\psi_-^\dagger(1)\rangle, \end{aligned} \tag{2.2}$$

where 1 is used as an abbreviaton for $\mathbf{r}_1t_1$ and $\pm$ indicates the spin. The operators in (2.2) are in the Heisenberg representation and the angular brackets indicate an ensemble average. Thus

$$\langle Q\rangle=\mathrm{Tr}Q\exp[-\beta(\mathcal{H}_0-\mu N)]/\mathrm{Tr}\exp[-\beta(\mathcal{H}_0-\mu N)],$$

where $\mathcal{H}_0$ is the Hamiltonian (2.1) with the fields $\mathbf{A}$ and ϕ omitted and μ is the chemical potential. The equations of motion of the Green's functions (2.2) are

$$\begin{aligned} &\{i(\partial/\partial t_1)-(1/2m)[\mathbf{p}_1+(e/c)\mathbf{A}(1)]^2-V_p(1)+e\phi(1)\}G_+(1,1')+\Delta(1)F^\dagger(1,1')=\delta(1-1'),\\ &\{i(\partial/\partial t_1)+(1/2m)[\mathbf{p}_1-(e/c)\mathbf{A}(1)]^2+V_p(1)-e\phi(1)\}F^\dagger(1,1')+\Delta^\dagger(1)G_+(1,1')=0,\\ &\{i(\partial/\partial t_1)-(1/2m)[\mathbf{p}_1+(e/c)\mathbf{A}(1)]^2-V_p(1)+e\phi(1)\}F(1',1)-\Delta(1)G_-(1',1)=0,\\ &\{i(\partial/\partial t_1)+(1/2m)[\mathbf{p}_1-(e/c)\mathbf{A}(1)]^2+V_p(1)-e\phi(1)\}G_-(1',1)-\Delta^\dagger(1)F(1',1)=-\delta(1-1'), \end{aligned} \tag{2.3}$$

where

$$\Delta^\dagger(1)=-igF^\dagger(1,1);\quad \Delta(1)=-igF(1,1). \tag{2.4}$$

In (2.3) we have not included explicitly the Coulomb interaction. This is treated in the Hartree approximation and has the effect of screening the impurity potential, and we suppose that it is already included in the definition of $V_p(r)$. ϕ is also the total self-consistent potential in the metal.

For simplicity we begin by considering the case where there is only a transverse vector potential $\mathbf{A}_t$ acting on the superconductor. Thus, in (2.3) we set $\phi=0$ and replace $\mathbf{A}$ by $\mathbf{A}_t$. The modifications introduced by longitudinal potentials are considered below. To obtain the transport equation we use a procedure, similar to that of Kadanoff and Baym,[8] and add to Eqs. (2.3) the corresponding adjoint equations, introduce new variables $\mathbf{R}=\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_1')$, $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_1'$, and $t=\frac{1}{2}(t_1+t_1')$, and take the limit $t_1'\to t_1^+$. The Green's functions are then assumed to depend on the three variables $\mathbf{R}$, $\mathbf{r}$, and t and are entirely analogous to the well-known Wigner distribution functions. It is convenient to use a matrix notation and introduce a column vector

$$\mathbf{G}'(\mathbf{R},\mathbf{r},t)=\begin{bmatrix} G_+(\mathbf{r}_1t,\mathbf{r}_1't^+)\\ F^\dagger(\mathbf{r}_1t,\mathbf{r}_1't^+)\\ F(\mathbf{r}_1't^+,\mathbf{r}_1t)\\ G_-(\mathbf{r}_1't^+,\mathbf{r}_1t)\end{bmatrix}=\begin{bmatrix} G_+(\mathbf{R},\mathbf{r},t)\\ F^\dagger(\mathbf{R},\mathbf{r},t)\\ F(\mathbf{R},-\mathbf{r},t)\\ G_-(\mathbf{R},-\mathbf{r},t)\end{bmatrix}. \tag{2.5}$$

A 2×2 matrix notation leads to difficulties and so we use this expanded version. The order of the times in (2.5) should be noted. The equations thus obtained by combining (2.3) with the adjoint equations are (retaining terms linear in the fields only)

$$\begin{aligned} &\{\mathbf{M}(\mathbf{R},\mathbf{r},t)-(e/mci)[\mathbf{A}_t(1)\cdot\boldsymbol{\nabla}_1+\mathbf{A}_t(1')\cdot\boldsymbol{\nabla}_{1'}]\}\mathbf{G}'(\mathbf{R},\mathbf{r},t)\\ &\quad=[V_p(\mathbf{r}_1)P_1-V_p(\mathbf{r}_1')P_2]\mathbf{G}'(\mathbf{R},\mathbf{r},t). \end{aligned} \tag{2.6}$$

Here $\mathbf{M}$ is an operator [see below Eq. (2.8)] and P_1 and P_2 are the diagonal matrices

$$P_1=\begin{bmatrix}1&&&\\&-1&&\\&&1&\\&&&-1\end{bmatrix};\quad P_2=\begin{bmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{bmatrix}.$$

It is convenient to take the Fourier transform of (2.6) on the variable $\mathbf{r}$. We then find

$$\begin{aligned} &\{\mathbf{M}(\mathbf{R},\mathbf{k},t)-(e/mci)[\mathbf{A}_t(\mathbf{R}_k,t)\cdot(\tfrac{1}{2}\boldsymbol{\nabla}+i\mathbf{k})+\mathbf{A}_t(\mathbf{R}_k{}^*,t)\cdot(\tfrac{1}{2}\boldsymbol{\nabla}-i\mathbf{k})]\}\mathbf{G}'(\mathbf{R},\mathbf{k},t)\\ &\quad=(1/V)\textstyle\sum_q V_p(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{R}}[P_1\mathbf{G}'(\mathbf{R},\mathbf{k}-\frac{1}{2}\mathbf{q},t)-P_2\mathbf{G}'(\mathbf{R},\mathbf{k}+\frac{1}{2}\mathbf{q},t)]. \end{aligned} \tag{2.7}$$

Here $\mathbf{M}(\mathbf{R},\mathbf{k},t)$ is the matrix

$$\begin{bmatrix} i\dfrac{\partial}{\partial t}+\dfrac{i\mathbf{k}}{m}\cdot\boldsymbol{\nabla} & \Delta(\mathbf{R}_k,t) & -\Delta^\dagger(\mathbf{R}_k{}^*,t) & 0\\ \Delta^\dagger(\mathbf{R}_k,t) & i\dfrac{\partial}{\partial t}+2\epsilon_k-\dfrac{1}{4m}\nabla^2 & 0 & \Delta^\dagger(\mathbf{R}_k{}^*,t)\\ -\Delta(\mathbf{R}_k{}^*,t) & 0 & i\dfrac{\partial}{\partial t}-2\epsilon_k+\dfrac{1}{4m}\nabla^2 & -\Delta(\mathbf{R}_k,t)\\ 0 & \Delta(\mathbf{R}_k{}^*,t) & -\Delta^\dagger(\mathbf{R}_k,t) & i\dfrac{\partial}{\partial t}-\dfrac{i\mathbf{k}}{m}\cdot\boldsymbol{\nabla}\end{bmatrix}, \tag{2.8}$$

[8] L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin, Inc., New York, 1962).

and we have used the abbreviaton $\nabla=\nabla_R$, $\epsilon_k=k^2/2m$ and $\mathbf{R}_k=\mathbf{R}+(i/2)\nabla_k$, $\mathbf{R}_k{}^*=\mathbf{R}-(i/2)\nabla_k$. Any function of $\mathbf{R}_k$ can be understood in terms of its power-series expansion

$$\Delta(\mathbf{R}_k,t)=\Delta(\mathbf{R},t)+(i/2)(\nabla\cdot\nabla_k)\Delta(\mathbf{R},t)+\cdots. \quad (2.9)$$

The energy gap appearing in (2.8) may be regarded as a self-consistent field.

