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## Correlation Functions for a Heisenberg Ferromagnet\*

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It is shown that the ac longitudinal susceptibility of a Heisenberg ferromagnet is proportional to the Fourier transform of a dynamical  $zz$  correlation function of the spins,  $z$  being the direction of the spontaneous magnetization. The susceptibility can also be calculated by the Tyablikov decoupling approximation. Hence, the dynamical  $zz$  correlation function is obtained by taking the inverse Fourier transform of the latter result. This expression for the correlation function is an interpolation formula which agrees with the noninteracting spin-wave theory at very low temperatures and with the statistical theory at very high temperatures. The applications of these correlation functions to specific heat, susceptibility, and electrical conductivity are also discussed.

## I. INTRODUCTION

IT has been very fruitful to study the properties of ferromagnetic materials in terms of the Heisenberg model. The model consists of a regular array of localized spins that are coupled together by the so-called exchange interaction. In practice, the exchange coupling parameters are regarded as phenomenological constants whose values are determined by fitting the theory with a set of experimental results. Then one can use the model to explain other experimental results and thereby correlate a large number of related phenomena. It is clear that we must understand the statistical dynamics of the Heisenberg model in order to complete this program. This has not been entirely successful because of the mathematical difficulties in solving the model.

Considerable progress has been made toward approximate solutions of the model at high- and low-temperature regions. At temperatures very low compared with the Curie point, the normal modes of the spin system are of wave character. The dynamics of the model may be studied in terms of the spin-wave approximation.<sup>1-3</sup> At temperatures high above the Curie point, there is no long-range order and each spin interacts comparatively strongly only with its nearest

neighbors. Its interaction with other spins is progressively weaker when the latter are farther away. In this case, one can expand the partition function or the free energy into a power series of the inverse temperature.<sup>4,5</sup> One can then analyze the properties of the model in the paramagnetic temperature region. The Curie point corresponds to the temperature at which the power series for the susceptibility diverges. In recent years, numerical techniques have been developed to extrapolate the power series to the region near its circle of convergence.<sup>6-8</sup> This makes it possible to understand a number of critical phenomena that occur just above the Curie point.

At temperatures lower than but not negligibly small compared with the Curie point, both the above-mentioned methods of analysis fail. In this case, the Weiss molecular-field approximation<sup>9</sup> or the more refined Bethe-Peierl-Weiss<sup>10</sup> (BPW) approximation is useful. These methods ignore partly or wholly the correlation between the transverse components of the spins, and consequently, disagree with the spin-wave approximation at low temperatures. Above the Curie point, the Weiss approximation gives no short-range order.

<sup>4</sup> W. Opechowski, *Physica* **4**, 181 (1937).

<sup>5</sup> G. S. Rushbrooke and P. J. Wood, *Proc. Phys. Soc. (London)* **A70**, 765 (1957); *Mod. Phys.* **1**, 257 (1958).

<sup>6</sup> C. Domb and M. F. Sykes, *J. Math. Phys.* **2**, 63 (1961); *Phys. Rev.* **128**, 168 (1962).

<sup>7</sup> G. A. Baker, *Phys. Rev.* **124**, 768 (1961); **129**, 99 (1963).

<sup>8</sup> J. Gammel, W. Marshall, and L. Morgan, *Proc. Roy. Soc. (London)* **A275**, 257 (1963).

<sup>9</sup> P. Weiss, *J. Phys.* **6**, 667 (1907).

<sup>10</sup> P. R. Weiss, *Phys. Rev.* **74**, 1493 (1948).

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<sup>1</sup> T. Holstein and H. Primakoff, *Phys. Rev.* **58**, 1098 (1940); see also C. Kittel, *Quantum Theory of Solids* (John Wiley & Sons, Inc., New York, 1964), Chap. 4.

<sup>2</sup> F. J. Dyson, *Phys. Rev.* **102**, 1217 (1956).

<sup>3</sup> F. Keffer and R. Loudon, *J. Appl. Phys.* **32**, 2S (1961).

In 1959, Tyablikov<sup>11</sup> gave an interpolation theory derived from a study of two-time Green's functions for the transverse components of pairs of spins. The equation of motion of these Green's functions involve the Green's functions of three spins. If one makes a decoupling approximation to these higher order functions, one obtains a closed system of equations from which the two-spin Green's functions may be solved. A number of interesting results can be extracted from these approximate Green's functions, for instance, the magnetization and susceptibility as functions of the temperature, the transverse correlation functions of the spins, and the Curie temperature in terms of the coupling constants. An important feature of this theory is that it agrees with the noninteracting spin-wave theory at very low temperatures and with the statistical theory at very high temperatures. The predicted Curie point is very close to the other theories.<sup>6,10,12,13</sup> Although the original version of the theory was for spin  $\frac{1}{2}$  only, it has been extended to general integral or half-odd integral spins.<sup>13-16</sup> Other decoupling schemes have also been derived so that the results agree with the Dyson theory of spin waves at low temperatures.<sup>13,17</sup>

Since this theory agrees with the limiting theories at both temperature limits, there is hope that the results may not be far from the true solution of the Hamiltonian over the entire temperature range. This is clearly an improvement over the molecular field or BPW theories because the transverse correlation between the spins is taken into account. Nevertheless, it has not been possible to make a complete study of the Heisenberg model in this way because an essential quantity, the longitudinal or  $zz$  correlation function of the spins, does not follow directly from the theory,  $z$  being the direction of the magnetization. Kawasaki and Mori<sup>17,18</sup> succeeded in relating the  $zz$  correlation function above the Curie point with the static, wavelength-dependent susceptibility of the system. Tahir-Kheli and Callen<sup>19</sup> proposed a different form for the function in the entire temperature range. Both theories are incomplete because the former is not applicable below the Curie point, and the

latter is incorrect above the Curie point in predicting that the  $zz$  correlation is different from the  $xx$  or  $yy$  correlation for an isotropic model.

In this paper, we show that the wavelength and frequency-dependent susceptibility of the system in the direction of the spontaneous magnetization is proportional to the Fourier transform of a two-time  $zz$  correlation function of the spins. On the other hand, in analogy with Kawasaki and Mori, the ac susceptibility is solved by the Tyablikov decoupling approximation. Hence, one obtains expressions for the longitudinal dynamical correlation functions by taking the inverse Fourier transform of the latter result. The final expressions are shown to be interpolation formulas which agree with the high- and low-temperature theories. Thus, we have completed the task of calculating all the correlation functions for a Heisenberg ferromagnet in the Tyablikov approximation. The same technique may be used for the longitudinal correlation function in other decoupling schemes. In separate sections, we discuss the application of these correlation functions in the study of the thermodynamic and transport properties of Heisenberg ferromagnets.

