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Calculation of the Thermal Conductivity of Pure Superconducting Lead*

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A calculation of the thermal conductivity of normal and superconducting lead has been performed. A semiempirical model for the density of phonon states and the electron-phonon coupling constants is used. The calculation is based on the strong-coupling theory worked out by Tewordt and one of the authors (V.A.). It is found that the quasiparticle limit of this theory is valid for the calculation. The results are in fair agreement with experiment. A physical explanation for the anomalously large drop in the thermal conductivity of superconducting lead is suggested.

I. INTRODUCTION

THE thermal conductivity of pure superconducting lead has been considered anomalous for many years. The experimental situation is summarized in Fig. 1. For typical weak-coupling superconductors like tin and indium the curve of the reduced thermal conductivity (K_s/K_n) against reduced temperature (T/T_c) has a small limiting slope of about 1.5. For the strong-coupling superconductors lead and mercury the drop in the thermal conductivity is more precipitous. Recent experiments[1] on lead yield a limiting slope of about 9.

The strong-coupling superconductors are characterized by large electron-phonon matrix elements, and by peaks at low energies in the density of phonon states to which the electrons are coupled. Recently,[2] it has been possible to show unambiguously that these distinguishing characteristics are responsible for the anomalously large values in lead and mercury of the ratio of the energy gap at 0°K to the critical temperature, and for the anomalous thermodynamic properties of these metals. It has often been speculated that the smaller thermal conductivity of these strong-coupling superconductors is another consequence of their unusual electron-phonon interactions. However, just how this idea might explain the great reduction in thermal conductivity has heretofore been unclear.

In this work we report on a calculation, crude but containing all the essential physics, of the thermal conductivity of superconducting and normal lead. The results are in satisfactory agreement with experiment. Since the calculation is crude, the agreement is less important than the physical explanation, which we believe to be correct, that emerges from the calculation. This explanation is discussed below.

Our calculation is based on a theory worked out by L. Tewordt and one of the present authors.[3] The general theory is here supplemented by a specific model (discussed in the next section) for the phonon spectrum and electron-phonon coupling constants in lead. We find that, even near the critical temperature, long-lived particlelike excitations exist for the energies important in thermal conduction. In this quasiparticle limit the general formula [Eq. (2.17)] obtained in Ref. 3 reduces to[4]

$$K_s=\frac{A}{T^2}\int_{\Delta_1(\Delta_1,T)}^{\infty} d\omega\frac{\omega[\omega^2-\Delta_1^2(\omega,T)]^{1/2}}{Z_1\Gamma_s(\omega,T)}\operatorname{sech}^2\tfrac{1}{2}\beta\omega\,. \quad (1)$$

Above A is a constant discussed later, $\Delta_1(\omega,T)$ is the real part of the Eliashberg gap function,[3,5] T is the temperature, β is $1/k_BT$ where k_B is Boltzmann's constant, and $\Gamma_s(\omega,T)$ is the quasiparticle lifetime which is related to the parameters of the Eliashberg theory according to[3,6]

$$\omega Z_1\Gamma(\omega)=2Z_2(\omega^2-\Delta_1^2)-2\Delta_1\Delta_2Z_1\,. \quad (2)$$

Equation (1) has the same general form as is obtained from a phenomenological Boltzmann equation.[7] There is however an important difference inasmuch as the

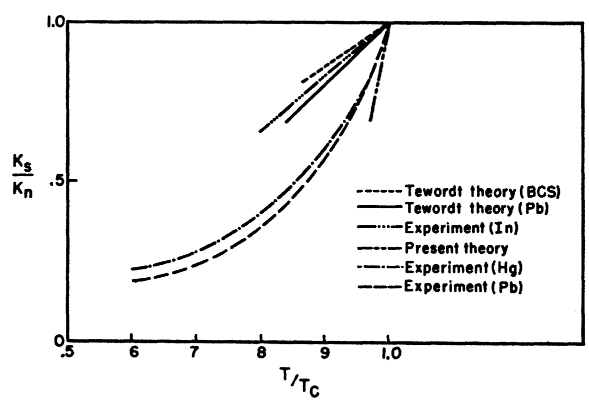

FIG. 1. Reduced conductivity versus reduced temperature. Experimental points for In from Tewordt (Ref. 6), that for Pb from Watson and Graham (Ref. 1).

