Our average value of $\Phi$ is seen to be lower than the limiting value of $\Phi = 0.375$ for spherical Fermi surfaces.

The values of $P_f$ and $\Phi$ reported in this work on the magnetoacoustic absorption in potassium are believed to represent, within the indicated experimental error, the trend in the dimensions of the Fermi surface in potassium. The adequacy of the independent-particle model in the nearly free-electron approximation for predicting the electronic structure of this alkali metal is clearly supported by this study.

# Spin-Disorder Thermal Resistivity of the Ferromagnetic Transition Metals*

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Data of White and Woods for the thermal resistivity $W$ of the transition metals show that $W$ for Fe, Co, and Ni displays an anomalous persistent temperature dependence in the region just below and through the Debye temperature. Since $W$ for the nonferromagnetic transition metals is approximately constant in this region, it is reasonable to attribute the additional temperature dependence to the well-known spin-disorder scattering of conduction electrons. It is shown, however, that a model which treats the magnetic $d$ electrons as localized and neglects $s$-$d$ transitions cannot account for the anomalous behavior of $W$, while a model which treats magnon-induced $s$-$d$ interband transitions presents a plausible explanation.

## 1. INTRODUCTION

THE band character of the electrons in the unfilled $d$ shells of the nonferromagnetic transition metals plays an essential role in electron transport processes.[1,2] At all but the very lowest temperatures the major contributor to the ideal electrical resistivity $\rho$ and the ideal thermal resistivity $W$ is phonon scattering of $s$ conduction electrons into holes in the $d$ band.[3] Even at the lowest temperatures there is evidence (although not conclusive) that the $d$ electrons are involved in Baber-type[4] scattering of conduction electrons.

The earliest treatment of the anomalous electrical resistivity of ferromagnetic transition metals was given by Mott who assumed, by analogy with the nonferromagnetic metals, that the role of the $d$ band was to act as a trap with a high density of states into which the $s$ conduction electrons could be scattered by phonons and lost from the current. As the $d$ electrons are responsible for the magnetic properties of these metals, the number of holes in the $d$ band is dependent on the magnetization. Although Mott's model had some of the qualitative features displayed by the real metals—namely,

that $\rho$ increases more rapidly, as a function of temperature, than for nonferromagnetic metals below the Curie temperature $T_c$, and above $T_c$ is linear—it could not account for some of the detailed behavior of $\rho$ near and above $T_c$.[5-7]

Vonsovski[8] recognized that an important scattering mechanism was being omitted and pointed out that the exchange interaction between $s$ and $d$ electrons may also give rise to a significant resistivity term. Kasuya[9] developed the theory of $s$-$d$ exchange interactions and subsequently calculated the contribution of this mechanism to $\rho$.[10] The $d$ electrons were, however, treated as localized, an approximation which excludes the possibility of interband transitions. We will refer to this as the single-band model. As the exchange interaction may be represented as an effective spin-dependent interaction between the conduction electrons and the localized $d$ electrons (magnetic system), the resulting contribution to $\rho$ is called the spin-disorder resistivity $\rho_{\text{mag}}$. Unfortunately, it is difficult to compare the detailed behavior of $\rho_{\text{mag}}$ as predicted by this theory with experimental results because it is difficult to

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separate the contribution due to phonon scattering of conduction electrons in any reliable way except in the high-temperature region where $\rho_{mag}$ is a large fraction of the total resistivity. Nonetheless, there is reasonable evidence of the importance of the s-d exchange interaction to the anomalous resistivity of the ferromagnetic transition metals. However, it is also clear that the d electrons are not strictly localized and whether or not interband transitions may be omitted with impunity remains a moot question.

In a recent study of $\rho$ for Fe, Co, Ni, and Gd at low temperatures, where spin waves are a good approximation of the magnetic spin system, Goodings[11] has shown that magnon-induced s-d interband transitions may comprise the major part of $\rho_{mag}$ above approximately 20°K.

In this paper we take up the effects of magnon-induced s-d transitions on the thermal resistivity. Although the effects of spin-disorder scattering on $\rho$ has been the subject of many investigations, little attention has been given to the associated phenomena of the thermal resistivity and it does not seem to have been realized that the single-band model is apparently inadequate to account for the experimentally observed behavior of $W$ for Fe, Co, and Ni in the region near the Debye temperature, $\Theta_D$.

A review of the data of White and Woods[12] shows that the temperature dependence of $W$ for the ferromagnetic metals is similar to that of the nonferromagnetic metals over the range 20°K to approximately $T = \Theta_D/2$, increasing roughly as $T^2$. However, above $T \approx \Theta_D/2$, $W$ for the nonferromagnetic metals undergoes a change of slope, becoming constant and reaching its full value near $T = \Theta_D$. $W$ for Fe, Co, and Ni, on the other hand, undergoes a change of slope near $T = \Theta_D/2$ but maintains a persistent temperature dependence through the Debye temperature. Above this region the situation is not clear because there does not seem to be data for $W$ extending to high temperatures. As the only difference between these two types of systems is that the ferromagnetic metals have an additional scattering mechanism, the difference in the behavior of $W$ may be reasonably attributed to the spin-disorder scattering. A rough estimate of the persistent temperature dependence above $T/\Theta_D = 0.5$ may be given by the parameter

$$\Delta W = [W(T/\Theta_D = 1) - W(T/\Theta_D = 0.5)] / W(T/\Theta_D = 0.5). \quad (1)$$

For the nonferromagnetic transition metals $\Delta W$ is approximately 0.05 as compared to 0.3 for Fe, Co, and Ni. It is easily shown, however, that $W_{mag}$ for these ferromagnetic metals computed on the single-band model is constant over the range 20 to 300°K and cannot therefore account for the persistent temperature

dependence observed experimentally. Moreover, a representative calculation for Ni shows that $W_{mag} = 0.02$ cm deg/W which is only 3% of $W_{total}(T/\Theta_D = 0.5)$. See Appendix A. Clearly, then, a modification of the single-band model must be considered if the anomalous behavior of $W$ is to be understood.

It is informative to briefly inspect the approximations made in the development of the s-d interaction theory.[9]

In the Hamiltonian of the system of conduction electrons (in our case s electrons), tightly bound electrons (in our base d electrons), and ions, the terms which give rise to the spin-disorder resistivities are the exchange interaction between s and d electrons. In second quantization notation these terms are represented by

$$\sum_{t_1} \sum_{t_2} \sum_{t_3} \sum_{t_4} \sum_{\nu} \sum_{\mu} a_{t_1,}{}^* a_{t_2,\mu}{}^* a_{t_3,\mu} a_{t_4,\nu} \left\langle t_1\nu, t_2\mu \left| \frac{e^2}{r} \right| t_3\mu, t_4\nu \right\rangle ,$$

where $t_1$, $t_2$, $t_3$, and $t_4$ refer to s and d states and $\nu$ and $\mu$ to the corresponding spin states.