## II. LONGITUDINAL SUSCEPTIBILITY

The Hamiltonian for the Heisenberg ferromagnet may be written as

$$H = -h \sum_i S_i^z - \sum_{i,j} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j, \quad (1)$$

where  $i, j$  label the locations of the spins  $\mathbf{S}_i, \mathbf{S}_j$  on a crystal lattice,  $J_{ij}$  are the exchange coupling constants,  $h$  is an external static field expressed in energy units. The spontaneous magnetization of the system lies in the positive  $z$  direction. We adopt the unit system such that  $\hbar=1$ . The longitudinal susceptibility is defined as the linear response to a small external field in the  $z$  direction. With a field of wave vector  $\mathbf{q}$  and frequency  $\omega$  applied on the system, the total Hamiltonian becomes

$$H' = H + H_1 e^{-i\omega t}, \quad (2)$$

where

$$H_1 = -f \sum_i S_i^z e^{i\mathbf{q} \cdot \mathbf{R}_i} = -f N S_{-q}^z, \quad (3)$$

and  $f$  is a small parameter. The Fourier transform of the spin operators is defined by

$$\mathbf{S}_q = (1/N) \sum_i \mathbf{S}_i e^{-i\mathbf{q} \cdot \mathbf{R}_i}, \quad (4)$$

and  $\mathbf{R}_i$  is the position vector of the  $i$ th lattice point. We study the response of the system by calculating the expectation value of the operator  $S_j^z$ , with  $j$  an arbitrary lattice point. By definition,

$$\langle S_j^z \rangle = \text{Tr}(\rho' S_j^z), \quad (5)$$

where  $\rho'$  is the density operator whose equation of

<sup>11</sup> N. N. Bogolyubov and S. V. Tyablikov, Dokl. Akad. Nauk SSSR, **126**, 53 (1959) [English transl.: Soviet Phys.—Doklady **4**, 604 (1959)]; see also V. L. Bonch-Bruевич and S. V. Tyablikov, *The Green Function Method in Statistical Mechanics* (North-Holland Publishing Company, Amsterdam, 1962), Chap. 7; D. N. Zubarev, Usp. Fiz. Nauk SSSR, **71**, 71 (1960) [English transl.: Soviet Phys.—Usp. **3**, 320 (1960)].

<sup>12</sup> H. A. Brown and J. M. Luttinger, Phys. Rev. **100**, 685 (1955).  
<sup>13</sup> For a comparison of these results, see H. B. Callen, Phys. Rev. **130**, 890 (1963).

<sup>14</sup> Yu. A. Izyumov and E. N. Yakovlev, Fiz. Met. i Metalloved **9**, 667 (1960).

<sup>15</sup> K. Kawasaki and H. Mori, Progr. Theoret. Phys. (Kyoto) **25**, 1045 (1961).

<sup>16</sup> R. A. Tahir-Kheli and D. ter Haar, Phys. Rev. **127**, 88 (1962).

<sup>17</sup> H. Mori and K. Kawasaki, Progr. Theoret. Phys. (Kyoto) **28**, 690 (1962).

<sup>18</sup> K. Kawasaki and H. Mori, Progr. Theoret. Phys. (Kyoto) **25**, 1043 (1961).

<sup>19</sup> R. A. Tahir-Kheli and H. B. Callen, J. Appl. Phys. **35**, 956 (1964).

motion is

$$i(d\rho'/dt)=[H',\rho']. \quad (6)$$

We regard the field as being turned on adiabatically so that the boundary condition is  $\rho'(-\infty)=\rho$ , and

$$\rho=e^{-\beta H}/\text{Tr}e^{-\beta H}. \quad (7)$$

It was shown by Kubo<sup>20</sup> that the above equation has an iterative solution

$$\rho'=\rho+\rho_1+O(f^2), \quad (8)$$

with

$$\rho_1=iNf\int_{-\infty}^t[S_{-q^z}(t'-t),\rho]e^{-i\omega t'}dt', \quad (9)$$

and  $S_{q^z}(t)=e^{iHt}S_{q^z}e^{-iHt}$ . If we expand the expectation value in Eq. (5) into a power series of  $f$ , we have

$$\langle S_j^z \rangle = \sigma + f\sigma_j^{(1)}(\omega)e^{-i\omega t} + O(f^2), \quad (10)$$

where  $\sigma=\text{Tr}(\rho S_j^z)$  is the spontaneous magnetization per spin in absence of the ac field, and

$$\sigma_j^{(1)}(\omega)=-iN\int_{-\infty}^0\langle[S_{-q^z}(t),S_j^z(0)]\rangle e^{-i\omega t}dt. \quad (11)$$

It will be convenient to define the Fourier components of  $\sigma_j^{(1)}(\omega)$ :

$$\sigma_q^{(1)}(\omega)=(1/N)\sum_j\sigma_j^{(1)}(\omega)e^{-i\mathbf{q}\cdot\mathbf{R}_j}. \quad (12)$$

The quantity  $\sigma_q^{(1)}(\omega)$  has the same expression as  $\sigma_j^{(1)}(\omega)$  except that  $S_j^z$  is replaced by  $S_{q^z}$ .

The two-spin correlation function in Eq. (11) has a spectral representation

$$\langle[S_{-q^z}(t)S_{q^z}(0)]\rangle=\int\frac{d\omega'}{2\pi}J_z(\mathbf{q},\omega')e^{-i\omega't}, \quad (13)$$

where

$$J_z(\mathbf{q},\omega')=2\pi\sum_{mn}\rho_m\langle m|S_{-q^z}|n\rangle\langle n|S_{q^z}|m\rangle\times\delta(E_m-E_n+\omega'), \quad (14)$$

and the sum is taken over the complete set of eigenstates  $|m\rangle, |n\rangle$  of  $H$ . It is then easy to show that

$$\langle[S_{-q^z}(t),S_{q^z}(0)]\rangle=\int\frac{d\omega'}{2\pi}J_z(\mathbf{q},\omega')(1-e^{-\beta\omega'})e^{-i\omega't},$$

and

$$\sigma_q^{(1)}(\omega)=N\int\frac{d\omega'}{2\pi}J_z(\mathbf{q},\omega')\frac{1-e^{-\beta\omega'}}{\omega'+\omega+i\delta}, \quad \delta=0^+, \quad (15)$$

after a simple integration. In later calculations, it is more convenient to work in the imaginary time formal-

ism. We define a Green's function<sup>21</sup>

$$G_q^{(z)}(\tau)=\langle TS_{-q^z}(\tau)S_{q^z}(0)\rangle, \quad -\beta\leq\tau\leq\beta, \quad (16)$$

where  $S_{-q^z}(\tau)=e^{\tau H}S_{-q^z}e^{-\tau H}$  and  $T$  is the  $\tau$ -ordering operator; then we can extend the definition of  $G_q^{(z)}(\tau)$  outside the range  $(-\beta,\beta)$  as a periodic function of period  $2\beta$ . For  $\beta>\tau>0$ , we can easily show that

$$G_q^{(z)}(\tau-\beta)=G_q^{(z)}(\tau).$$

This enables us to expand

$$G_q^{(z)}(\tau)=(1/\beta)\sum_n\mathcal{G}^{(z)}(\mathbf{q},\omega_n)e^{-i\omega_n\tau}, \quad (17)$$

where  $\omega_n=2n\pi/\beta$ . We can also represent  $\mathcal{G}^{(z)}(\mathbf{q},\omega_n)$  in terms of the spectral representation