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† Alfred P. Sloan Foundation Fellow.
[1] J. H. P. Watson and G. M. Graham, Can. J. Phys. **41**, 1738 (1963).
[2] J. C. Swihart, D. J. Scalapino, and Y. Wada, Phys. Rev. Letters **14**, 106 (1965).
[3] V. Ambegaokar and L. Tewordt, Phys. Rev. **134**, A805 (1964).
[4] We omit for the moment the effect of the "scattering in" terms discussed in Sec. 3 of Ref. 3 and in the Appendix of this paper.
[5] G. M. Eliashberg, Zh. Eksperim. i Teor. Fiz. **38**, 960 (1960) [English transl.: Soviet Phys.—JETP **9**, 1385 (1959)].
[6] L. Tewordt, Phys. Rev. **128**, 12 (1962).
[7] J. Bardeen, G. Rickayzen, and L. Tewordt, Phys. Rev. **129**, 657 (1963).

virtual effects of phonons in causing the superconducting transition have not been approximated by a model potential but treated on the same footing as the real transitions that scatter quasiparticles. As a result the large value of the energy gap (in units of k_BT_c) is in principle, and practice, contained in Eq. (1).

In order to bring out the physical origins of the large limiting slope of the reduced thermal conductivity of lead it is helpful to have a formal expression for this slope. Such an expression follows at once from (1). The conductivity in the normal state is obtained from this equation by setting Δ equal to zero. Forming the ratio K_s/K_n and taking the derivative at the critical temperature, one easily obtains

$$(t\equiv T/T_c;\ \bar{\Gamma}\equiv Z_1\Gamma)$$

$$\frac{d}{dt}\left(\frac{K_s}{K_n}\right)_{t=1}=-\tfrac{1}{2}\beta_c^2\frac{\partial\Delta_1^2}{\partial t}\bigg|_{t=1}\times\frac{\int_0^\infty d\omega\bar{\Gamma}^{-1}(\omega,T_c)\operatorname{sech}^2\tfrac{1}{2}\beta_c\omega}{\int_0^\infty d\omega(\beta_c\omega)^2\bar{\Gamma}^{-1}(\omega,T_c)\operatorname{sech}^2\tfrac{1}{2}\beta_c\omega}+\frac{\int_0^\infty d\omega\omega^2\bar{\Gamma}^{-1}(\omega,T_c)(\operatorname{sech}^2\tfrac{1}{2}\beta_c\omega)\partial/\partial t\ln[\bar{\Gamma}_n(\omega,T)/\bar{\Gamma}_s(\omega,T)]_{t=1}}{\int_0^\infty d\omega\omega^2\bar{\Gamma}^{-1}(\omega,T_c)\operatorname{sech}^2\tfrac{1}{2}\beta_c\omega},\quad(3)$$

where we have taken the temperature derivative of Δ_1^2 outside the integration because it is essentially constant in the relevant region of ω. The following three factors appear to be responsible for the large slope in lead as contrasted with weak coupling materials:

(1) The larger value of the ratio $2\Delta(0)/k_BT_c$ (4.3 for lead as opposed to 3.5 for materials well described by the BCS theory) has as its corollary a larger value of the slope $-\partial/\partial t(\beta_c\Delta_1)^2|_{t=1}$ (14.1 for lead, 9.4 for BCS). The more rapid opening up of the energy gap in lead means physically that the heat-carrying quasiparticles are more rapidly frozen out. This is the most obvious cause of the reduced thermal conductivity but we see that taken by itself it by no means suffices to explain the large effect.

(2) The quasiparticle lifetime (Γ^{-1}) decreases more rapidly with frequency in lead than in weak coupling materials. This effect has its origin in the small density of low frequency phonons in lead and is discussed further in the last section. Here it suffices to note that the ratio of integrals multiplying $\partial/\partial t(\Delta_1^2)$ in (3) is the larger the more rapidly Γ^{-1} decreases with frequency. In loose physical terms one can say that in all materials the advent of the energy gap suppresses the carriers that are most weakly damped, and are thus most efficient in carrying energy. In lead our calculations indicate that this suppression is particularly effective. For the ratio of integrals mentioned we find a value of about 1.1. For the model of Debye phonons and "jellium" matrix elements worked out by Tewordt[6] the ratio is about 0.5.