Restricting our consideration to terms which give rise to an effective s-electron-spin–d-electron-spin interaction, only the terms for which at least one $t$ refer to an s state and one $t$ to a d state are important. In the Ref. 9 two important approximations were made:

(a) the d states are represented by localized atomic functions,

(b) terms which represent transitions of conduction electrons into unfilled d states were neglected.

The terms for which $t_1$ and $t_2$ are s and d states and $t_3$ and $t_4$ are s and d states give rise to the well-known Kasuya s-d interaction Hamiltonian. In this paper we consider the relaxation of the above constraints in two separate cases.

*Case 1*: In order to assess the importance of the localized d approximation we attempt to take account of the band character of the d electrons by use of an artifice. The d electrons are assumed to be separable into two types, localized and nearly free. Transitions of the type $s \rightarrow d_{free}$ are admitted but not $s \rightarrow d_{localized}$. In this way we construct a two-band model of the conduction electrons and allow Mott type $s \rightarrow d$ transitions. In this paper, however, we will only consider magnon-induced transitions.

*Case 2*: The localized d electron approximation is retained but $s \rightarrow d$ localized transitions are admitted. These transitions give rise to Anderson-Clogston covalent admixtures.[13] Such a model has been considered by Kondo[14] in an investigation of the $g$ shift and anomalous Hall effect in Gd.

Clearly an accurate treatment of the problem lies somewhere between the two cases sited above. However, it is our concern in this investigation only to

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point out what we feel is the cause for the anomalous behavior of $W$.

In Sec. 2 the method by which the thermal resistivity is obtained is discussed, namely by the application of the variational principle to the Boltzmann equation. Although this method is more completely described in another work,[11] we present it here in the interest of continuity but principally in order to introduce a systematized notation developed by Goodings that we shall adopt. Although we shall be concerned principally with the thermal resistivity at low temperatures (which requires the consideration of inelastic scattering processes) we also present the results for the electrical resistivity.

In Sec. 3 the results of Sec. 2 are applied to the two-band model of the conduction electron and $\rho_{mag}$ and $W_{mag}$ are computed at low temperatures where spin waves are a good approximation of the magnetic system. Our results for $\rho_{mag}$ are different from those of Goodings in that inelastic scattering effects are included. $W_{mag}$ is shown to reduce to Kasuya's result at lowest temperatures—as it must when $s \rightarrow d$ transitions are frozen out—but increases by an order of magnitude as the temperature is increased above a critical value determined by the momentum gap between the $s$ and $d$ Fermi surfaces. Using this momentum gap as a parameter, $W_{mag}$ made be made to reproduce the temperature dependence of $W$ observed experimentally near the Debye temperature.

In Sec. 4, $\rho_{mag}$ and $W_{mag}$ are computed for case II of our approximation. It is shown that although the Anderson-Clogston $s$-$d$ admixtures may effect the necessary temperature dependence of $W_{mag}$ to reproduce the slope of $W$ observed near $T = \Theta_D$, the estimates of the $s$-$d$ exchange interaction require are an order of magnitude too large.

## 2. THE VARIATIONAL METHOD

The Boltzmann equation may be written in the general linear form

$$\mathfrak{D}(\mathbf{k}) + \sum_{\mathbf{k}'} L(\mathbf{k} \rightarrow \mathbf{k}')[\xi(\mathbf{k}') - \xi(\mathbf{k})] = 0, \qquad (2)$$

where $\mathfrak{D}(\mathbf{k})$ represents the rate of change of the distribution function of conduction electrons due to external fields and the second term represents the rate of change of the distribution function due to scattering processes. $\xi(\mathbf{k})$ is a measure of the change of the distribution function from its zero field, equilibrium value, $f_0(k)$, and is defined by the equation

$$f(\mathbf{k}) = f_0(k) - kT\xi(\mathbf{k})\frac{\partial f_0(k)}{\partial E_k}. \qquad (3)$$

If we define the quantity

$$\mathcal{J}_0 \equiv \sum_{\mathbf{k}} \mathfrak{D}(\mathbf{k})\xi(\mathbf{k}) \qquad (4)$$

and substitute the explicit dependence of $\mathfrak{D}(\mathbf{k})$ on the external fields, we obtain the relation

$$kT\mathcal{J}_0 = \sum_{\mathbf{k}} e\mathbf{\vartheta}(\mathbf{k})f_1(\mathbf{k}) \cdot \mathbf{\mathcal{E}}$$

$$-\frac{1}{T}\sum_{\mathbf{k}} [E(\mathbf{k}) - E(k_F)]\mathbf{\vartheta}(\mathbf{k})f_1(\mathbf{k}) \cdot \nabla(T), \qquad (5)$$

where $f_1(\mathbf{k}) = f(\mathbf{k}) - f_0(k)$, $E(\mathbf{k})$, and $\mathbf{\vartheta}(\mathbf{k})$ are the energy and velocity of a electron with momentum $\hbar\mathbf{k}$; $\mathbf{\mathcal{E}}$ and $\nabla T$ are the external electric field and thermal gradient.

Equation (5) may further be written as

$$kT\mathcal{J}_0 = V\mathbf{J} \cdot \mathbf{\mathcal{E}} - V(1/T)\mathbf{Q} \cdot \nabla T, \qquad (6)$$

when $\mathbf{J}$ and $\mathbf{Q}$ are the current density and heat flux, respectively. Thus, the electrical and thermal conductivities are given by

$$\sigma = (kT/V\mathcal{E}^2)\mathcal{J}_0, \qquad (7)$$

$$K = kT^2\mathcal{J}_0/V|\mathrm{grad}_r T|^2. \qquad (8)$$

If we define an integral invariant $\mathcal{J}$ in terms of a variational function $\Phi(\mathbf{k})$ as

$$\mathcal{J} \equiv 2 \sum_{\mathbf{k}} \mathfrak{D}(\mathbf{k})\Phi(\mathbf{k}) - \tfrac{1}{2} \sum_{\mathbf{k},\mathbf{k}'} L(\mathbf{k} \rightarrow \mathbf{k}')[\Phi(\mathbf{k}') - \Phi(\mathbf{k})]^2. \quad (9)$$

It can be shown that $\mathcal{J}$ is stationary with respect to small variations $\delta\Phi(\mathbf{k})$ provided $\Phi(\mathbf{k})$ satisfies the Boltzmann Eq. (2) and $L(\mathbf{k} \rightarrow \mathbf{k}') = L(\mathbf{k}' \rightarrow \mathbf{k})$. Thus $\mathcal{J} \leq \mathcal{J}_0$ and has its maximum value when $\Phi(\mathbf{k}) = \xi(\mathbf{k})$. The stationary property of $\mathcal{J}$ insures a reasonable approximation of $\sigma$ and $k$ when the function $\Phi(\mathbf{k})$ is chosen to have a form which is approximately the same as the exact solution. For the electrical conductivity, the variational function is chosen to have the form

$$\Phi(\mathbf{k}) = \Phi(E_k)\bar{\cos\theta}, \qquad (10)$$

where $\theta$ is the angle between $\mathbf{\mathcal{E}}$ and $\mathbf{k}$. For the thermal resistivity, the appropriate variational function is taken as

$$\Phi(\mathbf{k}) = \Phi(E)[E(k) - E(k_F)]\cos\phi, \qquad (11)$$

where $\phi$ is the angle between $\nabla T$ and $\mathbf{k}$.