$$\mathcal{G}^{(z)}(\mathbf{q},\omega_n)=\int\frac{d\omega'}{2\pi}J_z(\mathbf{q},\omega')\frac{1-e^{-\beta\omega'}}{\omega'-i\omega_n}, \quad (18)$$

where  $J_z(\mathbf{q},\omega')$  was given in Eq. (14). If we define

$$s_q^{(1)}(\omega_n)=N\mathcal{G}^{(z)}(\mathbf{q},\omega_n), \quad (19)$$

then, by comparison with Eq. (15), we find

$$\sigma_q^{(1)}(\omega)=s_q^{(1)}(i\omega-\delta). \quad (20)$$

Hence  $\sigma_q^{(1)}(\omega)$  and  $s_q^{(1)}(\omega_n)$  are related by analytical continuation. In the next section, we shall calculate  $s_q^{(1)}(\omega_n)$  by the Tyablikov approximation. Then, by Eqs. (18) and (19), one can show that

$$J_z(\mathbf{q},\omega)=\frac{1}{N}\frac{s_q^{(1)}(-i\omega+\delta)-s_q^{(1)}(-i\omega-\delta)}{1-e^{-\beta\omega}}. \quad (21)$$

Putting this result in Eq. (13), we obtain the dynamical  $zz$  correlation function of the spins.

### III. TYABLIKOV APPROXIMATION

We calculate in this section the longitudinal susceptibility by the Tyablikov approximation. For simplicity, we exhibit the procedure for spin- $\frac{1}{2}$  case only. The general spin problem is briefly sketched and the results displayed at the end of this section.

We consider a Green's function

$$G_{ij}(t)=i\langle TS_i^+(t)S_j^-(0)\rangle, \quad (22)$$

where  $T$  is the time ordering operator of Dyson. The Hamiltonian of the system is  $H'$  in Eq. (2). Since  $H'$  is time dependent, we must be more careful in defining the thermal average. Consider the time dependent part of  $H'$  as being turned on adiabatically, then at  $t=-\infty$  the system has the Hamiltonian  $H$ . The subsequent motion is described by the time development operator  $U(t,t')$

<sup>21</sup> See A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963), Chap. 3.

<sup>20</sup> R. Kubo, J. Phys. Soc. Japan **12**, 570 (1957).

which satisfied the equation of motion

$$i(\partial/\partial t)U(t,t')=H'(t)U(t,t') \quad (23)$$

and the boundary condition

$$U(t,t')=1. \quad (24)$$

In terms of  $U$  the Green's function is defined as

$$\begin{aligned} G_{ij}(t) &= i \operatorname{Tr}[\rho(-\infty)U(-\infty, t) \\ &\quad \times S_i^+ U(t, 0) S_j^- U(0, -\infty)], \quad t > 0 \\ &= i \operatorname{Tr}[\rho(-\infty)U(-\infty, 0) \\ &\quad \times S_j^- U(0, t) S_i^+ U(t, -\infty)], \quad t < 0, \end{aligned} \quad (25)$$

where  $\rho(-\infty)$  is the density operator at  $t = -\infty$  and is identical to the  $\rho$  in Eq. (7). It is straight forward to derive the equation of motion of  $G_{ij}(t)$  by use of the properties of  $U$  as given in Eqs. (23) and (24). The equation is

$$i(d/dt)G_{ij}(t) = 2\delta_{ij}\sigma_i(t)\delta(t) - i\langle T[S_i^+(t), H'(t)]S_j^-(0) \rangle, \quad (26)$$

where

$$\sigma_i(t) = \operatorname{Tr}[\rho(-\infty)U(-\infty, t)S_i^z U(t, -\infty)].$$

This quantity appeared before in Eq. (5) because it can be readily verified that  $\rho' = U(t, -\infty)\rho(-\infty)U(-\infty, t)$ . The second term in Eq. (26) may be explicitly evaluated; the result is

$$\begin{aligned} (h + f e^{i\mathbf{q}\cdot\mathbf{R}_i - i\omega t})G_{ij}(t) \\ + 2 \sum_k J_{ik} \{ i \langle T S_i^z(t) S_k^+(t) S_j^-(0) \rangle \\ - i \langle T S_k^z(t) S_i^+(t) S_j^-(0) \rangle \}. \end{aligned} \quad (27)$$

After the decoupling approximation, Eq. (26) reduces to

$$i(d/dt)G_{ij}(t) = 2\delta_{ij}\sigma_i(t)\delta(t) + (h + f e^{i\mathbf{q}\cdot\mathbf{R}_i - i\omega t})G_{ij}(t) + 2 \sum_k J_{ik} \{ \sigma_k(t)G_{ij}(t) - \sigma_i(t)G_{kj}(t) \}. \quad (28)$$

We now make analytical continuation to the imaginary time domain by replacing  $t$  by  $-i\tau$ ,  $\omega$  by  $i\omega_n$ ,  $G_{ij}(t)$  by  $-i\mathcal{G}_{ij}(\tau)$ , and  $\sigma_i(t)$  by  $s_i(\tau)$ . The quantity  $\omega_n = 2\pi n/\beta$ , and  $n$  is an integer. This gives the following equation of motion for  $\mathcal{G}_{ij}(\tau)$ :

$$\begin{aligned} (d/d\tau)\mathcal{G}_{ij}(\tau) &= 2\delta_{ij}s_i(\tau)\delta(\tau) - (h + f e^{i\mathbf{q}\cdot\mathbf{R}_i - i\omega_n\tau})\mathcal{G}_{ij}(\tau) \\ &\quad - 2 \sum_k J_{ik} \{ s_k(\tau)\mathcal{G}_{ij}(\tau) - s_i(\tau)\mathcal{G}_{kj}(\tau) \}. \end{aligned} \quad (29)$$

We solve this equation by perturbation. If we expand all the quantities in power series of  $f$ , we may write

$$\mathcal{G}_{ij}(\tau) = \mathcal{G}_{ij}^{(0)}(\tau) + f\mathcal{G}_{ij}^{(1)}(\tau) + O(f^2). \quad (30)$$

For  $s_i(\tau)$ , we analytically continue Eq. (10) to obtain

$$s_i(\tau) = \sigma + f s_i^{(1)}(\omega_n) e^{-i\omega_n\tau} + O(f^2). \quad (31)$$

Putting these results into Eq. (29) and collecting terms

independent of  $f$ , we find

$$\begin{aligned} \frac{d}{d\tau}\mathcal{G}_{ij}^{(0)}(\tau) &= 2\delta_{ij}\sigma\delta(\tau) - h\mathcal{G}_{ij}^{(0)}(\tau) \\ &\quad - 2 \sum_k J_{ik}\sigma[\mathcal{G}_{ij}^{(0)}(\tau) - \mathcal{G}_{kj}^{(0)}(\tau)]. \end{aligned} \quad (32)$$

This is just the Tyablikov equation of motion in the imaginary time representation. Equation (32) may be solved by making space and time Fourier expansions, i.e.,

$$\mathcal{G}_{ij}^{(0)}(\tau) = (1/N) \sum_{\mathbf{q}} (1/\beta) \sum_m \mathcal{G}^{(0)}(\mathbf{q}, \omega_m) e^{i\mathbf{q}\cdot\mathbf{R}_i - i\omega_m\tau}; \quad (33)$$