(3) The ratio $\Gamma_n(\omega,T)/\Gamma_s(\omega,T)$ decreases for lead when T decreases below T_c so that the last ratio of integrals in (3) is positive. The sign of this term appears to be connected with the coherence factors that go into a calculation of the relaxation rate for a quasiparticle in a superconductor. In our model the dominant relaxation process is one in which two quasiparticle excitations annihilate, emitting a phonon. This gives a positive sign. The other two kinds of processes (scattering of quasiparticles with phonon emission and absorption) give the opposite sign, as is discussed in more detail later. Our calculation gives for the term in question the value 3.5. Working backwards from the final slope obtained in Ref. 6 one can conclude that for the model used in this reference the ratio is negative and approximately -0.9.

Although no one of the three factors discussed above is large enough to account for the effect, taken together they change the slope (of 1.6) obtained in Tewordt's model (in which the virtual processes are accounted for by the BCS model and the real processes by a Debye spectrum of longitudinal phonons coupled to the electrons by "jellium" matrix elements) to the large value 11. As will be seen there is no reason to trust our model of lead to better than 20% so that the agreement must be regarded as satisfactory.

In the next section we briefly discuss and criticize the model we have used. The calculations are the topic of Sec. III. The last section contains some further discussion of the results. In the Appendix, corrections to the scattering rate due to the disequilibrium of other modes (so called "scattering in" corrections) are estimated and found to be small.

II. THE MODEL

The calculation has as its primary ingredients the frequency distributions of phonons of polarization $\lambda[F_\lambda(\omega)]$ and the coupling constants $\alpha_\lambda(\omega)$. We use, with no further adjustments, a model introduced recently by Swihart, Scalapino, and Wada.[2] This model is

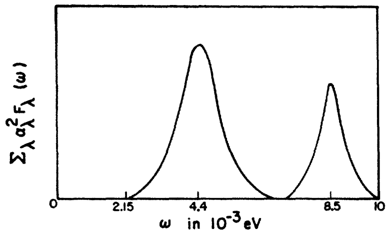


FIG. 2. Phonon spectrum used in present calculation.

a slight modification of that used by Schrieffer, Scalapino, and Wilkins.[8] The distribution functions are taken to be cut-off Lorentzians whose positions and widths are chosen semiempirically. (Details are given in the Appendix.) The coupling constants are taken to be independent of frequency and equal. The value is adjusted to fit the gap edge of 0°K in absolute units. This model has been shown in Ref. 8 to give excellent agreement with tunneling data. The critical temperature has been calculated in Ref. 2 by numerically solving the Eliashberg equations using the model for $F_\lambda(\omega)$ and $\beta_\lambda(\omega)$ and extrapolating the curve of $[\Delta_1(\Delta_1,T)]^2$ to zero. The critical temperature so calculated is in excellent agreement with experiment.

Figure 2 shows the phonon spectrum used in these calculations. We note that there are no longitudinal phonons below 7 millielectron volts (meV) and no phonons whatever below 2.15 meV. Although the model is extremely crude we believe it to have a certain validity even for our region of interest which is energies of a few times the critical temperature (in energy units 0.6 meV) of lead. McMillan and Rowell[9] have been able to invert the Eliashberg gap equations so that, using the electron density of states from tunneling, they can empirically determine $\sum_\lambda \alpha_\lambda^2(\omega)F_\lambda(\omega)$. They find that their result generally agrees with the model. In particular it is essentially zero for small ω. This is explained as follows.[9] Because of the large sound velocity of the longitudinal mode, the density of longitudinal phonons is smaller than that of transverse phonons by a factor of 30 for low frequencies. However, transverse phonons can interact with electrons only by means of umklapp processes. Since small q phonons cannot participate in umklapp processes, $\alpha_t^2(\omega)F_t(\omega)$ (t for transverse) is zero for small ω. The point of deviation of $\sum_\lambda \alpha_\lambda^2(\omega)F_\lambda(\omega)$ from zero may thus be interpreted as the frequency of onset of umklapp processes. McMillan's curve is of course not smooth. However for our purposes the additional details are not likely to be relevant. Only the average effect of high energy virtual phonons enters into the parameter Δ_1. The other important quantity, the width $\bar{\Gamma}$ does depend on the coupling to low frequency phonons. These have, however, not been resolved by McMillan's method, at least in the results so far published.