### A. The Scattering Mechanism and Notation

In Secs. 3 and 4 the above results are applied to scattering of conduction electrons through an exchange interaction of the form

$$\mathcal{H} = -2 \sum_{n} G(\mathbf{r} - \mathbf{R}_n)\mathbf{s} \cdot \mathbf{S}_n, \qquad (12)$$

where $\mathbf{s}$ is the spin of a conduction electron at $\mathbf{r}$ and $\mathbf{S}_n$ is the spin of a magnetic ion at $\mathbf{R}_n$. It is assumed that below approximately $T = 300°K$, spin waves are a good approximation of the magnetic system and that the important scattering processes are those in which

a conduction electron is scattered from the Bloch state $\psi_k$ to $\psi_{k'}$ with an accompanying spin flip and absorption or emission of a single magnon.

As we will be concerned with conduction electrons in two bands, an $s$ and a $d$ band, in Sec. 3, it is convenient at this point to introduce a band index $\alpha$. The integral invariant $\mathcal{I}$ defined by Eq. (9) is rewritten as

$$\mathcal{I} = 2 \sum_{k,\alpha} \mathfrak{D}(k\alpha+)\Phi(k\alpha+)$$
$$-\tfrac{1}{2} \sum_{k,k',\alpha,\alpha'} L(k\alpha+ \to k'\alpha'-)$$
$$\times [\Phi(k'\alpha'-)-\Phi(k\alpha+)]^2$$
$$+2 \sum_{k,\alpha} \mathfrak{D}(k\alpha-)\Phi(k\alpha-)$$
$$-\tfrac{1}{2} \sum_{k,k',\alpha,\alpha'} L(k\alpha- \to k'\alpha'+)$$
$$\times [\Phi(k'\alpha'+)-\Phi(k\alpha-)]^2, \quad (13)$$

where the $\pm$ represent the spin quantum number of the conduction electrons.

Kasuya took account of the spin dependence of the energy of conduction electrons in a molecular-field approximation. This spin dependence entered his results through the parameter $x_0$ which is defined in Appendix A and indicated as being negligible above approximately $T=20°K$. Hence, we assume it is a good approximation to replace each pair of spin-dependent functions by the single spin-independent functions $\Phi(k,\alpha)$, $\xi(k,\alpha)$, $E(k,\alpha)$, $f_0(k,\alpha)$ and $\vartheta(k,\alpha)$. $L(k\alpha\pm \to k'\alpha'\mp)$, however, has been shown by Goodings to be given by

$$L(k\alpha\pm \to k'\alpha'\mp) = \sum_q \frac{kT}{2}\left(\frac{\partial f_0}{\partial E_k}\frac{\partial f_0}{\partial E_{k'}}\right)^{1/2} \text{csch}\left(\frac{E_q}{2kT}\right)$$
$$\times \frac{4\pi S}{Nh}|G_{\alpha\alpha'}(q)|^2 \delta_{k',k\pm q}\delta[E(k)-E(k')\pm E_q], \quad (14)$$

where $E_q$ is the energy of a spin wave of wave vector $q$ and $G_{\alpha\alpha'}(k-k')$ is given by

$$G_{\alpha\alpha'}(k-k') = \frac{1}{\Omega}\int \exp[i(k-k')\cdot r]$$
$$\times u_{k'\alpha'}^*(r)u_{k\alpha}(r)G(r)dr, \quad (15)$$

with $\Omega$ representing the atomic volume and $u_k(r)$ the Bloch character of the conduction electrons. The important feature of the Eq. (14) is that the spin dependence of $L(k\alpha\pm \to k'\alpha'\mp)$ is in the delta functions of momentum and energy. Consequently, Eq. (13) may be written in terms of the spin independent functions as

$$\mathcal{I} = 4 \sum_{k\alpha} \mathfrak{D}(k\alpha)\Phi(k\alpha) - \tfrac{1}{2} \sum_{k,k',\alpha,\alpha',\text{spin}} L(k\alpha \to k'\alpha')$$
$$\times [\Phi(k'\alpha')-\Phi(k\alpha)]^2. \quad (16)$$

The point of this development is that if we define the quantities

$$\mathfrak{D}_\alpha = 2 \sum_k \mathfrak{D}(k\alpha)\Phi(k\alpha), \quad (17)$$

$$2\mathcal{L}_{\alpha\alpha} = \sum_{k,k',\text{spin}} L(k\alpha \to k'\alpha)[\Phi(k',\alpha)-\Phi(k\alpha)]^2, \quad (18)$$

$$\mathfrak{M}_{\alpha\alpha} = \sum_{k,k',\text{spin}} L(k\alpha \to k'\alpha')[\Phi(k,\alpha)]^2, \quad (19)$$

$$\mathfrak{M}_{\alpha\alpha'} = \sum_{k,k',\text{spin}} L(k\alpha \to k'\alpha')\Phi(k,\alpha)\Phi(k',\alpha'), \quad (20)$$

then Eq. (16), becomes

$$\mathcal{I} = 2\mathfrak{D}_s + 2\mathfrak{D}_d - (\mathcal{L}_{ss}+\mathfrak{M}_{ss}) - (\mathcal{L}_{dd}+\mathfrak{M}_{dd}) + 2\mathfrak{M}_{sd} \quad (21)$$

and the problem of computing the electrical and thermal resistivities from Eqs. (7) and (9) is reduced to evaluating the above defined "basic integrals."

Finally, it is necessary to write Eq. (21) in a form such that $\mathcal{I}$ is unchanged if $\Phi(k,s)$ and $\Phi(k,d)$ are multiplied by arbitrary constants $c_s$ and $c_d$. In this way, if we replace $\Phi(k,s)$ and $\Phi(k,d)$ in Eq. (21) by $c_s\Phi(k,s)$ and $c_d\Phi(k,d)$ and require that $(\partial\mathcal{I}/\partial c_s)=(\partial\mathcal{I}/\partial c_d)=0$ we obtain $\mathcal{I}$ in the form

$$\mathcal{I} = \frac{(\mathcal{L}_{ss}+\mathfrak{M}_{ss})\mathfrak{D}_d^2 + 2\mathfrak{M}_{sd}\mathfrak{D}_s\mathfrak{D}_d + (\mathcal{L}_{dd}+\mathfrak{M}_{dd})\mathfrak{D}_s^2}{(\mathcal{L}_{ss}+\mathfrak{M}_{ss})(\mathcal{L}_{dd}+\mathfrak{M}_{dd})-\mathfrak{M}_{sd}^2}. \quad (22)$$

## 3. THE ITINERANT $d$-ELECTRON MODEL

The mechanics of the calculation of the electrical and thermal resistivities has been set up in Sec. 2 so that all that remains is to describe the model to be used and discuss the approximations employed to evaluate the basic integrals.

