Electrostatic Instabilities in a Plasma with Anisotropic Velocity Distribution

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An anisotropic magnetized plasma, in which the distribution of particle velocities is a bi-Maxwellian characterized by the temperatures $T_\perp$ and $T_{\parallel}$, is considered in order to delimit as much as possible the location of possibly unstable roots to the dispersion equation. We find that marginally unstable roots of the dispersion equation can occur only if both $[l+1/2]<\omega<[l+1-(T_{\parallel}/T_\perp)]$ and $\omega<(T_\perp-T_{\parallel})/T_{\parallel}$, where l is an integer and ω is the real part of the frequency in units of the cyclotron frequency. Thus, in particular, the system is stable if $T_\perp<2T_{\parallel}$.

IN this paper we wish to extend previous work[1] to delimit as much as possible the location of possibly unstable roots to the dispersion equation for electrostatic waves in an infinite plasma situated in a constant magnetic field.

The dispersion equation may be written in the form

$$1+\sum_j \omega_{pj}^2\Omega_j^{-2}F_j=0\,,$$

where F_j is given in Ref. 1 for the bi-Maxwellian distribution function.

The theorems given in the previous article, which were based on an examination of $\mathrm{Im}(F_i)$, are here extended to show that marginally unstable solutions $[\mathrm{Im}(\omega)=0^+]$ of the dispersion equation can occur only if both

$$l+\tfrac{1}{2}<\omega<l+1-T_{\parallel}/T_\perp \quad\text{and}\quad \omega<T_\perp/T_{\parallel}-1\,, \tag{1}$$

where l is an integer. Hence, in particular, the system is stable if $T_{\parallel}/T_\perp>\frac{1}{2}$.

Following the same arguments as in Ref. 1, it is sufficient to show that $\mathrm{Im}(F)$ is non-negative in the regime of interest located in the first quadrant of the complex ω plane, where

$$F=n_{\parallel}^2F_C+n_\perp^2F_B=\int_0^\infty dx\,(n_{\parallel}^2x+n_\perp^2\sin x)\times\exp[i\omega x-\mu x^2-\lambda(1-\cos x)]$$

and $\mathrm{Im}(F_B)\geq 0$. When $\omega=l+\epsilon$ is real, $0\leq\epsilon<1$,

$$\mathrm{Im}(F_C)=Ae^{-\lambda}\sum_{k=-\infty}^{\infty}g(k+\omega)I_k(\lambda)\,,$$

where

$$A=\frac{1}{2}\left(\frac{\pi}{\mu}\right)^{1/2}\frac{1}{\lambda}\frac{T_{\parallel}}{T_\perp}>0 \quad\text{and}\quad g(x)=x\exp\left(-\frac{x^2}{4\mu}\right).$$

Thus, since $I_{k-l}(\lambda)>I_{k+l}(\lambda)>0$ for $k,\,l>0$,

$$\mathrm{Im}(F_C)>Ae^{-\lambda}\{g(\epsilon)I_l(\lambda)+\sum_{k=1}^{\infty}[g(k+\epsilon)-g(k-\epsilon)]I_{k+l}(\lambda)\}\,.$$

When $x>a>0$, notice that $g(x)-g(x-a)$ is positive for $x>x_a$, and negative thereafter. Therefore, because $I_k(\lambda)$ is a positive, decreasing function of k, there is an $m\geq 0$ such that, for all k,

$$[g(k+\epsilon)-g(k-\epsilon)][I_{k+l}(\lambda)-I_{m+l}(\lambda)]\geq 0\,.$$

Therefore,

$$\begin{aligned}\mathrm{Im}(F_C)&>Ae^{-\lambda}\{g(\epsilon)I_l(\lambda)+\sum_{k=1}^{\infty}[g(k+\epsilon)-g(k-\epsilon)]I_{m+l}(\lambda)\}\\&>Ae^{-\lambda}I_{m+l}(\lambda)\sum_{k=-\infty}^{\infty}g(k+\epsilon)\\&=-2\mu Ae^{-\lambda}I_{m+l}\frac{d}{d\omega}\left\{\sum_{k=-\infty}^{\infty}\exp\left[-\frac{(k+\omega)^2}{4\mu}\right]\right\}.\end{aligned}$$

The infinite sum is just

$$e^{-\omega^2/4\mu}\theta_3(-i\omega/4\mu,\,e^{-1/4\mu})=(4\pi\mu)^{1/2}\theta_3(\pi\omega,e^{-4\pi^2\mu})\,,$$

where $\theta_3(z,q)$ is a theta function.[2] Moreover, one can show[3]

$$(d/dz)[\ln\theta_3(z,q)]=-B(z,q)\sin(2z)\,,$$

where $B(z,q)$ is positive if $z,\,q>0$.