[8] J. R. Schrieffer, D. J. Scalapino, and J. W. Wilkins, Phys. Rev. Letters **10**, 336 (1963).

[9] W. L. McMillan and J. M. Rowell, Phys. Rev. Letters **14**, 108 (1965).

For the reasons given above we believe that the physical arguments given in the Introduction will survive a better model. When a better spectrum is available, further calculations to test this belief are planned.

III. CALCULATIONS

The distribution functions and coupling constants described above, together with a pseudopotential to simulate the screened Coulomb repulsion, have been used in Ref. 2 to calculate the functions $\Delta_1(\omega,T)$, $\Delta_2(\omega,T)$, $Z_1(\omega,T)$, $Z_2(\omega,T)$. Dr. J. C. Swihart has very kindly supplied us with tabulated values of these functions as well as with calculated values in the normal state of $Z_{1n}(\omega,T)$ and $Z_{2n}(\omega,T)$. We have used these functions for several values near the critical temperature to evaluate the quadratures [Eq. (2.17) of I]:

$$K(T)=-\frac{A}{T^2}\int_0^\infty d\omega g(\omega,T)\omega\,\mathrm{sech}^2\tfrac{1}{2}\beta\omega\,, \tag{4}$$

where

$$g_n(\omega,T)=-2\pi[Z_{2n}(\omega,T)]^{-1} \tag{5}$$

and

$$g_s(\omega,T)=-\pi\omega[\mathrm{Im}Z(\omega,T)[\omega^2-\Delta^2]^{1/2}]^{-1}\times(1+(\omega^2-|\Delta|)^2/|\omega^2-\Delta^2|)\,. \tag{6}$$

In the above, $\Delta=\Delta_1+i\Delta_2$ and $Z=Z_1+iZ_2$. The quantity A in (4), which cancels out of the reduced thermal conductivity but is needed for absolute comparisons with experiment, is given by

$$A=N(0)v_F^2/24\pi k_B\,, \tag{7}$$

with $N(0)$ the density of states at the Fermi surface for one spin, and v_F the Fermi velocity. In principle, band effects, as opposed to electron phonon effects, should be included in (7).

The result of the numerical calculations for the reduced conductivity is given in Fig. 1. As mentioned in the Introduction we obtain a limiting slope of 11.

It is interesting to calculate in absolute units the value of the thermal conductivity given by the present model. Using free electron values appropriate to the density of lead in (7) we obtain from (4) and (5) the value 9 W/cm °K. Recent theoretical work by Ashcroft and Wilkins[10] leads to a band mass $m^*=1.12m$ and a

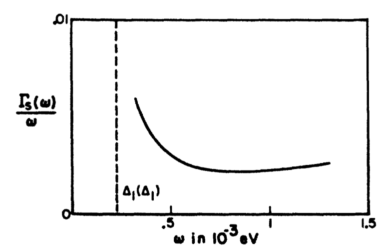


FIG. 3. $\Gamma(\omega)/\omega$ at $0.978T_c$.

[10] N. W. Ashcroft and J. W. Wilkins (to be published).

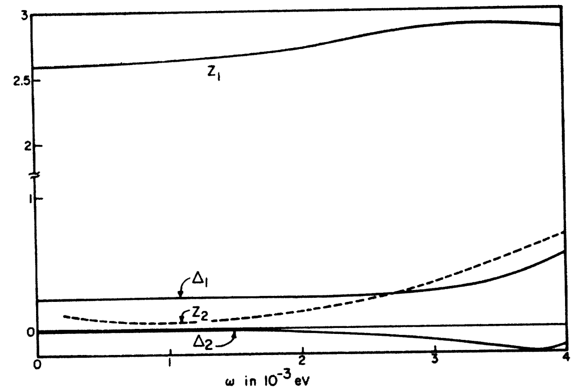


FIG. 4. $\Delta(\omega)$, $Z(\omega)$ at $0.989T_c$. Δ in 10^{-3} eV; Z dimensionless.