In this section we treat a fraction of the $d$ electrons to be itinerant and the conduction electrons are represented by two overlapping, spherical bands at the Fermi energy, an $s$ band and a $d$ band with effective masses $m_s$ and $m_d$, respectively, that is

$$E(k,s) = \hbar^2 k^2/2m_s, \quad (23)$$

$$E(k,d) = \hbar^2 k^2/2m_d. \quad (24)$$

The standard approximations are made that the exchange interaction defined in Eq. (12) is replaced by a delta function and the energy of spin wave of wave vector $q$ is $E_q=Dq^2$. One of the major approximations made in this study is that umklapp processes are not considered. Using these approximations, the basic integrals may be easily evaluated. In Appendix B $\mathcal{L}_{\alpha\alpha}$ is computed for the thermal resistivity as an illustrative example. The other integrals are computed in a similar fashion and the results are as follows.

Defining the functions

$$\mathcal{Q}_n{}^m \equiv \frac{a^{4-2n}}{6} \int_0^Q \left(\frac{E_q}{2kT}\right)^m \operatorname{csch}^2\!\left(\frac{E_q}{2kT}\right) q^{3-2n} dq , \qquad (25)$$

$$\bar{R}_n{}^m \equiv \frac{a^{4-2n}}{6} \int_{|k_{F_s}-k_{F_d}|}^Q \left(\frac{E_q}{2kT}\right)^m \operatorname{csch}^2\!\left(\frac{E_q}{2kT}\right) q^{3-2n} dq , \qquad (26)$$

where $Q$ is the radius of a sphere whose volume is equal to the first Brillouin zone, $k_{F_s}$ and $k_{F_d}$ are the Fermi momenta for $s$ and $d$ electrons, respectively, and $a$ is the nearest-neighbor atomic distance, we obtain the expressions for the electrical resistivity.

$$\mathcal{L}_{\alpha\alpha} = \left(\frac{2\pi S V^2 kT}{N\hbar k_F{}^4 a^4}\right) G_{\alpha\alpha}{}^2 n_\alpha{}^2 \Phi_\alpha{}^2 \mathcal{Q}_0{}^1 , \qquad (27)$$

$$\mathfrak{M}_{\alpha\alpha} = \left(\frac{4\pi S V^2 kT}{N\hbar k_{F_\alpha} k_{F_{\alpha'}} a^2}\right) G_{\alpha\alpha'}{}^2 n_\alpha n_{\alpha'} \Phi_\alpha{}^2 \bar{R}_1{}^1 , \qquad (28)$$

$$\mathfrak{M}_{\alpha\alpha'} = \left(\frac{2\pi S V^2 kT}{N\hbar k_{F_\alpha}{}^2 k_{F_{\alpha'}}{}^2 a^4}\right) G_{\alpha\alpha'}{}^2 n_\alpha n_{\alpha'} \Phi_\alpha \Phi_{\alpha'}$$

$$\times [-\bar{R}_0{}^1 + a^2 (k_{F_s}{}^2 + k_{F_d}{}^2)\bar{R}_1{}^1] , \qquad (29)$$

$$\mathfrak{D}_\alpha = \tfrac{2}{3} e\mathcal{E} V \vartheta_\alpha n_\alpha \Phi_\alpha , \qquad (30)$$

and for the thermal resistivity

$$\mathcal{L}_{\alpha\alpha} = (kT)^2 \frac{2\pi S V^2 kT}{N\hbar k_{F_\alpha}{}^4 a^4} G_{\alpha\alpha}{}^2 n_\alpha{}^2 \Phi_\alpha{}^2$$

$$\times \left[ 4\mathcal{Q}_1{}^3 a^2 k_{F_\alpha}{}^2 - \frac{2}{3}\mathcal{Q}_0{}^3 + \frac{\pi^2}{3}\mathcal{Q}_0{}^1 \right] , \qquad (31)$$

$$\mathfrak{M}_{\alpha\alpha'} = (kT)^2 \frac{4\pi S V^2 kT}{N\hbar k_{F_\alpha} k_{F_{\alpha'}} a^2} G_{\alpha\alpha'}{}^2 n_\alpha n_{\alpha'} \Phi_\alpha{}^2$$

$$\times \left[ \frac{4}{3}\bar{R}_1{}^3 + \frac{\pi^2}{3}\bar{R}_1{}^1 \right] , \qquad (32)$$

$$\mathfrak{M}_{\alpha\alpha'} = (kT)^2 \frac{2\pi S V^2 kT}{N\hbar k_{F_\alpha} k_{F_{\alpha'}} a^2} G_{\alpha\alpha'}{}^2 n_\alpha n_{\alpha'} \Phi_\alpha \Phi_{\alpha'}$$

$$\times \left[ \frac{2}{3}\bar{R}_0{}^3 - \frac{2}{3}a^2(k_{F_s}{}^2 + k_{F_d}{}^2)\bar{R}_1{}^3 - \frac{\pi^2}{3}\bar{R}_0{}^1 \right.$$

$$\left. + \frac{\pi^2}{3}a^2(k_{F_s}{}^2 + k_{F_d}{}^2)\bar{R}_1{}^1 \right] , \qquad (33)$$

$$\mathfrak{D}_\alpha = (2\pi^2/9)k^2 T (\operatorname{grad}_r T) V n_\alpha \vartheta_\alpha \Phi_\alpha , \qquad (34)$$

where $n_\alpha$ is the density of states at the Fermi energy, $\vartheta_\alpha$ the Fermi velocity of electrons in the $\alpha$ band, $V$ is the sample volume, and $N$ the number of atoms in the sample.

Before making an analysis of the experimental data, it is of interest to compare our results with those of

Kasuya. This can be done by requiring $n_d=0$ which removes the itinerant $d$ electrons from the conduction band. From Eqs. (7) and (22) $\rho$ is then given by

$$\rho = (V\mathcal{E}^2/kT)(\mathcal{L}_{ss}/\mathfrak{D}_s{}^2) . \qquad (35)$$

Substituting for $\mathcal{L}_{ss}$ and $\mathfrak{D}_s$ from Eqs. (27) and (30), Eq. (35) becomes

$$\rho = \frac{3\pi}{8} \frac{V}{N} \frac{S m_s}{e^2 \hbar} \frac{G_{ss}{}^2}{E_F} \left(\frac{kT}{Dk_F{}^2}\right)^2 J_2\!\left(\frac{kT}{DQ^2}\right) , \qquad (36)$$

where

$$J_2(x) = \int_0^x \frac{z^2 dz}{(e^z-1)(1-e^{-z})} . \qquad (37)$$

This expression is identical to that obtained by Kasuya using the total balance method of solving the Boltzmann equation. When Goodings reduced his expression for $\rho$ to a single-band, conduction-electron model he found that his expression differed from Kasuya's by a factor of $\frac{3}{2}$, a fact which he attributed to the variational method. As the method used in this investigation is the same as that of Goodings except that inelastic scattering effects have been included, it is now clear that this factor arose as a result of having omitted inelastic scattering.