For the purpose of determining the collision term in the transport equation, we will neglect the external field and regard $|\Delta|$ as a constant. After we have found the collision term, these omitted driving terms will be reinstated in the transport equation. As the scattering potential is time independent the only time dependence in (2.7) (omitting $\mathbf{A}_t$) arises from the phases of Δ and F. From the definitions (2.2) (see Gorkov[3])

$$\begin{aligned} &F^\dagger(\mathbf{R},\mathbf{r},t)=e^{2i\mu t}\hat{F}^\dagger(\mathbf{R},\mathbf{r},t);\quad F(\mathbf{R},\mathbf{r},t)=e^{-2i\mu t}\hat{F}(\mathbf{R},\mathbf{r},t),\\ &\Delta^\dagger(\mathbf{R},t)=e^{2i\mu t}|\Delta|;\qquad\qquad \Delta(\mathbf{R},t)=e^{-2i\mu t}|\Delta|, \end{aligned} \quad (2.10)$$

and it is convenient to define a new distribution function $\mathbf{G}(\mathbf{R},\mathbf{r},t)$ analogous to (2.5) except that $F^\dagger$ and F are replaced by $\hat{F}^\dagger$ and $\hat{F}$, respectively. The only effect of this time dependence is to replace ϵ_k in (2.8) by $\tilde{\epsilon}_k=\epsilon_k-\mu$. So that Eq. (2.7) becomes

$$\mathbf{N}(\mathbf{R},\mathbf{k},t)\mathbf{G}(\mathbf{R},\mathbf{k},t)=(1/V)\textstyle\sum_q V_p(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{R}}\\ \times[P_1\mathbf{G}(\mathbf{R},\mathbf{k}-\tfrac{1}{2}\mathbf{q},t)-P_2\mathbf{G}(\mathbf{R},\mathbf{k}+\tfrac{1}{2}\mathbf{q},t)], \quad (2.11)$$

where $\mathbf{N}$ differs from $\mathbf{M}$ only in that $\tilde{\epsilon}_k$ replaces ϵ_k and $|\Delta|$ replaces Δ and $\Delta^\dagger$. Thus

$$\mathbf{N}(\mathbf{R},\mathbf{k},t)=\mathbf{M}(\mathbf{R},\mathbf{k},t,|\Delta|)+(P_1-P_2)\mu.$$

Then regarding $|\Delta|$ as a constant we take the Fourier transform of (2.11) with respect to $\mathbf{R}$ and t. It is convenient to formally retain the t dependence in (2.11).

$$\mathbf{N}(\mathbf{K},\mathbf{k},W)\mathbf{G}(\mathbf{K},\mathbf{k},W)\\ =(1/V)\textstyle\sum_q V_p(\mathbf{q})[P_1\mathbf{G}(\mathbf{K}-\mathbf{q},\mathbf{k}-\tfrac{1}{2}\mathbf{q},W)\\ -P_2G(\mathbf{K}-\mathbf{q},\mathbf{k}+\tfrac{1}{2}\mathbf{q},W)]. \quad (2.12)$$

The collision term is obtained by iterating (2.12) in powers of the scattering potential V.[9] Thus

$$\mathbf{G}(\mathbf{K},\mathbf{k},W)=\mathbf{N}^{-1}(\mathbf{K},\mathbf{k},W+i\delta)\\ \times(1/V)\textstyle\sum_q V_p(\mathbf{q})[P_1\mathbf{G}(\mathbf{K}-\mathbf{q},\mathbf{k}-\tfrac{1}{2}\mathbf{q},W)\\ -P_2\mathbf{G}(\mathbf{K}-\mathbf{q},\mathbf{k}+\tfrac{1}{2}\mathbf{q},W)], \quad (2.13)$$

where we have chosen on physical grounds the retarded value of the inverse matrix $\mathbf{N}^{-1}$ by replacing W by $W+i\delta$, where δ is an infinitesimal constant. Substituting (2.13) into the right-hand side of (2.12), we obtain

$$\mathbf{N}(\mathbf{K},\mathbf{k},W)\mathbf{G}(\mathbf{K},\mathbf{k},W)=(1/V^2)\sum_{q,q'} V_p(\mathbf{q})V_p(\mathbf{q}')\{P_1\mathbf{N}^{-1}(\mathbf{K}-\mathbf{q},\mathbf{k}-\tfrac{1}{2}\mathbf{q},W+i\delta)\\ \times[P_1\mathbf{G}(\mathbf{K}-\mathbf{q}-\mathbf{q}',\mathbf{k}-\tfrac{1}{2}\mathbf{q}-\tfrac{1}{2}\mathbf{q}',W)-P_2\mathbf{G}(\mathbf{K}-\mathbf{q}-\mathbf{q}',\mathbf{k}-\tfrac{1}{2}\mathbf{q}+\tfrac{1}{2}\mathbf{q}',W)]-P_2N^{-1}(\mathbf{K}-\mathbf{q},\mathbf{k}+\tfrac{1}{2}\mathbf{q},W+i\delta)\\ \times[P_1\mathbf{G}(\mathbf{K}-\mathbf{q}-\mathbf{q}',\mathbf{k}+\tfrac{1}{2}\mathbf{q}-\tfrac{1}{2}\mathbf{q}',W)-P_2\mathbf{G}(\mathbf{K}-\mathbf{q}-\mathbf{q}',\mathbf{k}+\tfrac{1}{2}\mathbf{q}+\tfrac{1}{2}\mathbf{q}',W)]\}. \quad (2.14)$$

We now average this equation over all positions of the impurities. The averaging on the right-hand side should actually include the function $\mathbf{G}$ which depends implicitly on the impurities. If we neglect interference between different scattering events we can average over $V_p{}^2$ and $\mathbf{G}$ separately using

$$[V_p(\mathbf{q})V_p(\mathbf{q}')]_{\text{average}}=n_0V\delta_{\mathbf{q}+\mathbf{q}',0}|V(\mathbf{q})|^2,$$

where n_0 is the number of impurities in unit volume. This procedure should be valid if the density of impurities is small and range of the potential short.[10] After averaging the function $\mathbf{G}$ will not depend on the variables $\mathbf{R}$ and t and we write

$$\mathbf{G}(\mathbf{K},\mathbf{k},W)=(2\pi)^4\delta(\mathbf{K})\delta(W)\mathbf{G}(\mathbf{k}),$$

and (2.14) becomes

$$\mathbf{N}(0,\mathbf{k},0)\mathbf{G}(\mathbf{k})\\ =(n_0/V)\textstyle\sum_q|V(\mathbf{q})|^2\{[P_1\mathbf{N}^{-1}(\mathbf{q},\mathbf{k}+\tfrac{1}{2}\mathbf{q},i\delta)P_1\\ +P_2\mathbf{N}^{-1}(-\mathbf{q},\mathbf{k}+\tfrac{1}{2}\mathbf{q},i\delta)P_2]\\ \mathbf{G}(\mathbf{k})-[P_1\mathbf{N}^{-1}(\mathbf{q},\mathbf{k}+\tfrac{1}{2}\mathbf{q},i\delta)P_2\\ +P_2\mathbf{N}^{-1}(-\mathbf{q},\mathbf{k}+\tfrac{1}{2}\mathbf{q},i\delta)P_1]\mathbf{G}(\mathbf{k}+\mathbf{q})\}. \quad (2.15)$$

The matrices $P_i\mathbf{N}^{-1}P_j$ appearing here are easily calculated. We will only retain the imaginary part which leads to relaxation. In the presence of driving terms, we assume that the collision term has the same form as in (2.15) except that $\mathbf{G}(\mathbf{k})$ is replaced by its local value $\mathbf{G}(\mathbf{R},\mathbf{k},t)$ and that $\mathbf{N}(0,\mathbf{k},0)$ is replaced by the transport operator appearing on the left-hand side of (2.7). The transport equation in the presence of a transverse field $\mathbf{A}_t$ then becomes

$$\{\mathbf{N}(\mathbf{R},\mathbf{k},t)-(e/mci)[\mathbf{A}_t(\mathbf{R}_k,t)\cdot(\tfrac{1}{2}\nabla+i\mathbf{k})\\ +\mathbf{A}_t(\mathbf{R}_k{}^*,t)\cdot(\tfrac{1}{2}\nabla-i\mathbf{k})]\}\mathbf{G}(\mathbf{R},\mathbf{k},t)\\ =[\dot{\mathbf{G}}(\mathbf{R},\mathbf{k},t)]_{\text{coll}}. \quad (2.16)$$

$\mathbf{N}(\mathbf{R},\mathbf{k},t)$ differs from $\mathbf{M}(\mathbf{R},\mathbf{k},t)$ in (2.8) only in that $\tilde{\epsilon}_k$ replaces ϵ_k and $|\Delta(\mathbf{R}_k,t)|$ replaces $\Delta^\dagger(\mathbf{R}_k,t)$ and $\Delta(\mathbf{R}_k,t)$.