$N$  is the total number of spins. The solution for  $\mathcal{G}^{(0)}(\mathbf{q}, \omega_m)$  is found to be

$$\mathcal{G}^{(0)}(\mathbf{q}, \omega_m) = 2\sigma/(\omega_q - i\omega_m), \quad (34)$$

where

$$\omega_q = h + 2\sigma J(0, \mathbf{q}), \quad (35)$$

and

$$J(\mathbf{p}, \mathbf{q}) = \sum_k J_{ik} [e^{-i\mathbf{p}\cdot\mathbf{R}_{ik}} - e^{-i\mathbf{q}\cdot\mathbf{R}_{ik}}]. \quad (36)$$

As shown in standard treatments,  $\sigma$  is found by the following self-consistency conditions:

$$\langle S_i^- S_i^+ \rangle_{f=0} = \frac{1}{2} - \sigma,$$

and

$$\begin{aligned} \langle S_i^- S_i^+ \rangle_{f=0} &= (1/N\beta) \sum_m \sum_{\mathbf{q}} \mathcal{G}^{(0)}(\mathbf{q}, \omega_m) e^{-i\omega_m\delta} \\ &= (2\sigma/N) \sum_{\mathbf{q}} N(\omega_q), \end{aligned} \quad (37)$$

where  $\delta = 0^+$ ,  $N(\omega_q) = [e^{\beta\omega_q} - 1]^{-1}$ . Hence, the equation for  $\sigma$  is

$$\frac{1}{2\sigma} = \frac{1}{N} \sum_{\mathbf{q}} \coth \frac{\beta\omega_q}{2}. \quad (38)$$

If we collect terms proportional to  $f$ , we obtain

$$\begin{aligned} (d/d\tau)\mathcal{G}_{ij}^{(1)}(\tau) &= 2\delta_{ij}s_i^{(1)}(\tau)\delta(\tau) - e^{i\mathbf{q}\cdot\mathbf{R}_i} e^{-i\omega_n\tau} \mathcal{G}_{ij}^{(0)}(\tau) \\ &\quad - 2\sigma \sum_k J_{ik} [\mathcal{G}_{ij}^{(1)}(\tau) - \mathcal{G}_{kj}^{(1)}(\tau)] \\ &\quad + 2 \sum_k J_{ik} [s_i^{(1)}(\tau)\mathcal{G}_{kj}^{(0)}(\tau) - s_k^{(1)}(\tau)\mathcal{G}_{ij}^{(0)}(\tau)]. \end{aligned} \quad (39)$$

We make the following space and time Fourier expansions:

$$\begin{aligned} \mathcal{G}_{ij}^{(1)}(\tau) &= \frac{1}{N^2} \sum_{\mathbf{q}_1 \mathbf{q}_2} \frac{1}{\beta} \sum_m \mathcal{G}^{(1)}(\mathbf{q}_1 \mathbf{q}_2, \omega_m) e^{i(\mathbf{q}_1 \cdot \mathbf{R}_i - \mathbf{q}_2 \cdot \mathbf{R}_j) - i\omega_m\tau}, \\ s_i^{(1)}(\tau) &= (1/N) \sum_{\mathbf{q}_1} s_{\mathbf{q}_1}^{(1)}(\omega_n) e^{i\mathbf{q}_1 \cdot \mathbf{R}_i - i\omega_n\tau}. \end{aligned} \quad (40)$$

In terms of these Fourier coefficients, the equation is

easily solvable; the result is

$$\mathcal{G}^{(1)}(\mathbf{q}_1, \mathbf{q}_2, \omega_m) = \frac{1}{\omega_{q_1} - i\omega_m} \times \left\{ 2s_{q_1-q_2}^{(1)}(\omega_n) - \frac{2\sigma\delta_{q_1-q_2, q}}{\omega_{q_1-q} - i\omega_m + i\omega_n} + \frac{2(\omega_{q_1-q_2} - \omega_{q_2})s_{q_1-q_2}^{(1)}(\omega_n)}{\omega_{q_2} - i\omega_m + i\omega_n} \right\}. \quad (41)$$

The self-consistency condition now reads

$$\mathcal{G}_{ij}(\tau) |_{\tau=0^-} = \langle S_i^- S_i^+ \rangle_{\tau=0} = \frac{1}{2} - s_i(\tau) |_{\tau=0}. \quad (42)$$

Collecting the first-order terms in  $f$ , we find

$$\begin{aligned} -s_i^{(1)}(\omega_n) &= \frac{1}{N^2\beta} \sum_m \sum_{\mathbf{q}_1 \mathbf{q}_2} \mathcal{G}^{(1)}(\mathbf{q}_1, \mathbf{q}_2, \omega_m) e^{i(\mathbf{q}_1 - \mathbf{q}_2) \cdot \mathbf{R}_i + i\omega_m \delta} \\ &= \frac{1}{N^2} \sum_{\mathbf{q}_1 \mathbf{q}_2} \left\{ 2s_{q_1-q_2}^{(1)}(\omega_n) N(\omega_{q_1}) \right. \\ &\quad - \frac{2\sigma\delta_{q_1-q_2, q}}{i\omega_n + \omega_{q_1-q} - \omega_{q_1}} [N(\omega_{q_1}) - N(\omega_{q_1-q})] \\ &\quad + \frac{2(\omega_{q_1-q_2} - \omega_{q_2})s_{q_1-q_2}^{(1)}(\omega_n)}{i\omega_n + \omega_{q_2} - \omega_{q_1}} \\ &\quad \left. \times [N(\omega_{q_1}) - N(\omega_{q_2})] \right\} e^{i(\mathbf{q}_1 - \mathbf{q}_2) \cdot \mathbf{R}_i}. \quad (43) \end{aligned}$$

After some straightforward manipulations, we obtain

$$\begin{aligned} s_{\mathbf{q}}^{(1)}(\omega_n) &= \frac{4\sigma^2}{N} \sum_{\mathbf{k}} \frac{N(\omega_{\mathbf{k}}) - N(\omega_{\mathbf{k}-\mathbf{q}})}{i\omega_n + \omega_{\mathbf{k}-\mathbf{q}} - \omega_{\mathbf{k}}} \\ &\div \left\{ 1 + \frac{4\sigma}{N} \sum_{\mathbf{k}} \frac{(\omega_{\mathbf{q}} - \omega_{\mathbf{k}-\mathbf{q}}) [N(\omega_{\mathbf{k}}) - N(\omega_{\mathbf{k}-\mathbf{q}})]}{i\omega_n + \omega_{\mathbf{k}-\mathbf{q}} - \omega_{\mathbf{k}}} \right\}. \quad (44) \end{aligned}$$