Further, we find that the Lorentz number is given by

$$L = \frac{k^2 \pi^4 Q^2}{3^2 e^2 k_{F_s}{}^2} t \left\{ \frac{J_3(t^{-1})}{J_2(t^{-1})} + t\left(\frac{Q}{k_{F_s}}\right)^2 \left[ \frac{\pi^2}{3} \frac{1}{6} \frac{J_4(t^{-1})}{J_2(t^{-1})} \right] \right\}^{-1} , \qquad (38)$$

which reduces to Kasuya's expression in the low-temperature region and to $L_0 = (\pi^2/3)(k^2/e)$ in the high-temperature limit.
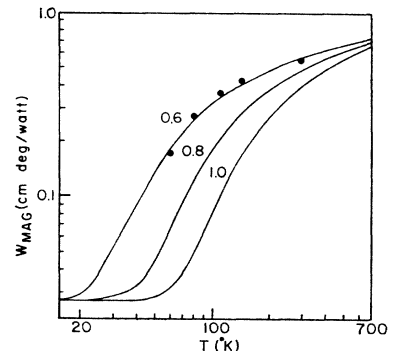
## A. Comparison with Experimental Results

This section is restricted to consideration of the thermal resistivity as our results for the electrical resistivity are essentially the same as those obtained by Gooding except that we have considered the effects of inelastic scattering as well as elastic scattering.

Calculations of $W$ have been performed for Fe, Co, and Ni using the Eqs. (9), (22), (25), (26), and (31) then (34) and the results are displayed in Figs. 1 and 2.



FIG. 1. Computed curves of $W_{mag}$ for Fe on a two-band model along with experimental results of White and Woods of the ideal thermal resistivity near one-half the Debye temperature. The values given to the left of the curves are the parameter
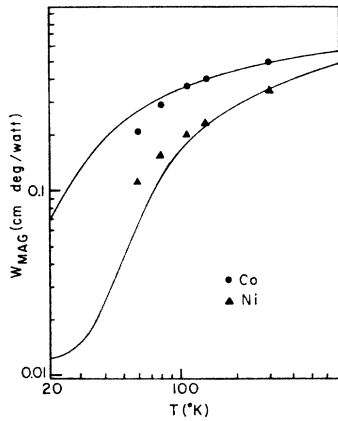
$$a|k_{F_s} - k_{F_d}| .$$

Fig. 2. Computed curves of $W_{mag}$ for Co and Ni on a two-band model along with experimental results of White and Woods of the ideal thermal resistivity near one-half the Debye temperature. The values given to the left of the curve are the parameter $a|k_{F_s}-k_{F_d}|$.

The values for the physical parameters used are given in Table I. The values for $k_{F_s}$ and $E_F$ were obtained assuming one $s$ electron per atom,[15] the ratio of the density of states $n_d/n_s$ from electronic specific-heat data,[16] the values of the effective spin $s$, and $D/k$ were taken from low-temperature magnetization data.[17] Reliable estimates of the parameters $G_{ss}$, $G_{sd}$, and $G_{dd}$ are difficult to obtain. Atomic spectra data[18] indicate values in the range $0.2 \to 0.5$ eV while estimates obtained by assuming the Heisenberg exchange Hamiltonian as given by the indirect coupling theories of Kasuya and Yosida[19] give $G_{ss} \sim 1-3$ eV for $G(\mathbf{r}-\mathbf{R}_n)$ taken to be a square well of radii $3.0/k_F$ and $5.0/k_F$ [the $\delta$-function approximation of $G(\mathbf{r}-\mathbf{R}_n)$ gives a negative nearest-neighbor exchange integral]. Estimates obtained from measurements of $\rho$ above the Curie temperature by using a single-band model yield values of $G_{ss} \approx 0.7$ eV.[7,10] Clearly because of the approximations in the theory, $G_{ss}$ is best treated as a variable parameter. In this work we take, for convenience,

$$G_{ss}=G_{sd}=G_{dd}=0.5 \text{ eV}.$$

The essential features of the thermal resistivity on the two band model may be simply summed up although $W$ is obtained from a somewhat unwieldy expression. Just as with Mott's original model, the $d$ band acts as a trap with a high density of states into which the $s$ electrons are scattered. Above some characteristic temperature $T_m$ which depends on the momentum gap between the $s$ and $d$ band Fermi spheres, the contri-

[15] L. R. Walker, G. K. Wertheim, and V. Jaccarino, Phys. Rev. Letters 6, 98 (1961).

[16] See, for example, J. M. Ziman, Electrons and Phonons (Oxford University Press, New York, 1960), p. 127.

[17] F. Keffer, Handbuch der Physik, edited by S. Flügge (Springer-Verlag, Berlin, to be published). The values of $D/k$ used in this calculation were taken from an earlier edition and differ slightly from the published values.

[18] Charlotte E. Moore, Natl. Bur. Std. (U. S.) Circ. No. 467 (1952), Vol. 2. For example, for Fe, $2G_{ss}S'$ is approximately the difference in energy between the $3d^74s^3F$ and $3d^74S^5F$ terms of Fe I, $S'$ being the unpaired $d$-electron spin in these terms.

[19] K. Yosida, Phys. Rev. 106, 893 (1957).

bution to the thermal resistivity due to magnon-induced $s$-$d$ transitions, $W_{mag}(s$-$d)$, is constant and an order of $n_d/n_s$ larger than the contribution from intraband processes. Below $T_m$, $W_{mag}(s$-$d)$ decays exponentially as $s$-$d$ transitions become frozen out and at the lowest temperatures $W_{mag} \approx W_{mag}(s$-$s)$, the contribution from intraband $s \to s$ scattering. For the purposes of discussion it is convenient to define a cross-over region in which $W_{mag} \propto T^n$ with $n \approx 2$. The position of this region on the temperature scale is shown in Fig. 1 to depend on the parameter $a(k_{F_s}-k_{F_d})$. This fact may be used to explain the persistent temperature dependence of $W$ observed experimentally in the region $T/\Theta_D > 0.5$ where the contribution from phonon scattering of conduction electrons is constant. By fitting the "knee" of the calculated $W_{mag}$ curves into this region an upper limit for the parameters $a(k_{F_s}-k_{F_d})$ and $m_d/m_s$ may be established. The results are presented in Table II. Because of the neglect of umklapp processes and the assumption of spherical energy bands, these estimates are regarded as no more than some average Fermi-surface property.