[9] This was suggested by Professor H. Suhl.

[10] W. Kohn and J. M. Luttinger, Phys. Rev. **108**, 590 (1957); S. F. Edwards, Phil. Mag. **33**, 1020 (1958).

The collision term is given by

$$[\dot{\mathbf{G}}(\mathbf{R},\mathbf{k},t)]_{\text{coll}}=-(2\pi i n_0/V)\sum_{k'}|V(\mathbf{k}-\mathbf{k}')|^2[\delta(E_k-E_{k'})/(E_k+E_{k'})^2]$$
$$\times\{(2E_kE_{k'}+2\tilde{\epsilon}_k\tilde{\epsilon}_{k'})B_1[\mathbf{G}(\mathbf{k})-\mathbf{G}(\mathbf{k}')]+(2E_kE_{k'}-2\tilde{\epsilon}_k\tilde{\epsilon}_{k'})B_2[\mathbf{G}(\mathbf{k})+\mathbf{G}(\mathbf{k}')]$$
$$+|\Delta|(\epsilon_k-\epsilon_{k'})[(B_3+B_4)\mathbf{G}(\mathbf{k})+(B_3-B_4)\mathbf{G}(\mathbf{k}')]+2|\Delta|^2[(B_5-B_6)\mathbf{G}(\mathbf{k})-(B_5+B_6)\mathbf{G}(\mathbf{k}')]\}. \quad (2.17)$$

On the right-hand side of (2.17) we have omitted the coordinates $\mathbf{R}$ and t which appear in $\mathbf{G}(\mathbf{R},\mathbf{k},t)$ and $E_k=(\tilde{\epsilon}_k^2+|\Delta|^2)$ and is the usual quasiparticle energy appearing in the theory of superconductivity. The $\delta(E_k-E_{k'})$ appearing in (2.17) expresses the conservation of the quasiparticle energies in the collisions. Momentum is not conserved as the impurities are fixed. The 4×4 matrices B in (2.17) are given by

$$B_1=\begin{pmatrix}1&&&\\&0&&\\&&0&\\&&&1\end{pmatrix},\quad B_2=\begin{pmatrix}0&&&\\&1&&\\&&1&\\&&&0\end{pmatrix},\quad B_3=\begin{pmatrix}0&1&1&0\\0&0&0&0\\0&0&0&0\\0&1&1&0\end{pmatrix},$$
$$B_4=\begin{pmatrix}0&0&0&0\\1&0&0&1\\1&0&0&1\\0&0&0&0\end{pmatrix},\quad B_5=\begin{pmatrix}&&&1\\&0&&\\&&0&\\1&&&\end{pmatrix},\quad B_6=\begin{pmatrix}&&&0\\&&1&\\&1&&\\0&&&\end{pmatrix}. \quad (2.18)$$

Before considering the modifications of (2.10) in the presence of longitudinal fields, we make the following observations. We first note that if we set $\Delta=0$ and $F=0$ in (2.16), then the first row of this equation reduces to the familiar Boltzmann equation for a normal metal except for the terms in the fields. These terms do not have the familiar form because $\mathbf{k}$ is the canonical momentum and not the mechanical momentum. Secondly, we neglect the fields in (2.16) and substitute the BCS equilibrium values of $\mathbf{G}$, which are independent of $\mathbf{R}$ and t, into (2.16). The BCS value of $\mathbf{G}$ is

$$\mathbf{G}_0(\mathbf{k})=\begin{pmatrix}G_{0+}(k)\\F_0^\dagger(k)\\F_0(k)\\G_{0-}(k)\end{pmatrix}=i\begin{pmatrix}u_k^2f_k+v_k^2(1-f_k)\\u_kv_k(1-2f_k)\\u_kv_k(1-2f_k)\\-u_k^2(1-f_k)-v_k^2f_k\end{pmatrix}, \quad (2.19)$$

where

$$\frac{u_k^2}{v_k^2}=\frac{1}{2}\left(1\pm\frac{\tilde{\epsilon}_k}{E_k}\right)$$

and $f_k=(1+e^{\beta E_k})^{-1}$. It is found that the left-hand side vanishes identically because of the relations

$$|\Delta|F_0^\dagger(k)-|\Delta|F_0(k)=0,$$
$$2\tilde{\epsilon}_kF_0^\dagger(k)+[G_{0+}(k)+G_{0-}(k)]=0. \quad (2.20)$$

The right-hand side may also be shown to vanish by using the fact that $G_0(k)$ depends only on $\tilde{\epsilon}_k$ and writing

$$\delta(E_k-E_{k'})/(E_k+E_{k'})^2=(4E_k|\tilde{\epsilon}_k|)^{-1}$$
$$\times[\delta(\tilde{\epsilon}_k-\tilde{\epsilon}_{k'})+\delta(\tilde{\epsilon}_k+\tilde{\epsilon}_{k'})]. \quad (2.21)$$

This is a feature of all transport equations in that they describe the evolution of the distribution from a given initial distribution.

In the case where the field and gap vary slowly we can expand these terms in (2.16) about the point $\mathbf{R}$. Retaining terms up to the first derivatives in the fields (2.16) becomes

$$\{\mathbf{N}(\mathbf{R},\mathbf{k},t)-(e/mci)[\mathbf{A}_t\cdot\nabla-\mathbf{k}\cdot(\nabla\cdot\nabla_k\mathbf{A}_t)]\}\mathbf{G}(\mathbf{R},\mathbf{k},t)$$
$$=[\dot{\mathbf{G}}(\mathbf{R},\mathbf{k},t)]_{\text{coll}}. \quad (2.22)$$

The argument of $\mathbf{A}$ is $\mathbf{R}$, t. This expansion in powers of $\nabla_R\cdot\nabla_k$ is equivalent to an expansion in ξ/δ, where ξ is the coherence distance and δ is a parameter giving the spatial variation of the field. Thus, for London superconductors it is sufficient to use Eq. (2.22).

3. INCLUSION OF LONGITUDINAL FIELDS

In the presence of longitudinal fields in order that Eqs. (2.3) be gauge invariant the gap will in general have a phase in addition to the factor $2\mu t$ in (2.10).[11] We thus define new functions

$$G_+(1,1')=e^{-(ie/c)[W(1)-W(1')]}g_+'(1,1');\quad G_-(1',1)=e^{-(ie/c)[W(1')-W(1)]}g_-'(1',1),$$
$$F^\dagger(1,1')=e^{(ie/c)[W(1)+W(1')]}f^\dagger(1,1');\quad F(1',1)=e^{-(ie/c)[W(1)+W(1')]}f(1',1), \quad (3.1)$$
$$\Delta^\dagger(1)=e^{(2ie/c)W(1)}|\Delta(\mathrm{L})|;\quad \Delta(1)=e^{-(2ie/c)W(1)}|\Delta(1)|,$$

where $|\Delta|$ and W are real. If $W(r,t)$ is of the form $(e/c)W(\mathbf{r},t)-\mu t=-\mathbf{r}\cdot\mathbf{k}_0$, we would pair states in the BCS model with momentum $\mathbf{k}+\mathbf{k}_0$ and $-\mathbf{k}+\mathbf{k}_0$, i.e., $2\mathbf{k}_0$ is the center-of-mass momentum of a condensed pair. Introducing these new functions, the Gorkov Eq. (2.3) for the new functions g' and f are exactly as in (2.3) except that the potentials $\mathbf{A}$ and ϕ are replaced by the gauge-invariant potentials

$$\mathbf{A}_g(\mathbf{r},t)=\mathbf{A}(\mathbf{r},t)-\nabla W(\mathbf{r},t);$$
$$\phi_g(\mathbf{r},t)=\phi(\mathbf{r},t)+(1/c)(\partial W(\mathbf{r},t)/\partial t) \quad (3.2)$$

and Δ and $\Delta^\dagger$ are replaced by $|\Delta|$. By analogy with

[11] V. Ambegaokar and L. P. Kadanoff, Nuovo Cimento 22, 914 (1961).