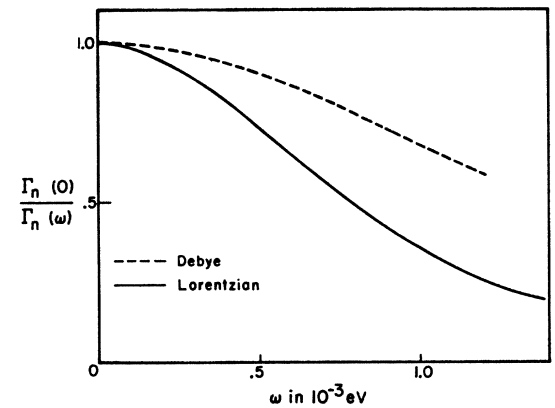
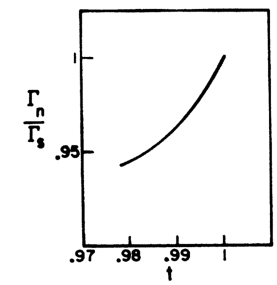


FIG. 6. Lifetime as function of ω for $T=0.989T_c$.

density of states 70% of the free value. Making corrections for these band effects we obtain a value of 5.6 W/cm °K. The experimental value is 5.1 W/cm °K. The degree of agreement is no doubt fortuitous.[11] What is relevant is that there is no disagreement as to orders of magnitude. It is also reasonable that the calculated conductivity should be too high, as some phonons which limit the conductivity have certainly been omitted.

For the superconductor the calculation was performed using the form (6). Because Z_2 and Δ_2 are small in the region of interest, Eqs. (4) and (6) are accurately approximated by Eqs. (1) and (2) as was verified by explicit calculation at one temperature. The goodness of the quasiparticle approximation is illustrated in Fig. 3. For purposes of reference a plot of the functions Δ_1, Δ_2, Z_1, and Z_2 at a temperature of $0.989T_c$ is given in Fig. 4. One feature that is worth noting is that Z^{-1}, which is the amplitude for the quasiparticle state to be contained in the bare state, is quite small. Thus even though the quasiparticles dominate the long time behavior which is relevant for the thermal conductivity, the renormalization effects are not small. These effects are of course automatically included in our calculation. In Fig. 5 the ω dependence of the ratio Γ_s/Γ_n at the temperature $T=0.989T_c$ is displayed. It is seen that the ratio is approximately constant and equal to unity. The absence of ω dependence of the ratio Γ_s/Γ_n near T_c was also a feature of Tewordt's calculation as is illustrated in the figure. An important difference [as was emphasized in the paragraph marked (2) in the Introduction] is the ω dependence of $[\Gamma(\omega)]^{-1}$. This is shown (again for $T=0.99T_c$) in Fig. 6. As can be seen this function drops off considerably more rapidly in our model than in the Debye model. The temperature dependence of

FIG. 7. Temperature dependence of Γ_n/Γ_s at $\omega=0.6\times10^{-3}$ eV.



the ratio $\bar{\Gamma}_n/\bar{\Gamma}_s$ is illustrated in Fig. 7. The positive slope near T_c was noted in the paragraph marked (3) in the Introduction.

IV. DISCUSSION

As has already been emphasized, the effect we are concerned with in this paper hinges on the frequency and temperature dependence of the relaxation rate of a quasiparticle. The relaxation processes have been extensively discussed by Tewordt. Figure 8 illustrates the possible processes. The processes marked 1 and 2 correspond to the scattering of a quasiparticle into another quasiparticle state with the emission or absorption of a phonon. As is well known, a sum of one-body operators can also cause two quasiparticles to annihilate,

FIG. 5. $\Gamma_s(\omega)/\Gamma_n(\omega)$ $0.989T_c$ for present spectrum and for Debye spectrum.