At the lowest temperatures, the fact that $W_{mag} \approx W_{mag}(s$-$s)$ can be used to obtain an upper limit for the parameter $G_{ss}$. The experimental data of White and Woods show that at temperatures as low as 20°K, $W \propto T^2$ and we may conclude that $W > W_{mag}(s$-$s)$. Using the values in Table I for the physical parameters and expressions (36) and (38) we obtain the limits

$$G_{ss} < 0.52, 0.95, 0.80$$

for Fe, Co, and Ni, respectively.

These results are significant and very important to an interpretation of the electrical resistivity at low temperatures and shed some light on the localized $d$ electron approximation. At temperatures less than 10°K $\rho$ is proportional to $T^2$ for Fe, Co, and Ni[12]. This behavior has been attributed to the spin-disorder resistivity. However, there are other transition metals, viz., W, Pd, Pt, Nd, which also display a $T^2$ dependence in this region. These metals are not ferromagnetic and the $T^2$ dependence is probably due to electron-electron scattering.[4] It is believed[3] that it is a good approximation to treat the $d$ electrons in these metals as being in a band. This is important to the electron-electron scattering processes because significant contributions to $\rho$ are made when $s$ electrons are scattered by the more "massive" $d$ electrons conserving momentum while decreasing the electron current. It has been

TABLE I. Values of physical parameters used in this calculation.

| | $V/N$ ($\times 10^{-23}$ cm³) | $k_{F_s}a$ | $E_F$ (eV) | $n_d/n_s$ | $D/ka^2$ (°K) | $S$ | $G_{ss}$ (eV) | $G_{sd}$ (eV) | $G_{dd}$ (eV) |
|---|---|---|---|---|---|---|---|---|---|
| Fe | 1.18 | 3.37 | 7.1 | 10.0 | 545 | 1.06 | 0.5 | 0.5 | 0.5 |
| Co | 1.10 | 3.47 | 7.3 | 10.3 | 638 | 0.77 | 0.5 | 0.5 | 0.5 |
| Ni | 1.09 | 3.46 | 7.3 | 15.3 | 761 | 0.27 | 0.5 | 0.5 | 0.5 |

TABLE II. Estimates of upper limits for the physical parameters obtained from thermal-resistivity data.

| Metal | $a\|k_{F_s}-k_{F_d}\|$ | $m_d/m_s$ |
|-------|-------------------------|-----------|
| Fe | 0.6 | 12.2 |
| Co | 0.4 | 11.6 |
| Ni | 0.6 | 18.5 |

shown[20] that in order that $\rho_{mag}$ be the sole cause of the $T^2$ dependence in Fe, Co, and Ni, $G_{ss}$ must be 3.2, 2.0, and 9.0 eV,[21] respectively. As the estimates for the upper limit for $G_{ss}$ from the low-temperature $W$ data are less, another process must then contribute to the $T^2$ dependence of $\rho$. If these processes are taken to be electron-electron scattering, this is evidence of the correctness of the two band model for the conduction electrons.

## 4. THE EFFECTS OF s-d ADMIXTURES ON A SINGLE BAND MODEL

In this section, the localized $d$ electron approximation is retained and the conduction electrons are taken to lie in a spherical $s$ band. The processes which give rise to $s$-$d$ transitions on a two-band model are treated here by second-order perturbation theory. They give rise to a shift in energy of the $s$ electrons through processes in which an $s$ electron is transferred to a $d$ orbital and back into a different $s$ state. Kondo[14] has shown that in second quantization notation this energy may be represented by the effective spin-spin Hamiltonian

$$\mathcal{3C}_{AD}=\sum_{k,k'} AP_2(\cos\theta_{kk'})\mathbf{s}_c\cdot\mathbf{S}_d a_{k'}^+ a_k^-, \qquad (39)$$

where $A$ is a measure of the $s$-$d$ admixtures, $P_2$ is a Legendre polynomial of second order, $\theta_{kk'}$ is the angle between the wave vector $\mathbf{k}$ and $\mathbf{k}$, $\mathbf{s}_c$ and $\mathbf{S}_d$ are the spins of a conduction electron and a magnetic ion, respectively, and $a_k^-$ and $a_k^+$ are destruction and creation operators for the $s$ electron Bloch states $\psi_k$ and $\psi_{k'}$, respectively.

This term has the same form as the $s$-$d$ exchange interaction. Thus, with the inclusion of $s$-$d$ admixtures, the calculation for $\rho$ and $W$ proceeds as in Sec. 3 with $G_{ss}^{(0)}$ replaced by $G_{ss}^{(0)}-AP_2(\cos\theta_{kk'})$. The resulting expressions for $\rho$ and $W$ are given by

$$\rho_{mag}=\frac{3\pi}{4}\frac{VmSG_{ss}^2}{N\hbar e^2 E_F}\left(\frac{2kT}{k_F^2 D}\right)^2 \sum_{p=0} C_p\left(\frac{2kT}{Dk_F^2}\right)^p$$

$$\times\frac{\Gamma(3+p)}{2^{3+p}}Z(2+p), \qquad (40)$$

[20] D. A. Goodings, J. Appl. Phys. 34, 1370 (1963).
[21] In Ref. 20 the estimate of $G_{ss}$ for N quoted as 0.9 eV is in error and should be 9.0 eV.

$$W_{mag}=\frac{27VmSG_{ss}^2}{2\pi^3\hbar N k^2 TE_F}\frac{kT}{k_F^2 D}\sum_{p=0}^{4} C_p\left(\frac{2kT}{Dk_F^2}\right)^p$$

$$\times\left\{\frac{4\Gamma(4+p)}{2^{4+p}}Z(3+p)+\frac{2kT}{Dk_F^2}\left[\frac{\pi^2}{3}\frac{\Gamma(3+p)}{2^{3+p}}\right.\right.$$

$$\left.\left.\times Z(2+p)-\frac{2}{3}\frac{\Gamma(5+p)}{2^{5+p}}Z(4+p)\right]\right\}, \qquad (41)$$

where $\Gamma$ and $Z$ are gamma and Riemann zeta functions, respectively, and

$$C_0=(1-A/G_{ss})^2,$$
$$C_1=3A/G_{ss}(1-A/G_{ss}),$$
$$C_2=A/4G_{ss}(-3+11A/G_{ss}), \qquad (42)$$
$$C_3=-(9/8)(A/G_{ss})^2,$$
$$C_4=(9/64)(A/G_{ss})^2.$$

In obtaining Eqs. (40) and (41) the zone boundary corrections were neglected and $Q$ replaced by $\infty$ as this is a good approximation below $T=300°K$.