He II we define the superfluid velocity by

$$\mathbf{v}_s=(e/mc)\mathbf{A}_g=(e/mc)(\mathbf{A}-\nabla W). \tag{3.3}$$

As before we introduce the vector distribution function

$$\mathbf{g}'(\mathbf{R},\mathbf{r},t)=\begin{bmatrix} g_+'(\mathbf{r}_1t,\mathbf{r}_1't^+) \\ f^\dagger(\mathbf{r}_1t,\mathbf{r}_1't^+) \\ f(\mathbf{r}_1't^+,\mathbf{r}_1t) \\ g_-'(\mathbf{r}_1't^+,\mathbf{r}_1t)\end{bmatrix}=\begin{bmatrix} g_+'(\mathbf{R},\mathbf{r},t) \\ f^\dagger(\mathbf{R},\mathbf{r},t) \\ f(\mathbf{R},-\mathbf{r},t) \\ g_-'(\mathbf{R},-\mathbf{r},t)\end{bmatrix}. \tag{3.4}$$

In this section we will confine ourselves to the case where the fields $\mathbf{A}$, ϕ, and W are slowly varying functions of the coordinates. A more careful analysis shows that only the gauge-invariant potentials (3.2) need be slowly varying, but for simplicity in presentation we assume that they are separately slowly varying. Thus we have using $W(1)-W(1')=\mathbf{r}\cdot\nabla W(\mathbf{R},t)$

$$g_\pm'(\mathbf{R},\mathbf{r},t)=e^{\pm(ie/c)\mathbf{r}\cdot\nabla W}G_\pm(\mathbf{R},\mathbf{r},t), \tag{3.5}$$

and then the Fourier transform of $\mathbf{g}'(\mathbf{R},\mathbf{r},t)$ is

$$\mathbf{g}'(\mathbf{R},\mathbf{k},t)=\begin{bmatrix} G_+(\mathbf{R},\mathbf{k}-(e/c)\nabla W,t) \\ f^\dagger(\mathbf{R},\mathbf{k},t) \\ f(\mathbf{R},-\mathbf{k},t) \\ G_-(\mathbf{R},-\mathbf{k}-(e/c)\nabla W,t)\end{bmatrix}. \tag{3.6}$$

The equation for the distribution function $\mathbf{g}'$ obtained by combining (2.3) with the adjoint equations is

$$\begin{aligned}&[\mathbf{M}(\mathbf{R},\mathbf{k},t|\Delta|)-(e/mci)[\mathbf{A}_g\cdot\nabla-\mathbf{k}\cdot(\nabla\cdot\nabla_k\mathbf{A}_g)] \\ &\quad+ei[\nabla\phi_g-(e/2mc^2)(\nabla\mathbf{A}_g^2)]\cdot\nabla_k\tfrac{1}{2}(P_1+P_2)+(2e\phi_g-(e^2/mc^2)\mathbf{A}_g^2)\tfrac{1}{2}(P_1-P_2)]\mathbf{g}'(\mathbf{R},\mathbf{k},t) \\ &\quad=(1/V)\textstyle\sum_q V_p(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{R}}[P_1\mathbf{g}'(\mathbf{R},\mathbf{k}-\tfrac{1}{2}\mathbf{q},t)-P_2\mathbf{g}'(\mathbf{R},\mathbf{k}+\tfrac{1}{2}\mathbf{q},t)],\end{aligned} \tag{3.7}$$

where $\mathbf{M}(\mathbf{R},\mathbf{k},t,|\Delta|)$ is the operator (2.8) with $|\Delta|$ replacing $\Delta^\dagger$ and Δ. We can now determine the collision term as before by inverting the left-hand side of (3.7) and substituting the result in the right-hand side. In the collision term we will neglect terms depending on the derivatives of $\mathbf{A}_g$ and ϕ_g. However, the terms in (3.7) proportional to $(e/mc)\mathbf{A}_g\cdot\nabla=\mathbf{v}_s\cdot\nabla$ and $e\phi_g-(e^2/2mc^2)\mathbf{A}_g^2=e\phi_g-\frac{1}{2}m\mathbf{v}_s^2$ must be retained as they lead to the following changes in the collision term. The term $\mathbf{v}_s\cdot\nabla$ arises because in the presence of superfluid flow the quasiparticle energies are $E_{k1,2}=E_k\pm\mathbf{k}\cdot\mathbf{v}_s$ and it is these quasiparticle energies that are conserved in the collisions with the impurities. The term $e\phi_g-\frac{1}{2}m\mathbf{v}_s^2$ is the chemical potential (see below). This is clear in the absence of fields from Eq. (2.10). The collision term in the presence of superfluid flow replacing (2.17) is now

$$\begin{aligned}[\dot{\mathbf{g}}'(\mathbf{R},\mathbf{k},t)]_{\text{coll}}=&-\frac{\pi in_0}{V}\sum_{k'}|V(\mathbf{k}-\mathbf{k}')|^2\left[\frac{\delta(E_{k1}-E_{k'1})+\delta(E_{k2}-E_{k'2})}{(E_{k1}+E_{k'2})(E_{k2}+E_{k'1})}\right] \\ &\times\Bigg\{\left[2E_kE_{k'}+2\bar{\epsilon}_k\bar{\epsilon}_{k'}-\frac{2(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}_s}{(E_k-E_{k'})}(E_k\bar{\epsilon}_{k'}+\bar{\epsilon}_kE_{k'})\right]B_1[\mathbf{g}'(\mathbf{k})-\mathbf{g}'(\mathbf{k}')]+(2E_kE_{k'}-2\bar{\epsilon}_k\bar{\epsilon}_{k'})B_2[\mathbf{g}'(\mathbf{k})+\mathbf{g}'(\mathbf{k}')] \\ &+|\Delta|[\bar{\epsilon}_k-\bar{\epsilon}_{k'}-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}_s][(B_3'+B_4')\mathbf{g}'(\mathbf{k})+(B_3'-B_4')\mathbf{g}'(\mathbf{k}')]+|\Delta|[\bar{\epsilon}_k-\bar{\epsilon}_{k'}+(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}_s] \\ &\times[(B_3''+B_4'')\mathbf{g}'(\mathbf{k})+(B_3''-B_4'')\mathbf{g}'(\mathbf{k}')]+2|\Delta|^2[(B_5-B_6)\mathbf{g}'(\mathbf{k})-(B_5+B_6)\mathbf{g}'(\mathbf{k}')]\Bigg\},\end{aligned} \tag{3.8}$$

where

$$B_3'=\begin{bmatrix}0&1&1&0\\0&0&0&0\end{bmatrix},\quad B_3''=\begin{bmatrix}0&0&0&0\\0&1&1&0\end{bmatrix},$$

$$B_4'=\begin{bmatrix}0&0\\1&0\\1&0\\0&0\end{bmatrix},\qquad B_4''=\begin{bmatrix}0&0\\0&1\\0&1\\0&0\end{bmatrix}. \tag{3.9}$$

The B are 4×4 matrices and only the nonzero row or column has been indicated. The other matrices B are defined in (2.18). Equation (3.8) reduces to (2.17) when $\mathbf{v}_s=0$. Note that the quasiparticle energies $E_{k1,2}=E_k\pm\mathbf{k}\cdot\mathbf{v}_s$ are conserved in the collisions. The form of this collision term is such that superfluid flow will not be damped by collisions with impurities.