The spectral density function can be found from Eq. (21) to be

$$J_z(\mathbf{q}, \omega) = \frac{8\sigma^2}{N(1 - e^{-\beta\omega})} \times \frac{B(\mathbf{q}, \omega)C(\mathbf{q}, \omega) - A(\mathbf{q}, \omega)D(\mathbf{q}, \omega)}{C^2(\mathbf{q}, \omega) + D^2(\mathbf{q}, \omega)}, \quad (45)$$

where

$$\begin{aligned} A(\mathbf{q}, \omega) &= \frac{1}{N} \sum_{\mathbf{k}} \frac{N(\omega_{\mathbf{k}}) - N(\omega_{\mathbf{k}-\mathbf{q}})}{\omega + \omega_{\mathbf{k}-\mathbf{q}} - \omega_{\mathbf{k}}}, \\ B(\mathbf{q}, \omega) &= \frac{\pi}{N} \sum_{\mathbf{k}} [N(\omega_{\mathbf{k}-\mathbf{q}}) - N(\omega_{\mathbf{k}})] \delta(\omega + \omega_{\mathbf{k}-\mathbf{q}} - \omega_{\mathbf{k}}), \\ C(\mathbf{q}, \omega) &= 1 + \frac{4\sigma}{N} \sum_{\mathbf{k}} \frac{(\omega_{\mathbf{q}} - \omega_{\mathbf{k}-\mathbf{q}}) [N(\omega_{\mathbf{k}}) - N(\omega_{\mathbf{k}-\mathbf{q}})]}{\omega + \omega_{\mathbf{k}-\mathbf{q}} - \omega_{\mathbf{k}}}, \\ D(\mathbf{q}, \omega) &= \frac{4\pi\sigma}{N} \sum_{\mathbf{k}} (\omega_{\mathbf{q}} - \omega_{\mathbf{k}-\mathbf{q}}) [N(\omega_{\mathbf{k}-\mathbf{q}}) - N(\omega_{\mathbf{k}})] \\ &\quad \times \delta(\omega + \omega_{\mathbf{k}-\mathbf{q}} - \omega_{\mathbf{k}}). \quad (46) \end{aligned}$$

Thus, all the correlation functions may be calculated by using this expression for the spectral density.

The general spin problem is solved in an analogous way. Following Callen,<sup>13</sup> we define the Green's function

$$G_{ija}(t) = -i \langle TS_i^+(t) e^{aS_i^z} S_j^-(0) \rangle. \quad (47)$$

After decoupling, the equation of motion of this Green's function is exactly like Eq. (28), except that the first term on the right-hand side is replaced by

$$\delta_{ij} \delta(t) \Theta_i(a, t),$$

where

$$\Theta_i(a, t) = \langle [S_i^+, e^{aS_i^z} S_i^-] \rangle. \quad (48)$$

Again, we analytically continue into the imaginary time domain and expand  $\mathcal{G}_{ija}(\tau)$ ,  $\Theta_i(a, \tau)$ , and  $s_i(\tau)$  into power series of  $f$ . Then we may solve for  $\mathcal{G}_{ija}^{(0)}(\tau)$  and  $\mathcal{G}_{ija}^{(1)}(\tau)$  in terms of  $\sigma$ ,  $s_i^{(1)}(\tau)$ ,  $\Theta_i^{(0)}(a, \tau)$ , and  $\Theta_i^{(1)}(a, \tau)$ . The self-consistency condition now takes the form

$$\begin{aligned} \mathcal{G}_{ija}(\tau) |_{\tau=0^-} &= \langle e^{aS_i^z} S_i^- S_i^+ \rangle_{\tau=0} \\ &= [S(S+1) - (d/da) - (d^2/da^2)] \\ &\quad \times \Omega_i(a, \tau) |_{\tau=0}, \quad (49) \end{aligned}$$

where

$$\Omega_i(a, \tau) = \langle e^{aS_i^z} \rangle. \quad (50)$$

The quantities  $s_i(\tau)$ ,  $\Theta_i(a, \tau)$  are related to  $\Omega_i(a, \tau)$  by

$$\begin{aligned} s_i(\tau) &= (d/da) \Omega_i(a, \tau) |_{a=0}, \\ \Theta_i(a, \tau) &= \{ S(S+1)(e^{-a} - 1) + (e^{-a} + 1)(d/da) \\ &\quad - (e^{-a} - 1)(d^2/da^2) \} \Omega_i(a, \tau). \quad (51) \end{aligned}$$

Solving explicitly for the Fourier transform of  $s_i^{(1)}(\tau)$ , we obtain

$$\begin{aligned} s_{\mathbf{q}}^{(1)}(\omega_n) &= \frac{2\sigma\eta}{N} \sum_{\mathbf{k}} \frac{N(\omega_{\mathbf{k}}) - N(\omega_{\mathbf{k}-\mathbf{q}})}{i\omega_n + \omega_{\mathbf{k}-\mathbf{q}} - \omega_{\mathbf{k}}} \\ &\div \left\{ 1 + \frac{2\eta}{N} \sum_{\mathbf{k}} \frac{\omega_{\mathbf{q}} - \omega_{\mathbf{k}-\mathbf{q}}}{i\omega_n + \omega_{\mathbf{k}-\mathbf{q}} - \omega_{\mathbf{k}}} [N(\omega_{\mathbf{k}}) - N(\omega_{\mathbf{k}-\mathbf{q}})] \right\}, \quad (52) \end{aligned}$$

where

$$\omega_{\mathbf{k}} = h + 2\sigma J(0, \mathbf{k}), \quad (53)$$

$$\sigma = \frac{S\Phi^S + (S+1)(1+\Phi)^{S+1}}{(1+\Phi)^{2S+1} - \Phi^{2S+1}} - (1+\Phi), \quad (54)$$

$$\Phi = (1/N) \sum_{\mathbf{q}} N(\omega_{\mathbf{q}}), \quad (55)$$

$$\eta = \frac{1}{S(S+1)} \left[ \sigma + 1 + \Phi - \frac{(2S+1)\lambda_S}{(1+\Phi)^{2S+1} - \Phi^{2S+1}} \right], \quad (56)$$

and

$$\lambda_S = \frac{1}{(2S+1)!} \left( \frac{d}{dx} \right)^{2S+1} \frac{x^{S+1}}{1+\Phi(1-x)} \Big|_{x=0}. \quad (57)$$

These last four quantities are functions of temperature

only. For integral spins,  $\lambda_S$  has a simple form

$$\lambda_S = \Phi^S (1 + \Phi)^{S+1}, \quad (58)$$

and for half-odd integral spins we may write  $\lambda_S$  alternatively as

$$\lambda_S = \sum_{m=0}^{2S+1} \binom{S+1}{2S-m+1} \Phi^m, \quad (59)$$

where  $\binom{n}{r} = [\Gamma(n+1)]/[\Gamma(r+1)\Gamma(n-r+1)]$  are the binomial coefficients of the order  $n$ .