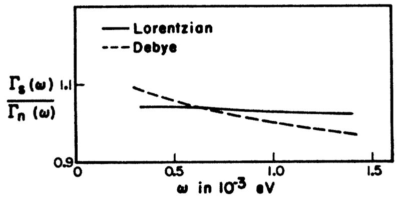
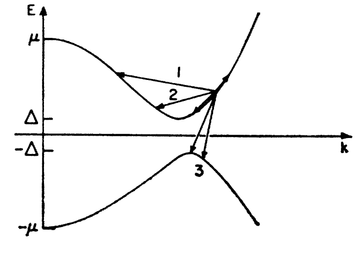


FIG. 8. Relaxation of quasiparticles by interaction with phonons. Processes 1 and 2 represent scattering of a quasiparticle into another by phonon emission or absorption. Process 3 represents annihilation of quasiparticles to form a ground state pair.



[11] The agreement can be made less embarassingly good by using the value determined semiempirically in Ref. 8 of $N(0) = 0.88N(0)_{\text{free}}$.

forming a ground state pair and emitting energy (in our case as a phonon). This process is represented by the arrows marked 3 in Fig. 8. (Here we have chosen to represent the initial second quasiparticle with energy E' as a final particle with energy $-E'$.) For our purposes we are interested in excitation energies of a few kT_c (i.e., in the neighborhood of 1 meV), and temperatures near T_c. At these temperatures very few phonons are thermally excited because the phonons have energies greater than 2 meV. Phonon absorption processes are therefore negligible. Phonon emission processes of the type 2 are also negligible because there are no phonons available with small enough energies. The dominant processes are thus of type 3. Let us first discuss the frequency dependence at the critical temperature. The analogue in the normal state of processes of type 3 are those in which particles above the Fermi surface are scattered into particle states below the Fermi surface. This process is governed by the availability of final states, i.e., we expect the frequency dependence roughly to correspond to a Fermi factor $f(\beta_c(\omega_0-\omega))$ where ω_0 is a characteristic phonon frequency and ω is the excitation energy. Taking $\omega_0\sim 2$ meV, $\beta_c\sim 0.5$ meV we see that the relaxation rate is expected to fall by one order of magnitude as ω is increased from zero to 1 meV. Reference to Fig. 5 shows that very roughly this is indeed so.

Now we consider the temperature dependence of the ratio Γ_s/Γ_n. Processes of type 3 have going with them the coherence factor $\frac{1}{2}[1-(\epsilon\epsilon'-\Delta^2)/EE']$ where $\epsilon=k^2/2m-\mu$ and $E=(\epsilon^2+\Delta^2)^{1/2}$ correspond to the decaying particle, and ϵ' (positive or negative) and $E'=+\epsilon'^2+\Delta^2$ to the second quasiparticle entering the process.[12] As a result of averaging over positive and negative ϵ' the coherence factor reduces[6] to $\frac{1}{2}(1+\Delta^2/EE')$. The temperature dependence of Γ_s/Γ_n is contained in distribution functions and in Δ. For the quantity $\partial/\partial t \ln[\Gamma_s/\Gamma_n]_{t=1}$, which enters the expression (3) for the limiting slope, only the dependence on Δ survives. Consider the effect of slightly reducing the temperature (or increasing Δ). Clearly the coherence factor increases. The quantity Δ also enters the expression for Γ_s through density of state factors and as the lower limit $(E+\Delta)$ for integration over phonon energies. Because the phonon energies are large on the scale of Δ these effects are unimportant. The increase in the coherence factor therefore implies an increase in Γ_s/Γ_n as T is decreased below T_c. This is in agreement with our detailed calculation. As mentioned in the introduction it appears that in the Debye model the opposite temperature dependence is obtained. In this connection it is probably relevant that the coherence factors for processes 1 and 2, after averaging over ϵ', are $\frac{1}{2}(1-\Delta^2/EE')$.

[12] For a physical discussion of the origin of the coherence factors, see J. R. Schrieffer, *Theory of Superconductivity* (W. A. Benjamin, Inc., New York, 1964), Secs. 3–5. In the quasiparticle limit of the strong coupling theory the coherence factors contain Z_1. [See Ref. 6, Eq. (3.5)]. The physical argument given above is however unchanged.