It is convenient to make one further modification in the transport equation corresponding to the introduction of the mechanical momentum instead of the kinetic momentum. We thus write

$$\begin{aligned}G_\pm(\mathbf{R},\pm\mathbf{k}-(e/c)\nabla W,t)&=G_\pm(\mathbf{R},\pm\mathbf{k}+m\mathbf{v}_s-(e/c)\mathbf{A},\mathrm{t}) \\ &=g_\pm(\mathbf{R},\pm\mathbf{k}+m\mathbf{v}_s,t),\end{aligned} \tag{3.10}$$

so that the $\mathbf{R}$, t dependence of $G_\pm$ occurring via $\mathbf{A}$ is now shown explicitly in $g_\pm$. The operators ∇ and $(\partial/\partial t)$ appearing on the left-hand side of (3.7) act on all the $\mathbf{R}$, t dependence of the functions $G_\pm(\mathbf{R},\pm\mathbf{k}-(e/c)\nabla W,t)$, i.e., not only on that explicitly appearing but also any dependence on the coordinates contained in ∇W. We now arrange that these operators act only on the explicit $\mathbf{R}$, t dependence appearing in $g_\pm$. Thus, for example,

$$\begin{aligned}(\partial/\partial t)G_+(\mathbf{R},\mathbf{k}+(e/c)\nabla W,t)&=(\partial/\partial t)g_+(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t) \\ &\quad+m(\partial\mathbf{v}_s/\partial t)\cdot\nabla_kg_+(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t).\end{aligned} \tag{3.11}$$

We now also define a new column vector

$$\begin{aligned}\mathbf{g}(\mathbf{R},\mathbf{k},t)=[&g_+(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t);\quad f^\dagger(\mathbf{R},\mathbf{k},t); \\ &f(\mathbf{R},\mathbf{k},t);\quad g_-(\mathbf{R},-\mathbf{k}+m\mathbf{v}_s,t)].\end{aligned} \tag{3.12}$$

The transport equation determining this distribution function is now

$$\{\mathbf{M}(\mathbf{R},\mathbf{k},t,|\Delta|)+i\mathbf{v}_s\cdot\nabla - eiR_1[\mathbf{E}+(1/mc)(R_1\mathbf{k}+m\mathbf{v}_s)\times\mathbf{H}]\cdot\nabla_k - R_2[iR_2\mathbf{k}\cdot(\nabla\cdot\nabla_k\mathbf{v}_s)+2e\phi_g-m\mathbf{v}_s^2]\}\mathbf{g}(\mathbf{R},\mathbf{k},t) = [\dot{\mathbf{g}}(\mathbf{R},\mathbf{k},t)]_{\text{coll}}, \quad (3.13)$$

where the fields $\mathbf{E}=-\nabla\phi-(1/c)(\partial\mathbf{A}/\partial t)$ and $\mathbf{H}=\text{curl}\mathbf{A}$ and R_1 and R_2 are the diagonal matrices

$$R_1=\begin{bmatrix}1 & & \\ & 0 & \\ & & -1\end{bmatrix},\quad R_2=\begin{bmatrix}0 & & \\ & 1 & \\ & & -1\end{bmatrix}.$$

The transport Eqs. (2.16) and (3.13) can be solved by the same techniques as are used for the Boltzmann equation for the normal metal. There is, however, the added complication that after the solution for g and f in terms of $\Delta(\mathbf{R},t)$ and the driving fields is obtained, $\Delta(\mathbf{R},t)$ must then be determined by the condition (2.4).

4. THE TWO-FLUID MODEL

In this section we consider the conservation relations for the particle density $\rho(\mathbf{R},t)$, momentum density $\mathbf{J}(\mathbf{R},t)$, and energy density $\mathcal{E}(\mathbf{R},t)$ and derive the two-fluid model. These quantities are given by

$$\rho(\mathbf{R},t)=-(im/V)\textstyle\sum_k[G_+(\mathbf{R},\mathbf{k},t)+G_-(\mathbf{R},-\mathbf{k},t)+i], \quad (4.1)$$

$$\mathbf{J}(\mathbf{R},t)=-(i/V)\textstyle\sum_k\{[\mathbf{k}+(e/c)\mathbf{A}(\mathbf{R},t)]G_+(\mathbf{R},\mathbf{k},t)-[\mathbf{k}-(e/c)\mathbf{A}(\mathbf{R},t)][G_-(\mathbf{R},-\mathbf{k},t)+i]\}, \quad (4.2)$$

$$\mathcal{E}(\mathbf{R},t)=-(i/V)\textstyle\sum_k\{(1/2m)[\mathbf{k}+(e/c)\mathbf{A}(\mathbf{R},t)]^2G_+(\mathbf{R},\mathbf{k},t) +(1/2m)[\mathbf{k}-(e/c)\mathbf{A}(\mathbf{R},t)]^2[G_-(\mathbf{R},-\mathbf{k},t)+i]\}-\Delta\Delta^\dagger/g. \quad (4.3)$$

The combination $G_-(\mathbf{R},-\mathbf{k},t)+i$ occurs because of the definition of G_- as the density of unoccupied states (2.2). These formulas take a simpler form if we introduce the functions as in (3.10)

$$G_\pm(\mathbf{R},\pm\mathbf{k}-(e/c)\mathbf{A}(\mathbf{R},t),t)=g_\pm(\mathbf{R},\pm\mathbf{k},t). \quad (4.4)$$

Then ρ remains unchanged except for the substitution of $g_\pm$ for $G_\pm$ and

$$\mathbf{J}(\mathbf{R},t)=-(i/V)\textstyle\sum_k\mathbf{k}[g_+(\mathbf{R},\mathbf{k},t)-g_-(\mathbf{R},-\mathbf{k},t)-i]. \quad (4.5)$$

$$\mathcal{E}(\mathbf{R},t)=-(i/V)\textstyle\sum_k\epsilon_k[g_+(\mathbf{R},\mathbf{k},t)+g_-(\mathbf{R},-\mathbf{k},t)+i]-(\Delta\Delta^\dagger/g). \quad (4.6)$$

In order to derive the two-fluid model we follow the method developed by Chapman and Enskog[12] for normal fluids. It is assumed that the system is in local thermodynamic equilibrium, i.e., equilibrium of the system is attained much more rapidly than any other process that we consider. The actual interactions between normal fluid and superfluid that lead to this equilibrium are not contained in our Boltzmann transport Eq. (3.13) as we have only considered collisions with fixed impurities. Owing to our ignorance of these terms we make certain plausible assumptions about the solutions of (3.13). It is assumed that the distribution functions g and f in (3.13) have the form for local equilibrium but depend on space and time through the eight quantities $\rho(\mathbf{R},t)$, $\mathbf{v}_n(\mathbf{R},t)$, $\mathbf{v}_s(\mathbf{R},t)$, and $\beta(\mathbf{R},t)$, where ρ is the density, $\mathbf{v}_n$ and $\mathbf{v}_s$ are the normal and superfluid velocities, and $\beta=(kT)^{-1}$ the inverse temperature. All these quantities are slowly varying functions of $\mathbf{R}$ and t. We have already indicated how the superfluid velocity is defined (3.3). It is related to the phase of the energy gap or the center-of-mass momentum of a condensed pair. The normal fluid velocity $\mathbf{v}_n$ corresponds to the thermal average velocity of the excitations in the superconductor. This velocity can be conveniently introduced into the theory by considering the Hamiltonian $\mathcal{H}-\mathbf{P}\cdot\mathbf{v}_n$ instead of $\mathcal{H}$ given by (2.1), $\mathbf{P}$ is the total momentum operator for the system. The solution assumed for Eq. (3.13) is