In Fig. 3, $W_{mag}$ is sketched for various values of the parameter $A/G_{ss}\equiv\zeta$, using the physical parameters in Table I. The most important feature of these results is that the inclusion of $s$-$d$ admixtures ($\zeta>0$) depresses the thermal resistivity below the value obtained in the case $\zeta=0$ because of the effective antiferromagnetic coupling caused by these admixtures. Consequently, although $s$-$d$ admixture effects the necessary temperature dependence of $W_{mag}$ at low temperatures, viz., $W_{mag}\alpha T^{1.5}$ with $0.75<\zeta<1.0$, the value of $G_{ss}$ must be an order of magnitude larger than the estimate obtained in Sec. 3 on a two-band model if $W_{mag}$ is to account for the temperature dependence observed near $T=0.5\Theta_D$. Values of $G_{ss}=5.0$ eV and $\zeta>0.75$ must, however, be ruled out for the following reasons. Above $T_c$, $s$-$d$ admixtures enhance the effects of the $s$-$d$ exchaneg interaction on $\rho$ and a value of 5 eV for $G_{ss}$ would
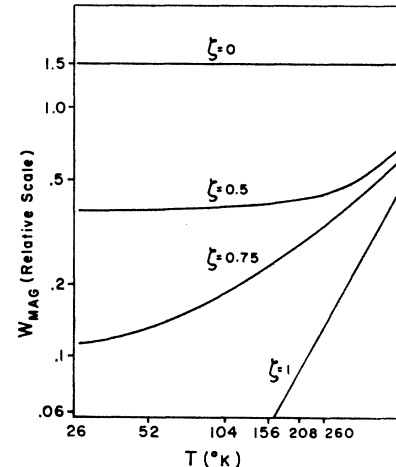


FIG. 3. Computed curves of $W_{mag}$ on a single-band model with the inclusion of $s$-$d$ admixtures for various values of the parameter $\zeta=A/J$, where $A$ and $J$ are the $s$-$d$ admixture and $s$-$d$ exchange parameters, respectively.

produce a spin-disorder electrical resistivity an order of magnitude larger than the total $\rho$ observed experimentally.[14] Moreover, in the lowest temperature region if one attributes the $T^2$ dependence of $\rho$ to $\rho_{mag}$, the parameter $G_{ss}$ of Kasuya's model must be replaced by $G_{ss}(1-A/G_{ss})$—call this$\bar{G}_{ss}$. As it was mentioned earlier, in order that $\rho_{Total} \approx \rho_{mag}$ at the lowest temperatures, $\bar{G}_{ss}$ must be 9 eV for Ni, while the estimates obtained above require

$$\bar{G}_{ss} = G_{ss}(1-\zeta) \approx 1 \text{ eV},$$

for $G_{ss} = 5$ eV and $\zeta = 0.7$.

While in the case $\zeta = 1$, Eqs. (40) and (42) show that $\rho_{mag}$ is proportional to $T^4$ at the lowest temperatures. Thus if $\zeta$ is in the range $0.75 < \zeta \le 1$, the $T^2$ dependence of $\rho_{Total}$ observed at the lowest temperatures becomes anomalous on the single-conduction-band model.

## ACKNOWLEDGMENTS

## APPENDIX A

In this section the spin-disorder thermal resistivity is computed on the single-band model of conduction electrons. Kasuya has obtained expressions for $\rho$ and the Lorentz number, $L$, due to spin-disorder scattering at low temperatures from which $W_{mag}$ may be obtained. Hence we take

$$\rho = \frac{3\pi}{8} \frac{m}{Ne^2} \frac{SG_{ss}^2}{\hbar E_F} \frac{q_0}{k_F} t^2 J_3(t^{-1}, x_0) \quad \text{(A1)}$$

and

$$L = \frac{k_B^2}{e^2} \frac{\pi^4}{9} \left(\frac{q_0}{k_F}\right)^2 t \left[\frac{J_3(t^{-1},x_0)}{J_2(t^{-1},x_0)} - \frac{J_2(t^{-1},x_0)}{J_1(t^{-1},x_0)}\right]^{-1}, \quad \text{(A2)}$$

where $q_0$ is the radius of a sphere whose volume is equal to the first Brillouin zone, $k_F$ the Fermi momentum and $E_F$ the Fermi energy of conduction electron, $Dq^2$ the energy of a spin wave of wave vector $q$, $t$ a reduced temperature $k_B T/Dq^2$, $x_0 = (1/t)[(H_c/E_F)(q_0/k_F)]^2$, $H_c$ the exchange energy of a conduction electron in the molecular field of the magnetic system, $N$ the number of atoms per unit volume, $S$ the spin of a magnetic ion, $G_{ss}$ the $s$-$d$ exchange parameter, and $J_n(t^{-1},x_0)$ the transport integrals defined by

$$J_n(t^{-1},x_0) = \int_{x_0}^{t^{-1}} \frac{x^n dx}{(e^x - 1)(1-e^{-x})}. \quad \text{(A3)}$$

The expression for the thermal resistivity is that obtained by using the definition of $L$ and thus,

$$W = \rho/LT. \quad \text{(A4)}$$

When Eqs. (A1) and (A2) are substituted into Eq. (A4) the expression for $W$ may at first appear formidable. However, in the temperature region of concern, viz., from 20 to 300°K, and using the parameter values appropriate to Ni it becomes quite tame. Using the values for the physical parameter from Table I we obtain

$$t^{-1} = 2280°\text{K}/T. \quad \text{(A5)}$$

Thus over the range of concern $t^{-1}$ is sufficiently larger than unity to be replaned by $\infty$ in the $J_n(t^{-1},x_0)$ functions. Similarly, if we follow Kasuya and set $H_c = SG_{ss}$ and take $G_{ss}$ to be approximately 0.5 eV and using table, we obtain

$$x_0 = 0.19°\text{K}/T, \quad \text{(A6)}$$

and $x_0$ is sufficiently less than unity to be replaced by zero in the $J_n$ functions. In this way it is found that $W_{mag} = 0.02$ cm deg/W for Ni.