We also write down the expressions for the transverse correlation functions for later use. These are

$$\langle S_{-q}^-(t) S_q^+(0) \rangle = \frac{1}{N} \frac{2\sigma e^{i\omega_q t}}{e^{\beta\omega_q} - 1}, \quad (60)$$

$$\langle S_{-q}^+(t) S_q^-(0) \rangle = \frac{1}{N} \frac{2\sigma e^{(\beta-it)\omega_q}}{e^{\beta\omega_q} - 1}, \quad (61)$$

for general spin. They are derived by an analytical continuation of  $\mathcal{G}^{(0)}(\mathbf{q}, \omega_m)$  into the real frequency domain and taking inverse Fourier transform. In terms of the spectral representation, we have

$$\langle S_{-q}^-(t) S_q^+(0) \rangle = \int \frac{d\omega}{2\pi} J_-(\mathbf{q}, \omega) e^{-i\omega t}, \quad (62)$$

$$\langle S_{-q}^+(t) S_q^-(0) \rangle = \int \frac{d\omega}{2\pi} J_+(\mathbf{q}, \omega) e^{-i\omega t}, \quad (63)$$

where

$$J_-(\mathbf{q}, \omega) = -[4\pi\sigma/N(1 - e^{-\beta\omega})]\delta(\omega + \omega_q), \quad (64)$$

$$J_+(\mathbf{q}, \omega) = [4\pi\sigma/N(1 - e^{-\beta\omega})]\delta(\omega - \omega_q). \quad (65)$$

#### IV. NATURE OF THE RESULT

Some properties of the  $zz$  correlation function as found in the last section are discussed here. As before, we use the spin- $\frac{1}{2}$  case as the example, although the properties under discussion are common to all spins.

At very low temperatures where few spin waves are excited, we may approximate  $\sigma \cong \frac{1}{2}$  and ignore  $N(\omega_q)$  compared with unity. Equation (44) then gives

$$s_q^{(1)}(\omega_n) = \frac{1}{N} \sum_{\mathbf{k}} \frac{N(\omega_{\mathbf{k}}) - N(\omega_{\mathbf{k}-\mathbf{q}})}{i\omega_n + \omega_{\mathbf{k}-\mathbf{q}} - \omega_{\mathbf{k}}}. \quad (66)$$

From Eqs. (21) and (13), we find

$$\langle S_{-q}^z(t) S_q^z(0) \rangle = \frac{1}{N^2} \sum_{\mathbf{k}} N(\omega_{\mathbf{k}-\mathbf{q}}) \times [1 + N(\omega_{\mathbf{k}})] e^{-i(\omega_{\mathbf{k}} - \omega_{\mathbf{k}-\mathbf{q}})t}. \quad (67)$$

This is in agreement with the result in the noninteracting spin-wave approximation. Above the Curie point,

we expand  $\sigma$  in powers of  $h$ . In the limit of  $h \rightarrow 0$ , the quantity

$$\sigma/h \rightarrow \chi, \quad (68)$$

where  $\chi$  is the paramagnetic susceptibility of the system. From Eq. (38) we can derive the following equation for  $\chi$ :

$$\frac{1}{2\chi} = \frac{1}{N\beta} \sum_{\mathbf{q}} \frac{2}{1 + 2\chi J(0, \mathbf{q})}. \quad (69)$$

At high temperatures where one can expand in powers of  $\beta$ , the susceptibility is found to agree with the statistical theory to the order  $\beta^2$ . In the paramagnetic temperature region, the transverse spin correlation function is found by taking the proper limits of Eqs. (60) and (61) for  $h \rightarrow 0$ . The results are, for  $S = \frac{1}{2}$ :

$$\langle S_{-q}^+(t) S_q^-(0) \rangle = \langle S_{-q}^-(t) S_q^+(0) \rangle = \frac{1}{N\beta} \frac{2\chi}{1 + 2\chi J(0, \mathbf{q})}. \quad (70)$$

For the longitudinal correlation function, we note that

$$s_q^{(1)}(\omega_n) = \frac{\chi}{1 + 2\chi J(0, \mathbf{q})} \delta_{n0}$$

in the limit. When continued to the real frequency domain the Kronecker  $\delta$  becomes  $\beta^{-1}\delta(\omega)$ . Hence,

$$\langle S_{-q}^z(t) S_q^z(0) \rangle = \frac{1}{N\beta} \frac{\chi}{1 + 2\chi J(0, \mathbf{q})}. \quad (71)$$

These results show that the  $xx$ ,  $yy$ , and  $zz$  correlation functions are all equal. This is to be expected, because, when  $h=0$  in the paramagnetic region, all directions in space must be equivalent. These formulas also agree with those of Kawasaki and Mori. The result of Tahir-Kheli and Callen for the  $zz$  correlation function is different from the transverse ones above the Curie point, and, consequently, the rotational invariance property is violated. For small  $q$ , the above correlation functions are in accord with the van Hove theory of critical fluctuation.<sup>22</sup>

There is a sum rule for the static correlation functions of the spins, namely,

$$\sum_{\mathbf{q}} \langle \mathbf{S}_{-\mathbf{q}} \cdot \mathbf{S}_{\mathbf{q}} \rangle = S(S+1). \quad (72)$$

The derivation of this rule is trivial. Above the Curie point, the left-hand side is

$$\frac{3}{N\beta} \sum_{\mathbf{q}} \frac{\chi}{1 + 2\chi J(0, \mathbf{q})},$$

which is equal to  $3/4$  by using Eq. (69). So the sum rule is checked out all right. In the spin-wave region,

<sup>22</sup> L. van Hove, Phys. Rev. **95**, 249 (1954).

one can also verify that the sum rule is approximately satisfied. However, there seems to be no easy way of checking it at a general temperature below the Curie point. Perhaps one should not even expect to do so, because the expressions for the correlation functions are only approximate.

In conclusion, we have shown that the static and dynamic  $zz$  correlation functions obtained in the last section are indeed interpolation formulas. The sum rule and the rotational invariance properties are verified above the Curie temperature.

### V. THERMODYNAMICS OF THE HEISENBERG FERROMAGNET

The internal energy of the Heisenberg ferromagnet is given by

$$\begin{aligned} E = \langle H \rangle &= -Nh\sigma - \sum_{(ij)} J_{ij} \langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle \\ &= -N[h\sigma + \sum_{\mathbf{q}} J(\mathbf{q}) \langle \mathbf{S}_{-\mathbf{q}} \cdot \mathbf{S}_{\mathbf{q}} \rangle], \end{aligned} \quad (73)$$

after a simple transformation. The quantity  $\mathbf{S}_{\mathbf{q}}$  was defined in Eq. (4) and

$$J(\mathbf{q}) = \sum_j J_{ij} e^{-i\mathbf{q} \cdot \mathbf{R}_{ij}}. \quad (74)$$

So the internal energy can be computed if the static correlation functions of the spins are known. Differentiating the internal energy with respect to the temperature, we obtain the specific heat due to spin disorder. From this we can calculate the entropy of the system. In this manner, all the thermodynamical functions may be computed.