$$\begin{aligned} g_+(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t)&=i\{u_k^2f(E_{k1}-\mathbf{k}\cdot\mathbf{v}_n)+v_k^2[1-f(E_{k2}+\mathbf{k}\cdot\mathbf{v}_n)]\},\\ f^\dagger(\mathbf{R},\mathbf{k},t)=f(\mathbf{R},-\mathbf{k},t)&=iu_kv_k[1-f(E_{k1}-\mathbf{k}\cdot\mathbf{v}_n)-f(E_{k2}+\mathbf{k}\cdot\mathbf{v}_n)],\\ g_-(\mathbf{R},-\mathbf{k}+m\mathbf{v}_s,t)&=-i\{u_k^2[1-f(E_{k2}+\mathbf{k}\cdot\mathbf{v}_n)]+v_k^2f(E_{k1}-\mathbf{k}\cdot\mathbf{v}_n)\}, \end{aligned} \quad (4.7)$$

where $E_{k1,2}=E_k\pm\mathbf{k}\cdot\mathbf{v}_s$. This approximation to the solution of (3.13) will be referred to as the hydrodynamic one. We now check by substitution to what order of approximation (4.7) is, in fact, a solution of (3.13). The first two equations of (3.13) are

$$\{i(\partial/\partial t)+(i/m)(\mathbf{k}+m\mathbf{v}_s)\cdot\nabla-ei[E+(1/mc)(k+m\mathbf{v}_s)\times H]\cdot\nabla_k\}g_+(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t) +|\Delta(\mathbf{R}_k,t)|f^\dagger(\mathbf{R},\mathbf{k},t)-|\Delta(\mathbf{R}_k{}^*,t)|f(\mathbf{R},-\mathbf{k},t)=(\dot{g}_+)_{\text{coll}}, \quad (4.8)$$

$$\{i(\partial/\partial t)+2\epsilon_k-(\nabla^2/4m)+i[\mathbf{v}_s\cdot\nabla-\mathbf{k}\cdot(\nabla\cdot\nabla_k\mathbf{v}_s)]-2e\phi_g+m\mathbf{v}_s^2\}f^\dagger(\mathbf{R},\mathbf{k},t) +|\Delta(\mathbf{R}_k,t)|g_+(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t)+|\Delta(\mathbf{R}_k{}^*,t)|g_-(\mathbf{R},-\mathbf{k}+m\mathbf{v}_s,t)=(\dot{f}^\dagger)_{\text{coll}}. \quad (4.9)$$

[12] S. Chapman and D. Enskog, in *Mathematical Theory of Nonuniform Gases*, edited by S. Chapman and T. Cowling (Cambridge University Press, New York, 1958).

Substituting (4.7) in (4.8) and neglecting derivative terms and the fields, the left-hand side vanishes owing to the identity

$$|\Delta(R,t)|[f^\dagger(\mathbf{R},\mathbf{k},t)-f(\mathbf{R},-\mathbf{k},t)]=0.$$

The collision term on the right in (4.8) vanishes except for terms proportional to $\mathbf{v}_n$. This is a consequence of the fact that the superfluid is not damped by collisions with the impurities. It will turn out that $\mathbf{v}_n$ is proportional to the driving fields so that we are justified in neglecting it. Turning to Eq. (4.9) the collision term on the right again vanishes except for terms proportional to $\mathbf{v}_n$. Using the identity, which is easily obtained from (4.7),

$$2(\epsilon_k-\mu)f^\dagger(\mathbf{R},\mathbf{k},t)+|\Delta(\mathbf{R},t)| \times[g_+(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t)+g_-(\mathbf{R},-\mathbf{k}+m\mathbf{v}_s,t)]=0 \quad (4.10)$$

and neglecting terms involving derivatives we see that for the left-hand side to vanish we require that

$$e\phi_g-\tfrac{1}{2}m\mathbf{v}_s^2-\mu=0 \quad (4.11)$$

or

$$(e/c)\partial W/\partial t=-e\phi+\mu+\tfrac{1}{2}m\mathbf{v}_s^2. \quad (4.12)$$

Taking the time derivative of (3.3) and using (4.12) to eliminate W gives[13]

$$m(\partial\mathbf{v}_s/\partial t)=-e\mathbf{E}-\nabla(\mu+\tfrac{1}{2}m\mathbf{v}_s^2). \quad (4.13)$$

By using the vector identity

$$\tfrac{1}{2}\nabla\mathbf{v}_s^2=\mathbf{v}_s\times(\nabla\times\mathbf{v}_s)+\mathbf{v}_s\cdot\nabla\mathbf{v}_s=(e/mc)\mathbf{v}_s\times\mathbf{H}+\mathbf{v}_s\cdot\nabla\mathbf{v}_s,$$

Eq. (4.13) can be written in the form

$$m(d\mathbf{v}_s/dt)=-e[\mathbf{E}+(1/c)(\mathbf{v}_s\times\mathbf{H})]-\nabla\mu. \quad (4.14)$$

This is the first hydrodynamical equation and shows that the superfluid accelerates freely under the applied fields. The remaining hydrodynamic equations are provided by the conservation relations for $\rho(\mathbf{R},t)$, $\mathbf{J}(\mathbf{R},t)$, and $\mathcal{E}(\mathbf{R},t)$. These relations follow simply by taking moments of (4.8) and (4.9).

(*a*) *Particle density.* Adding Eq. (4.8) to the corresponding equation for g_- and summing over $\mathbf{k}$ gives[14]

$$(\partial\rho(\mathbf{R},t)/\partial t)+\nabla\cdot\mathbf{J}(\mathbf{R},t)=0. \quad (4.15)$$

Substituting from (4.7) into (4.5) we find for the hydrodynamical approximation to the current

$$\mathbf{J}=\rho\mathbf{v}_s+\mathbf{J}^0, \quad (4.16)$$

where $\mathbf{J}^0$ is the current in the coordinate system moving with a velocity $\mathbf{v}_s$.

$$\mathbf{J}^0=-(2i/V)\textstyle\sum_k\mathbf{k}g_+(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t). \quad (4.17)$$

[13] A similar argument has been made by P. W. Anderson, J. M. Luttinger, and N. R. Werthamer (to be published) to obtain the linearized version of this equation.

[14] It should be noted that $\Sigma_k(\mathbf{k}+m\mathbf{v}_s)\cdot\nabla g_+(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t)=\nabla\cdot\Sigma_k(\mathbf{k}+m\mathbf{v}_s)g_+(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t)$, where on the left-hand side ∇ acts only on the $\mathbf{R}$ explicitly appearing in g_+, whereas on the right-hand side the ∇ acts on all the functions of $\mathbf{R}$ appearing to the right. In this result $\mathbf{k}+m\mathbf{v}_s$ may be replaced by any function of $(\mathbf{k}+m\mathbf{v}_s)$.

We will distinguish quantities in the moving coordinate system by the superscript ^{0}. The only vector appearing in $g_+(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t)$ is $(\mathbf{v}_n-\mathbf{v}_s)$ and we define the normal fluid density ρ_n by

$$\mathbf{J}^0=\rho_n(\mathbf{v}_n-\mathbf{v}_s). \quad (4.18)$$

To the first order in $\mathbf{v}_n-\mathbf{v}_s$ we find

$$\rho_n=-\frac{1}{3\pi^2}\int k^4dk\frac{\partial f_k}{\partial E_k}. \quad (4.19)$$

This result is similar to the Landau result for He II except that here the excitations obey Fermi statistics.[15] For consistency in the hydrodynamic equations it is necessary to retain terms of order $(\mathbf{v}_n-\mathbf{v}_s)^2$ in ρ_n.