## APPENDIX B: EVALUATION OF THE BASIC INTEGRAL $\mathcal{L}_{ss}$ FOR THE THERMAL RESISTIVITY

Using Eqs. (17) and (21) and the sums over $\mathbf{k}$ to integrals over the energy $E(\mathbf{k})$ and the solid angle $\Omega$, $L_{ss}$ may be written as

$$\mathcal{L}_{ss} = \left(\frac{\pi^2 s v k T}{N \hbar}\right) \sum_{\pm} \sum_q \int G_{ss}^2(0) \text{ csch}\left(\frac{E_q}{2kT}\right)$$

$$\times \delta(\mathbf{k}-\mathbf{k}'\pm\mathbf{q}) d\Omega d\Omega' \sum_{\pm} \int n_s(E) n_s(E')$$

$$\times \left(\frac{\partial f_0}{\partial E} \frac{\partial f_0}{\partial E'}\right)^{1/2} [\Phi(\mathbf{k}',s) - \Phi(\mathbf{k},s)]^2$$

$$\times \delta(E-E'\mp Eq) dE dE', \quad \text{(B1)}$$

where $n_s(E)$ is the density of states in the $s$ band and $\delta_{\mathbf{k}',\mathbf{k}\pm q}$ has been replaced by $(8\pi^3/v)\delta(\mathbf{k}-\mathbf{k}'\pm\mathbf{q})$. Integrating over $dE'$ restricts $E' = E \mp Eq$ and we may write

$$\left(\frac{\partial f_0}{\partial E} \frac{\partial f_0}{\partial E'}\right)^{1/2}_{E'=E\mp E_q} = \frac{1}{2kT}\left[\cosh\gamma + \cosh\frac{E_q}{2kT}\right]^{-1}, \quad \text{(B2)}$$

where

$$\gamma \equiv \frac{E - E_F}{kT} \mp \frac{E_q}{2kT}. \quad \text{(B3)}$$

It is convenient to define the function

$$F(\gamma, E_q) \equiv \frac{1}{2kT}\left[\cosh\gamma + \cosh\left(\frac{E_q}{2kT}\right)\right]^{-1} \quad \text{(B4)}$$

which has the following important properties: (a) It is even and peaked about $\gamma = 0$. (b) It has an approximate width of $E_q$ or $kT$, whichever is greater. It is reasonable to assume that $n_s(E)$ and $\Phi_s(E)$ are slowly varying with respect to $F[\gamma(E), E_q]$ in the neighborhood of $\gamma = 0$ and

may be approximated by a Taylor expansion about $E=E_F$ truncated after the second term. In this way the energy function in the integral of Eq. (B1) may be written as

$$n_s(E')n_s(E)[\Phi_s(E')-\Phi_s(E)]^2$$

$$=n_s{}^2\Phi_s{}^2\left[\tfrac14 E_q{}^2(\cos\theta'+\cos\theta)^2\pm\tfrac14 E_q{}^3\frac{\Phi_s{}'}{\Phi_s}\right.$$

$$\times(\cos^2\theta-\cos^2\theta')\Big]+\gamma^2k^2T^2\Big[(\cos\theta'-\cos\theta)^2$$

$$\pm E_q\frac{\Phi_s{}'}{\Phi_s}(\cos^2\theta-\cos^2\theta')\Big]+\text{terms odd in }\gamma$$

$$+\text{terms envolving products of }\Phi_s{}'\text{ and }n_s{}'.\quad(B5)$$

When the integration over $dE$ is performed, the terms in Eq. (B1) odd in $\gamma$ vanish and the terms with $\pm$ sign vanish after sum over spin directions. We further employ the approximation that terms envolving products of first derivation of $\Phi_s$ and $n_s$ are negligible with respect to the remaining terms. Thus Eq. (B1) reduces to

$$\mathcal{L}_{ss}=\frac{\pi^2v^2skT}{Nh}\sum_q G_{ss}{}^2(0)\,\mathrm{csch}\left(\frac{E_q}{2kT}\right)n_s{}^2\Phi_s{}^2\int F(\gamma,E_q)$$

$$\times[\tfrac14 E_q{}^2(I'+2J+I)+(\gamma kT)^2(J')]dE,\quad(B6)$$

where

$$I=-\int \cos^2\theta\delta(\mathbf{k}-\mathbf{k}'\mp\mathbf{q})d\Omega d\Omega'=\frac{2\pi}{3}\frac{1}{k_{F_s}{}^2}\frac{1}{q},\quad(B7)$$
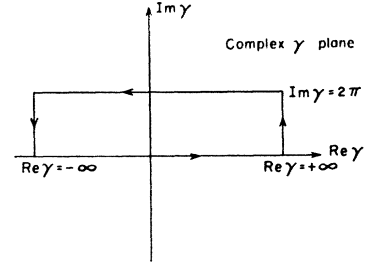
$$I'=-\int \cos^2\theta'\delta(\mathbf{k}-\mathbf{k}'\mp\mathbf{q})d\Omega d\Omega'=\frac{2\pi}{3}\frac{1}{k_{F_s}{}^2}\frac{1}{q},\quad(B8)$$

$$J=-\int \cos\theta\cos\theta'\delta(\mathbf{k}-\mathbf{k}'\mp\mathbf{q})d\Omega d\Omega'=\frac{\pi}{3}\frac{1}{k_{F_s}{}^2}$$

$$\times\left[-\frac{q}{k_{F_s}{}^2}+\frac{2}{q}\right],\quad(B9)$$

$$J'=-\int(\cos\theta'-\cos\theta)^2\delta(\mathbf{k}-\mathbf{k}'\mp\mathbf{q})d\Omega d\Omega'$$

$$=\frac{2\pi}{3}\frac{1}{k_{F_s}{}^4}q.\quad(B10)$$

Changing the variable $dE$ of integration in Eq. (B6) to



FIG. 4. Contour used for the evaluate of $\Gamma_m$ integrals.

$d\gamma$ leaves integrals of the type

$$\Gamma_m=\int_{-\infty}^{\infty}\frac{\gamma^m d\gamma}{[\cosh\gamma+\cosh E_q/2kT]},\quad(B11)$$

to be computed. These are easily evaluated by evaluating the integrals

$$\int_C\frac{\gamma^{m+2}d\gamma}{[\cosh\gamma+\cosh E_q/2kT]}$$

in the complex plane around the contour $C$ given in Fig. 4. In this way $\Gamma_0$ and $\Gamma_2$ are found to be given by

$$\Gamma_0=\frac{2(E_q/2kT)}{\sinh(E_q/2kT)},\quad(B12)$$

$$\Gamma_2=\frac{2}{3}\frac{(E_q/2kT)^3}{\sinh(E_q/2kT)}+\frac{\pi^2(E_q/2kT)}{\sinh(E_q/2kT)}.\quad(B13)$$

Substituting Eqs. (B12) and (B13) into Eq. (B6), converting sum over $q$ to integral and simplifying yields the result

$$\mathcal{L}_{ss}=\frac{\pi}{3}\frac{sv^2(kT)^3}{Nhk_{F_s}{}^4}G_{ss}{}^2(0)n_s{}^2\Phi_s{}^2$$

$$\times\left\{\int_0^Q\left(\frac{E_q}{2kT}\right)^3\mathrm{csch}^2\left(\frac{E_q}{2kT}\right)4k_{F_s}{}^2q\,dq\right.$$

$$+\frac{\pi^2}{3}\int_0^Q\left(\frac{E_q}{2kT}\right)\mathrm{csch}^2\left(\frac{E_q}{2kT}\right)q^3dq$$

$$\left.-\frac{2}{3}\int_0^Q\left(\frac{E_q}{2kT}\right)^3\mathrm{csch}^2\left(\frac{E_q}{2kT}\right)q\,dq\right\},$$

where $Q$ is the radius of a sphere whose volume is equal to the first Brillouin zone.