Although we have expressions for all the correlation functions, it is still difficult to carry out this study at a general temperature because of the numerical work involved. For the moment, we will discuss only the properties in the paramagnetic temperature region when  $h \rightarrow 0$ . In this case, it can be verified that

$$E = -3 \sum_{\mathbf{q}} \{ J(\mathbf{q}) \chi / \beta [1 + 2\chi J(0, \mathbf{q})] \}, \quad (75)$$

where  $\chi$  is given by Eq. (69). The specific heat is

$$C = \frac{3N}{2} \left[ 1 - \frac{\langle [1 + 2\chi J(0, \mathbf{q})]^{-1} \rangle^2}{\langle [1 + 2\chi J(0, \mathbf{q})]^{-2} \rangle} \right], \quad (76)$$

where

$$\langle [1 + 2\chi J(0, \mathbf{q})]^{-n} \rangle = \frac{1}{N} \sum_{\mathbf{q}} \frac{1}{[1 + 2\chi J(0, \mathbf{q})]^n}. \quad (77)$$

For convenience we set the Boltzmann constant equal to unity. At the Curie point  $T_c$  we have  $\chi \rightarrow \infty$ , so

$$C(T_c) = \frac{1}{2} 3N, \quad (78)$$

which is the same as in the molecular field theory. For  $T \gg T_c$ , it can be verified that

$$C \cong 6N\chi^2 \sum_j J_{ij}^2. \quad (79)$$

Since  $\chi \propto T^{-1}$  at these temperatures, this formula gives the familiar  $T^{-2}$  term for the specific heat. One can also study the dependence of  $C$  on the temperature by taking its first derivative:

$$\frac{dC}{dT} = \frac{3N}{4} \frac{\beta^4}{\chi^3} \left( \frac{d\chi}{d\beta} \right)^2 \left\{ \frac{\langle [1 + 2\chi J(0, \mathbf{q})]^{-2} \rangle}{\langle [1 + 2\chi J(0, \mathbf{q})]^{-1} \rangle} - \frac{\langle [1 + 2\chi J(0, \mathbf{q})]^{-3} \rangle}{\langle [1 + 2\chi J(0, \mathbf{q})]^{-2} \rangle} \right\}. \quad (80)$$

Since

$$\langle [1 + 2\chi J(0, \mathbf{q})]^{-2} \rangle - \langle [1 + 2\chi J(0, \mathbf{q})]^{-1} \rangle \times \langle [1 + 2\chi J(0, \mathbf{q})]^{-3} \rangle \leq 0 \quad (81)$$

for all  $\mathbf{q}$ , it follows that

$$dC/dT \leq 0. \quad (82)$$

Near but above  $T_c$ , the susceptibility  $\chi$  is large, and so only small values of  $\mathbf{q}$  are significant. It is well known that

$$J(0, \mathbf{q}) \cong \alpha q^2 \quad (83)$$

for small  $\mathbf{q}$ , so we can find

$$\langle [1 + 2\chi J(0, \mathbf{q})]^{-1} \rangle \sim \chi^{-1}, \quad (84)$$

$$\langle [1 + 2\chi J(0, \mathbf{q})]^{-2} \rangle = \frac{1}{8\pi} \frac{V_0}{(2\alpha\chi)^{3/2}}, \quad (85)$$

$$\langle [1 + 2\chi J(0, \mathbf{q})]^{-3} \rangle = \frac{1}{32\pi} \frac{V_0}{(2\alpha\chi)^{3/2}}, \quad (86)$$

where  $V_0$  is the volume of the unit magnetic cell. Similarly,

$$\begin{aligned} d\chi/d\beta &= \frac{1}{4} \langle [1 + 2\chi J(0, \mathbf{q})]^{-2} \rangle^{-1} \\ &= \frac{2\pi}{V_0} (2\alpha\chi)^{3/2}. \end{aligned} \quad (87)$$

Thus,

$$\left. \frac{dC}{dT} \right|_{T_c} = -6\pi^2 N \beta_c^4 \alpha^3 / V_0^2. \quad (88)$$

The specific-heat curve has a finite, negative slope at the high-temperature side of  $T_c$ . We will make a similar study below  $T_c$  by numerical analysis and will report the result in the future.

The susceptibility just above  $T_c$  depends only on the lower part of the spin-wave spectrum:

$$\chi \cong [(V_0^2 T_c^4) / (8\pi^2 \alpha^3)] (T - T_c)^{-2}. \quad (89)$$

The molecular field theory gives

$$\chi \propto (T - T_c)^{-1},$$

while the method of Padé approximants gives<sup>7,8</sup>

$$\chi \propto (T - T_c)^{-4/3}.$$

It is seen that the present theory over-corrects the molecular field theory.

## VI. TRANSPORT PROPERTIES OF FERROMAGNETIC METALS

In a ferromagnetic metal, the spin disorder gives rise to an additional mechanism for the random scattering of the conduction electrons. As a result, characteristic anomalies occur in all the transport properties of the material. We analyze here the effect on the electrical resistivity. Other transport properties may be studied in a parallel way.

The interaction between the conduction electrons and the core electrons is usually taken as the *s-d* or *s-f* interaction<sup>23,24</sup>:

$$H'' = \sum_{ij} I(\mathbf{r}_i - \mathbf{R}_j) \mathbf{s}_i \cdot \mathbf{S}_j, \quad (90)$$

where  $\mathbf{r}_i$ ,  $\mathbf{s}_i$  are the position and spin of the *i*th electron and  $\mathbf{R}_j$ ,  $\mathbf{S}_j$  the position and spin of the *j*th ion. The interaction potential is often approximated by a  $\delta$  function in space, i.e.,

$$I(\mathbf{r}_i - \mathbf{R}_j) = IV_0 \delta(\mathbf{r}_i - \mathbf{R}_j), \quad (91)$$

where  $V_0$  is the volume of the unit cell. For simplicity, we use the free-electron model for the conduction band. In the transport theory, we study the one-particle distribution function  $f_{\mathbf{k}}$  satisfying the Boltzmann equation

$$(\partial f_{\mathbf{k}} / \partial t)_f + (\partial f_{\mathbf{k}} / \partial t)_c = 0. \quad (92)$$

The first term is the field term whose expression is

$$(\partial f_{\mathbf{k}} / \partial t)_f = eE(\partial f_{\mathbf{k}} / \partial k_\gamma) \quad (93)$$

for an electric field  $E$  in the  $\gamma$  direction;  $e$  is the algebraic charge of the electron,  $k_\gamma$  is the  $\gamma$  component of the wave vector  $\mathbf{k}$ . The collision term may be calculated from the interaction Hamiltonian, Eq. (90) in the first Born approximation. If we average over the spins of the conduction electrons, we obtain<sup>25</sup>

$$\begin{aligned} \left( \frac{\partial f_{\mathbf{k}}}{\partial t} \right)_c = & -\frac{I^2}{4} \sum_{\mathbf{k}'} \int_{-\infty}^{\infty} e^{i(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'})t} dt \\ & \times \{ \langle \mathbf{S}_{-\mathbf{q}}(t) \cdot \mathbf{S}_{\mathbf{q}}(0) \rangle f_{\mathbf{k}}(1 - f_{\mathbf{k}'}) \\ & - \langle \mathbf{S}_{\mathbf{q}}(0) \cdot \mathbf{S}_{-\mathbf{q}}(t) \rangle f_{\mathbf{k}'}(1 - f_{\mathbf{k}}) \}, \quad (94) \end{aligned}$$

where  $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ . We write the dynamical correlation functions in the spectral representation

$$\langle \mathbf{S}_{-\mathbf{q}}(t) \cdot \mathbf{S}_{\mathbf{q}}(0) \rangle = \int \frac{d\omega}{2\pi} J(\mathbf{q}, \omega) e^{-i\omega t}, \quad (95)$$

where

$$J(\mathbf{q}, \omega) = \frac{1}{2} J_+(\mathbf{q}, \omega) + \frac{1}{2} J_-(\mathbf{q}, \omega) + J_z(\mathbf{q}, \omega), \quad (96)$$

and the quantities on the right-hand side are defined in Eqs. (13), (64), and (65). Then the collision term takes the form

$$\begin{aligned} \left( \frac{\partial f_{\mathbf{k}}}{\partial t} \right)_c = & -\frac{I^2}{4} \int_{-\infty}^{\infty} d\omega \sum_{\mathbf{k}'} \delta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'} - \omega) \\ & \times [f_{\mathbf{k}}(1 - f_{\mathbf{k}'}) - e^{-\beta\omega} f_{\mathbf{k}'}(1 - f_{\mathbf{k}})] J(\mathbf{q}, \omega). \quad (97) \end{aligned}$$