(*b*) *Momentum density.* Equation (4.8) and the corresponding equation for g_- are multiplied by $\pm\mathbf{k}+m\mathbf{v}_s$, respectively, and subtracted. After summing over $\mathbf{k}$ we find

$$(\partial\mathbf{J}/\partial t)+\nabla\cdot\mathbf{\Pi}+(e/m)[\rho\mathbf{E}+(1/c)(\mathbf{J}\times\mathbf{H})]=\mathbf{I}. \quad (4.20)$$

The stress tensor $\mathbf{\Pi}$ is given by

$$\Pi_{ij}=-(2i/V)\textstyle\sum_k(k_ik_j/m)g_+(\mathbf{R},\mathbf{k},t)-(\Delta\Delta^\dagger/g)\delta_{ij} \quad (4.21)$$

and

$$\mathbf{I}=-(2/V)\textstyle\sum_k\mathbf{k}(\dot{g}_+)_{\text{coll}}. \quad (4.22)$$

We first discuss Π_{ij}. Substituting from (4.7) we find

$$\Pi_{ij}=\rho v_{si}v_{sj}+J_i^0v_{sj}+J_j^0v_{si}+\Pi_{ij}^0, \quad (4.23)$$

$$\Pi_{ij}^0=-(2i/V)\textstyle\sum_k(k_ik_j/m)g_+(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t) -(\Delta\Delta^\dagger/g)\delta_{ij}. \quad (4.24)$$

The pressure P is defined by

$$\Pi_{ij}^0=\rho_n(v_n-v_s)_i(v_n-v_s)_j+P\delta_{ij}. \quad (4.25)$$

The damping term $\mathbf{I}$ is nonzero because momentum is not conserved in the collisions. Owing to the form of the collision term only the excitations are damped and substituting from (4.7) in (3.8) we find to first order in the velocity $\mathbf{v}_n$

$$\mathbf{I}=-\alpha\mathbf{v}_n, \quad (4.26)$$

where

$$\alpha=-\frac{1}{3\pi^2}\int k^4dk\frac{\partial f_k}{\partial E_k}\frac{1}{\tau_n}\frac{|\tilde{\epsilon}_k|}{E_k}, \quad (4.27)$$

and τ_n is the transport relaxation time in a normal metal

$$\frac{1}{\tau_n}=\tfrac{1}{2}n_0N_F\int d\Omega_{k'}|V(\mathbf{k}_F-\mathbf{k}_F')|^2(1-\cos\theta'). \quad (4.28)$$

N_F is the one-electron density of states at the Fermi surface and $d\Omega_k$ indicates an angular integration.

(*c*) *Energy density.* The conservation equation for $\mathcal{E}$ is found in the following way. The equations for g_+ and g_- are multiplied by $\epsilon_{k\pm mv_s}$, respectively, and added. The equations for $f^\dagger$ and f are multiplied by $|\Delta|$ and

[15] J. Bardeen, Phys. Rev. Letters **1**, 399 (1958).

added. The resulting two equations are then subtracted and summed over all momenta $\mathbf{k}$ giving

$$(\partial \mathcal{E}/\partial t)+\nabla\cdot\mathbf{J}_\varepsilon+(e/m)\mathbf{J}\cdot\mathbf{E}=0. \quad (4.29)$$

The energy current is given by

$$\mathbf{J}_\varepsilon=-(2i/V)\textstyle\sum_k(\mathbf{k}/m)\epsilon_k g_+(\mathbf{R},\mathbf{k},t)-2(|\Delta|^2/g)\mathbf{v}_s. \quad (4.30)$$

As the collisions conserve energy there is no collision term in (4.29). In the hydrodynamical model substituting (4.7) into (4.6) we find

$$\mathcal{E}=\tfrac{1}{2}\rho v_s^2+\mathbf{J}^0\cdot\mathbf{v}_s+\mathcal{E}^0, \quad (4.31)$$

where

$$\mathcal{E}^0=-(2i/V)\textstyle\sum_k\epsilon_k g_+(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t)-(|\Delta|^2/g). \quad (4.32)$$

In a similar manner the energy current is found to be

$$\mathbf{J}_\varepsilon=(\tfrac{1}{2}\rho v_s^2+\mathbf{J}^0\cdot\mathbf{v}_s+\mathcal{E}^0)\mathbf{v}_s+\mathbf{\Pi}^0\cdot\mathbf{v}_s+\tfrac{1}{2}\mathbf{v}_s^2\mathbf{J}^0+\mathbf{J}_\varepsilon^0, \quad (4.33)$$

where

$$\mathbf{J}_\varepsilon^0=-(2i/V)\textstyle\sum_k(\mathbf{k}/m)\epsilon_k g_+(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t). \quad (4.34)$$

This current may further be shown to have the form

$$\mathbf{J}_\varepsilon^0=[T\rho S+\mu\rho_n+\rho_n(\mathbf{v}_n-\mathbf{v}_s)^2](\mathbf{v}_n-\mathbf{v}_s), \quad (4.35)$$

where the entropy density ρS is defined by

$$\rho S=-(2k_B/V)\textstyle\sum_k\{f(E_{k1}-\mathbf{k}\cdot\mathbf{v}_n)\ln[f(E_{k1}-\mathbf{k}\cdot\mathbf{v}_n)] +[1-f(E_{k1}-\mathbf{k}\cdot\mathbf{v}_n)]\ln[1-f(E_{k1}-\mathbf{k}\cdot\mathbf{v}_n)]\}. \quad (4.36)$$

Using the definitions (4.25), (4.32), (4.36), and (4.18) we may also derive the thermodynamic identity

$$P+\mathcal{E}^0=T\rho S+(1/m)\mu\rho+\rho_n(\mathbf{v}_n-\mathbf{v}_s)^2. \quad (4.37)$$

The eight equations (4.14), (4.15), (4.20), and (4.29) together with the definitions of the fluxes $\mathbf{J}$ provide the set of hydrodynamical equations. By a simple thermodynamic argument we can also derive the equation for the entropy density

$$(\partial(\rho S)/\partial t)+\nabla\cdot(\rho S\mathbf{v}_n)=(\alpha/T)\mathbf{v}_n^2, \quad (4.38)$$

where α is given by (4.27). As in the case of He II the entropy flows with the normal fluid.

Except for the collision terms the above equations are exactly the two-fluid equations proposed by Landau[5] for He II extended to the case where the particles are charged. The propagation of plasma oscillations and second sound may be discussed by means of these equations. The calculations are similar to those of Landau for He II and we only give the results for the frequency ω_1 and damping ω_2 of waves with wave vector $\mathbf{k}$.

(1) *Plasma oscillations.*

$$\omega_1^2=(4\pi\rho e^2/m)+k^2u^2;\quad \omega_2=\alpha/2\rho, \quad (4.39)$$

where $u^2=(\partial P/\partial\rho)_S$ is the ordinary sound velocity.

(2) *Second sound.*

$$\omega_1^2=k^2(S^2\rho_s T/\rho_n C_V);\quad \omega_2=(\alpha/2\rho)(\rho_s/\rho_n), \quad (4.40)$$

where C_v is the specific heat. As discussed by Ginzburg[16] these latter waves will normally be too heavily damped to propagate. The real part ω_2 must also be limited by the condition $\omega_1<\Delta$.