In summary, the two effects described above together with the rapid increase of Δ_1^2, as the temperature is reduced below the critical temperature, appear to us to be responsible for the large limiting slope of the reduced thermal conductivity of lead. Although our model contains these effects it has certain obvious weaknesses. We have seen that the absolute value of the thermal conductivity at the critical temperature has approximately the correct value. At lower temperatures, however, the complete absence of low-frequency phonons will result in the thermal conductivity of the normal state not approaching the T^{-2} increase of the Bloch theory but instead increasing exponentially. In spite of such defects, there is no reason to suppose that the basic conclusions of this paper will be altered by a more accurate calculation. Indeed the crudeness of our model has had as a compensating advantage the absence of any adjustable parameters.

In closing, we would like to make two general comments. The first concerns the Martin-Kadanoff theory[13] of the thermal conductivity of pure superconductors. In this theory, which had the great merit of being the first to give a positive limiting slope, the temperature and frequency dependence of the lifetimes in the normal and superconducting states are neglected and the two lifetimes are set equal. These *ad hoc* approximations greatly simplify the evaluation of the limiting slope. Indeed from 3 we see that the second term of (3) is zero and the first (assuming a BCS dependence for Δ) gives the simple result

$$\frac{d}{dt}(K_s/K_n)_{t=1}=\tfrac{1}{2}(8\pi^2/7\zeta(3))(3/2\pi^2)\sim 1.4.$$

This is in excellent general agreement with the weak coupling superconductors. It is unfortunate but true that this agreement is fortuitous. As we have seen even in the weak coupling limit, the frequency and temperature dependence are separately important though their effects tend to cancel. Much of the effect for the strong materials appears to come from the fact that this cancellation no longer occurs.

Finally, a comment on the theory[3] on which this calculation is based seems appropriate. This theory was motivated to some extent by the feeling that the quasiparticle approximation (in the sense of lifetimes being small compared to excitation energies) might break down for thermal conduction in lead. The present calculations indicate that no such breakdown occurs. However, the virtual effects of high-energy phonons are important for thermal conduction. These are consistently treated by the theory of Ref. 3 and not by a phenomenological Boltzmann equation.

ACKNOWLEDGMENTS

We are very grateful to Dr. J. C. Swihart for sending us his numerical solutions of the Eliashberg equations

[13] L. P. Kadanoff and P. C. Martin, Phys. Rev. **124**, 678 (1961).

and for calculating Z_{2n} for us. Several conversations with Professor J. W. Wilkins about band effects in lead are gratefully acknowledged. One of the authors (V.A.) would like to thank the Research Institute for Theoretical Physics of the University of Helsinki for hospitality during the last stages of this work.

APPENDIX

We give here an estimate of the size of the "scattering in" contributions arising from electron-phonon scattering. The terms arising from electron-electron scattering have not been considered; however, we believe them to be small. We need to know the electron-phonon coupling matrix element and the phonon spectrum. In the calculations, these were the same as those used by Swihart, Scalapino, and Wada.[2] The coupling matrix element was represented by a constant, $\alpha^2=1.264$ and the phonon spectrum was given by

$$F_\lambda(\omega)=A_\lambda\left[\frac{1}{(\omega-\omega_1{}^\lambda)^2+(\omega_2{}^\lambda)^2}-\frac{1}{(\omega_3{}^\lambda)^2+(\omega_2{}^\lambda)^2}\right]; \quad |\omega-\omega_1{}^\lambda|<\omega_3{}^\lambda$$
$$=0; \quad |\omega-\omega_1{}^\lambda|>\omega_c{}^\lambda \quad \text{(A1)}$$

where A_λ normalized F_λ to unit, and ω^λ in millielectron volts are $\omega_1{}^t=4.4$, $\omega_2{}^t=0.75$, $\omega_1{}^l=8.5$, $\omega_2{}^l=0.5$, $\omega_3{}^\lambda=3\omega_2{}^\lambda$. These were chosen so as to give the correct gap at $T=0$ and to give agreement with results of tunneling experiments.