Following the method of Bloch,<sup>26</sup> we assume the following form for the distribution function:

$$f_{\mathbf{k}} = f_0(\epsilon_{\mathbf{k}}) - \alpha k_\gamma \frac{\partial f_0(\epsilon_{\mathbf{k}})}{\partial \epsilon_{\mathbf{k}}}, \quad (98)$$

and a similar expression for  $f_{\mathbf{k}'}$ . The quantity  $\alpha$  depends only on  $E$ . In this problem, the field is considered to be an infinitesimal. Substituting Eq. (98) into Eq. (97), we find the terms independent of  $\alpha$  to be

$$f_0(\epsilon_{\mathbf{k}})[1 - f_0(\epsilon_{\mathbf{k}'})] - e^{-\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'})} f_0(\epsilon_{\mathbf{k}'})[1 - f_0(\epsilon_{\mathbf{k}})] \equiv 0.$$

The terms proportional to  $\alpha$  are collected:

$$\begin{aligned} \left( \frac{\partial f_{\mathbf{k}}}{\partial t} \right)_c = & \frac{\alpha I^2}{4} \int_{-\infty}^{\infty} d\omega \sum_{\mathbf{k}'} \delta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'} - \omega) J(\mathbf{q}, \omega) \\ & \times \{ k_\gamma \beta e^{-\beta\omega} f_0(\epsilon_{\mathbf{k}} - \omega) [1 - f_0(\epsilon_{\mathbf{k}})] \\ & - k_\gamma' \beta f_0(\epsilon_{\mathbf{k}}) [1 - f_0(\epsilon_{\mathbf{k}} - \omega)] \}. \quad (99) \end{aligned}$$

The higher order terms of  $\alpha$  are neglected. We then transform the  $\mathbf{k}'$  sum into a sum over  $\mathbf{q}$  and average over the directions of  $\mathbf{q}$ . In general,  $J(\mathbf{q}, \omega)$  depends on the direction of  $\mathbf{q}$ , so we expand into spherical harmonics:

$$J(\mathbf{q}, \omega) = \sum_{l,m} J_{lm}(q, \omega) Y_{lm}(\theta, \phi), \quad (100)$$

where  $\theta$ ,  $\phi$  are the polar angles of  $\mathbf{q}$  with respect to some spatial coordinates. For the present discussion, we will only work out the contribution of the isotropic component of  $J(\mathbf{q}, \omega)$ . This is usually the most important contribution, and it corresponds roughly to the resistivity of a polycrystalline material. For simplicity, we will write  $J(q, \omega)$  in place of  $J_{lm}(q, \omega)$  for  $l = m = 0$ . Then

<sup>23</sup> T. Kasuya, Progr. Theoret. Phys. (Kyoto) **16**, 58 (1956).

<sup>24</sup> S. H. Liu, Phys. Rev. **121**, 451 (1961).

<sup>25</sup> S. H. Liu, Phys. Rev. **132**, 589 (1963).

<sup>26</sup> F. Bloch, Z. Phys. **50**, 208 (1930).

the scattering term becomes

$$\left(\frac{\partial f_k}{\partial t}\right)_c = \frac{mI^2V}{16\pi^2k}(\alpha k\gamma) \int d\omega \int q dq J(q,\omega) \\ \times \left\{ \beta e^{-\beta\omega} f_0(\epsilon_k - \omega) [1 - f_0(\epsilon_k)] \right. \\ \left. - \left[ 1 - \frac{q^2}{2k^2} - \frac{m\omega}{k^2} \right] \beta f_0(\epsilon_k) [1 - f_0(\epsilon_k - \omega)] \right\}. \quad (101)$$

To the lowest order in  $E$ , the field term is

$$(\partial f_k / \partial t)_f = - (eEk_\gamma / m) \beta f_0(\epsilon_k) [1 - f_0(\epsilon_k)]. \quad (102)$$

Substituting Eqs. (101) and (102) into Eq. (92), canceling  $k_\gamma$  from both sides, and integrating over  $\epsilon_k$ , we obtain

$$\frac{eE}{m} = \frac{\alpha m I^2 V}{16\pi k_F} \int d\omega \int \frac{q^3 dq}{2k_F^2} \frac{\beta\omega}{e^{\beta\omega} - 1} J(q,\omega), \quad (103)$$

where  $k_F$  is the Fermi wave vector. A term of the order  $\omega/\epsilon_F$  has been ignored in the above expression,  $\epsilon_F = k_F^2/2m$  being the Fermi energy. The current density can be calculated from the distribution function

$$j = \frac{2e}{V} \int v_\gamma f_k d^3k \\ = \frac{2e}{2\pi^2 m} \int (-\alpha) k_\gamma^2 \frac{\partial f_0(\epsilon_k)}{\partial \epsilon_k} k^2 dk \\ = \frac{ek_F^3 \alpha}{3\pi^2}. \quad (104)$$

Thus, the resistivity is found to be

$$\rho = \frac{E}{j} = \frac{3m^2 I^2 V}{32e^2 k_F^6} \int d\omega \int q^3 dq \frac{\beta\omega J(q,\omega)}{e^{\beta\omega} - 1}. \quad (105)$$

Below the Curie point, the resistivity expression is difficult to analyze. However, in the spin-wave region, the  $zz$  correlation function gives a negligible contribution, and the transverse part gives the  $T^2$  term in the resistivity.<sup>27</sup> Above the Curie temperature the resistivity has a simple form ( $S = \frac{1}{2}$ ):

$$\rho = \frac{9\pi m^2 I^2 V_0}{16\beta e^2 k_F^6} \int_0^{2k_F} dq \int \frac{d\Omega_q}{4\pi} \frac{\chi q^3}{1 + 2\chi J(0, \mathbf{q})}. \quad (106)$$

This is equivalent to the result of deGennes and Friedel.<sup>28</sup> We will report the numerical analysis of the complete resistivity curve in a later publication.

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<sup>27</sup> I. Mannari, Progr. Theoret. Phys. (Kyoto) **22**, 335 (1959); **26**, 51 (1961).

<sup>28</sup> P. G. deGennes and J. Friedel, J. Phys. Chem. Solids **4**, 71 (1958).