5. PHENOMENOLOGICAL RELATIONS AND THERMAL CONDUCTIVITY

In this section we investigate the response of the superconductor to various driving forces confining ourselves to linear response only. We will also only consider the case where all quantities vary slowly in space and time so that we make use of the transport Eq. (3.13). Omitting the nonlinear terms in this equation the first two members are

$$[i(\partial/\partial t)+(i/m)\mathbf{k}\cdot\nabla-ei\mathbf{E}\cdot\nabla_k]g_+(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t) +|\Delta(\mathbf{R},t)|[f^\dagger(\mathbf{R},\mathbf{k},t)-f(\mathbf{R},-\mathbf{k},t)]=(\dot{g}_+)_{\rm coll}, \quad (5.1)$$

$$[i(\partial/\partial t)+2\epsilon_k-2e\phi_g]f^\dagger(\mathbf{R},\mathbf{k},t) +|\Delta(\mathbf{R},t)|[g_+(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t) +g_-(\mathbf{R},-\mathbf{k}+m\mathbf{v}_s,t)]=(\dot{f}^\dagger)_{\rm coll}. \quad (5.2)$$

In solving these equations we proceed as in the solution of the normal Boltzmann equation and regard all driving terms as being of first order. The solution to these equations can then be taken in the form

$$g_\pm(\mathbf{R},\pm\mathbf{k},t)=g_{0\pm}(\mathbf{R},\pm\mathbf{k},t)+g_{1\pm}(\mathbf{R},\pm\mathbf{k},t), \quad (5.3)$$

$$f^\dagger(\mathbf{R},\mathbf{k},t)=f(\mathbf{R},-\mathbf{k},t)=f_0(\mathbf{R},\mathbf{k},t), \quad (5.4)$$

where $g_{0\pm}$ are the BCS values corresponding to a flow $\mathbf{v}_s$. Thus $g_\pm$ is modified by a small amount $g_{1\pm}$ due to the driving terms whereas f is unchanged. The BCS values $g_{0\pm}$ and f_0 depend on $\mathbf{R}$ and t through $\mathbf{v}_s$, $\beta(=1/k_BT)$ and μ and are given by

$$g_{0+}(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t)=i\{u_k^2f(E_{k1})+v_k^2[1-f(E_{k2})]\}, \quad (5.5)$$

$$f_0^\dagger(\mathbf{R},\mathbf{k},t)=f_0(\mathbf{R},\mathbf{k},t)=iu_kv_k[1-f(E_{k1})-f(E_{k2})], \quad (5.6)$$

$$g_{0-}(\mathbf{R},-\mathbf{k}+m\mathbf{v}_s,t) =-i\{u_k^2[1-f(E_{k2})]+v_k^2f(E_{k1})\}. \quad (5.7)$$

Equation (5.2) is satisfied to terms linear in g_{1+} and the driving terms if we choose

$$g_{1+}=-g_{1-} \quad (5.8)$$

because then $(\dot{f}^\dagger)_{\rm coll}$ vanishes. The left-hand side of Eq. (5.2) vanishes as before [see Eq. (4.10)] provided we choose

$$e\phi_g-\mu=0. \quad (5.9)$$

Again making use of the definition of $\mathbf{v}_s$ in (3.3) and eliminating W by means of (5.9) gives the linearized form of (4.13)

$$m(\partial\mathbf{v}_s/\partial t)=-e\mathbf{E}-\nabla\mu. \quad (5.10)$$

Turning now to (5.1) we assume that g_{1+} is of the form $\mathbf{k}\cdot\boldsymbol{\gamma}\times$(function of $\bar{\epsilon}_k$), where $\boldsymbol{\gamma}$ is a unit vector.

[16] V. L. Ginzburg, Zh. Eksperim. i Teor. Fiz. **41**, 828 (1961) [English transl.: Soviet Phys.—JETP **14**, 594 (1962)].

It is then not difficult to show that

$$(\dot{g}_{+})_{\text{coll}}=-(i/\tau_s)g_{1+}, \tag{5.11}$$

where τ_s is the relaxation time in the superconductor. This is related to the normal metal relaxation time τ_n given in Eq. (4.28) by

$$1/\tau_s=(|\tilde{\epsilon}_k|/E_k)(1/\tau_n). \tag{5.12}$$

A similar result was obtained by Bardeen, Rickayzen, and Tewordt.[4] Substituting (5.11) in (5.1) and assuming that all quantities, e.g., the field $\mathbf{E}$, $\nabla\mu$, the thermal gradient, and g_{1+} vary in time as $e^{i\omega t}$ we find

$$g_{1+}(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t)=-\tilde{\tau}_s[(\partial/\partial t)+(\mathbf{k}/m)\cdot\nabla-e\mathbf{E}\cdot\nabla_k]\times g_{0+}(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t), \tag{5.13}$$

where $\tilde{\tau}_s=((1/\tau_s)+i\omega)^{-1}$. From (5.5) we have that

$$\frac{\partial}{\partial t}g_{0+}(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t)=i\frac{\partial f}{\partial E_k}\mathbf{k}\cdot\frac{\partial\mathbf{v}_s}{\partial t}, \tag{5.14}$$

$$\frac{\mathbf{k}}{m}\cdot\nabla g_{0+}(\mathbf{R},\mathbf{k}+m\mathbf{v}_s,t)=-i\tilde{\epsilon}_k\frac{\partial f}{\partial E_k}\frac{\mathbf{k}}{m}\cdot\frac{\nabla T}{T}-(\nabla\mu)\cdot\nabla_k g_{0+}. \tag{5.15}$$

In (5.14), $\partial\mathbf{v}_s/\partial t$ can be replaced by $-(e/m)\mathbf{E}-(1/m)\nabla\mu$ from Eq. (5.10).

The induced current is given by

$$\mathbf{J}=-(2i/V)\textstyle\sum_k \mathbf{k}[g_{0+}(\mathbf{R},\mathbf{k},t)+g_{1+}(\mathbf{R},\mathbf{k},t)], \tag{5.16}$$

and substituting from (5.5) and (5.13) we find

$$\mathbf{J}=\rho_s\mathbf{v}_s+K_1((e/m)\mathbf{E}+(1/m)\nabla\mu)-K_2(\nabla T/T). \tag{5.17}$$

If, for simplicity, we regard $\tilde{\tau}_s$ as a constant independent of $\mathbf{k}$ it can be shown that

$$K_1=-\rho_n\tilde{\tau}_s;\quad K_2=\tilde{\tau}_s T\rho S, \tag{5.18}$$

where ρS is the entropy density defined in (4.36) (except that $\mathbf{v}_n=0$ in this case). The energy current is given by

$$\mathbf{J}_\varepsilon=-(2i/mV)\textstyle\sum_k \epsilon_k\mathbf{k}[g_{0+}(\mathbf{R},\mathbf{k},t)+g_{1+}(\mathbf{R},\mathbf{k},t)]-2(|\Delta|^2/g)\mathbf{v}_s. \tag{5.19}$$

Substituting from (5.5) and (5.13) we find

$$\mathbf{J}_\varepsilon=\mu\mathbf{J}-K_2((e/m)\mathbf{E}+(1/m)\nabla\mu)-K_4(\nabla T/T), \tag{5.20}$$

where

$$K_4=-\frac{1}{3m^2\pi^2}\int k^4dk\tilde{\epsilon}_k^2\frac{\partial f}{\partial E_k}\tilde{\tau}_s. \tag{5.21}$$

The phenomenological relations (5.17) and (5.20) can be interpreted by observing that the excitations in a superconductor behave in many ways like the electrons in a normal metal. Thus the superfluid contributes the terms $\rho_s\mathbf{v}_s$ and $\mu\rho_s\mathbf{v}_s$ in (5.17) and (5.20), respectively. The remaining terms are exactly what one would expect for a normal metal (assuming a constant relaxation time) except that the density of normal fluid occurs for the superconductor instead of the total density as in the normal metal. The superfluid has no entropy. These phenomenological relations are also exactly of the form suggested recently by Luttinger.[6]

Under stationary conditions from (5.10),

$$e\mathbf{E}+\nabla\mu=0. \tag{5.22}$$

In the case of the thermal conductivity the superfluid flow is determined by the boundary conditions. The most common condition is that of zero-mass flow, i.e., $\mathbf{J}=0$. Then from (5.17)

$$\rho_s\mathbf{v}_s=\tau_s\rho S\nabla T, \tag{5.23}$$

and from (5.21) the thermal conductivity is

$$\kappa=K_4/T. \tag{5.24}$$

When the explicit form of τ_s from (5.12) is substituted in K_4 this result agrees precisely with that of Ref. 4. Some further consequences of these phenomenological relations are discussed by Luttinger.[6]

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