The total contribution from both scattering out and scattering in terms is given by [Eq. (3.30) of I].

$$g(\omega)=g_0(\omega)\left[1-\frac{\alpha^2}{2\omega f(-\omega)}\sum_\lambda\int d\omega_\lambda F(\omega_\lambda)\times\{f(\omega_\lambda-\omega)(1+N\omega_\lambda)g(\omega-\omega_\lambda)+f(-\omega_\lambda-\omega)N(\omega_\lambda)g(\omega+\omega_\lambda)\}\right], \quad \text{(A2)}$$

where g_0 is g_n or g_s of Eq. (5) or (6). Let us consider at present only the normal state. We note that $g_0(\omega)$ is odd in ω. The calculation shows that for $|\omega|\gtrsim 0.7$, it decreases monotonically with ω. For $T=0.59\times10^{-3}$ eV, $g_0(1.25)\simeq-92.5$, $g_0(2)\simeq-45.2$, and $g_0(4)=-9.8$. The arguments of g_0 are in units of 10^{-3} eV. For the energies of importance to thermal conduction, the Lorentzian centered at $\omega=8.5$, contributes negligibly to the solution of (A2). Thus, the contribution from longitudinal phonons will be neglected. If we assume that $g(\omega)$ is not very different from $g_0(\omega)$, (i.e. $|[g(\omega)]/[g_0(\omega)]|\lesssim 1.5$), we see that the second term in integrand of (A2) is small compared with the first term in the integrand. Hence, we neglect it. Now, we approximate $F(\omega)$ by $\tilde{F}(\omega)$:

$$\tilde{F}(\omega_\lambda)=(2/2.3)[0.125\delta(3-\omega_\lambda)+0.5\delta(3.75-\omega_\lambda)+\delta(4.4-\omega_\lambda)+0.5\delta(5.05-\omega_\lambda)+0.125\delta(5.8-\omega_\lambda)], \quad \text{(A3)}$$

where the arguments are all measured in 10^{-3} eV. (A2) can now be written

$$\xi(\omega)=\xi_0(\omega)[1+\textstyle\sum_i A_i(\omega)\xi(\omega_i-\omega)]\equiv g(\omega)f(\omega), \quad \text{(A4)}$$

where $\xi_0(\omega)=g_0(\omega)f(\omega)$,

$$A_1(\omega)=\frac{\alpha^2(2/2.3)(0.125)}{2\omega f(-\omega)},$$

$\omega_1=3$, $\omega_2=3.75$, etc. The main correction to ξ comes from $A_1(\omega)\xi(\omega_1-\omega)$ since

$$\frac{\xi(\omega_2-\omega)}{\xi(\omega_1-\omega)}\lesssim\frac{1}{7}$$

for $\omega=1$ and less for $\omega<1$ assuming $|g(\omega)/g_0(\omega)|\lesssim 1.5$. The $i=3$, 4, 5 terms are smaller than $i=2$ term by an order of magnitude and so will be neglected. The $i=2$ term will be treated as a perturbation. (A3) can now be easily solved. To first order in $A_2(\omega)\xi_0(\omega_2-\omega)$,

$$\xi(\omega)=\xi_0(\omega)\left\{\frac{1+A_1(\omega)\xi_0(\omega_1-\omega)}{1-A_1(\omega)\xi_0(\omega_1-\omega)A_1(\omega_1-\omega)\xi_0(\omega)}+A_2(\omega)\xi_0(\omega_2-\omega)\right\}. \quad \text{(A5)}$$

For $\omega=1$, we find $\xi(1)\simeq0.96\xi_0(1)$ while $\xi(0.5)\simeq0.98\xi_0(0.5)$. In both cases, $A_2(\omega)\xi_0(\omega_2-\omega)\simeq-0.01$. Thus, we find that for the important energies $[\xi(\omega_1)]/[\xi(\omega_2)]\simeq[\xi_0(\omega_1)]/[\xi_0(\omega_2)]$. Now, (A1) may be solved by iteration. $g_0(\omega')$ never appears in this iteration series for $|\omega'|<10^{-3}$ eV for $\omega<1.25\cdot10^{-3}$ eV. $g_0(\omega)$ for such values of ω is approximately the same for superconducting and normal phases. Hence, we can also conclude that $[\xi_n(\omega)]/[\xi_s(\omega)]=[\xi_{n0}(\omega)]/[\xi_{s0}(\omega)]$ for the important frequencies. Thus, we see that the correction to K_s and K_n from the scattering in term is quite small.