In our case, $z=\pi\omega$, $q=e^{-4\pi^2\mu}$, and θ_3 is positive.

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[1] L. S. Hall and W. Heckrotte, Phys. Rev. **134**, A 1474 (1964).

[2] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, England, 1950), 4th ed., pp. 464 and 474.

[3] Reference 2, p. 489.

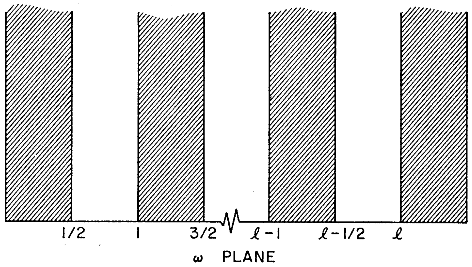

FIG. 1. Regions of stability (shaded).

Therefore,

$$\mathrm{Im}(F_C) > D\sin(2\pi\omega), \quad D>0, \tag{2}$$

which is non-negative whenever $l \leq \omega \leq l+\frac{1}{2}$.

Now, it was shown in Ref. 1 that $\mathrm{Im}(F_C)\geq 0$ when $\omega = l + i\sigma$, and, by a completely analogous argument, it is also easy to show that $\mathrm{Im}(F_C)\geq 0$ when $\omega = l+\frac{1}{2}+i\sigma$. Therefore, by Cauchy's theorem, $\mathrm{Im}(F_C)\geq 0$ whenever $l\leq \mathrm{Re}(\omega) \leq l+\frac{1}{2}$.

In order to complete the proof of (1), we write (for real ω)

$$\mathrm{Im}(F) = K\frac{T_\perp}{T_\parallel}\sum_{k=-\infty}^{\infty}\left[\omega + k\left(1-\frac{T_\parallel}{T_\perp}\right)\right] \times \exp\left[-\frac{(\omega+k)^2}{4\mu}\right] I_k(\lambda),$$

and restrict our attention to the case $T_\parallel/T_\perp < 1$, since $\mathrm{Im}(F)$ is obviously positive otherwise. Let $[x]$ denote the largest integer contained in x. Thus, if

$$n = [\omega(1-T_\parallel/T_\perp)^{-1}],$$

then $\omega = (1-T_\parallel/T_\perp)(n+\delta)$, where $0\leq\delta<1$. Hence,

$$\mathrm{Im}(F) > K\left(\frac{T_\perp}{T_\parallel}-1\right)\sum_{l=1}^{\infty} l\Bigg(I_{l-n}(\lambda) \times \exp\left\{-\left[l - n\frac{T_\parallel}{T_\perp}+\delta\left(1-\frac{T_\parallel}{T_\perp}\right)\right]^2\frac{1}{4\mu}\right\} - I_{l+n}(\lambda)\exp\left\{-\left[l+n\frac{T_\parallel}{T_\perp}-\delta\left(1-\frac{T_\parallel}{T_\perp}\right)\right]^2\frac{1}{4\mu}\right\}\Bigg),$$

and so

$$\mathrm{Im}(F)>0 \quad \text{if} \quad n(T_\parallel/T_\perp) > \delta(1-T_\parallel/T_\perp).$$

Rewriting, this becomes

$$[\omega(1-T_\parallel/T_\perp)^{-1}] \geq \omega, \quad \text{stable}. \tag{3}$$

Now if

$$\omega = l+\epsilon, \quad \text{then for} \quad 1-T_\parallel/T_\perp < \epsilon < 1,$$

$$\left[\omega\left(1-\frac{T_\parallel}{T_\perp}\right)^{-1}\right] \geq \left[l\left(1-\frac{T_\parallel}{T_\perp}\right)^{-1}\right]+1 \geq l+1 > \omega,$$

so that we have stability if

$$l + (1-T_\parallel/T_\perp) < \omega < l+1.$$

Moreover,

$$0 \leq \left[\omega\left(\frac{T_\perp}{T_\parallel}-1\right)^{-1}\right] \leq \left[\omega\left(1-\frac{T_\parallel}{T_\perp}\right)^{-1}\right] - [\omega] < \left[\omega\left(1-\frac{T_\parallel}{T_\perp}\right)^{-1}\right]+1-\omega$$

so that when

$$[\omega(T_\perp/T_\parallel - 1)^{-1}] \geq 1,$$

inequality (3) is satisfied, which completes the proof of (1).

In summary, if we refer to Fig. 1, no roots to the dispersion equation can be found for ω lying in the shaded portion of the complex ω plane. In addition, the condition of marginal instability $[\mathrm{Im}(\omega)=0^+]$ is possible only if both $l-\frac{1}{2}<\omega<l-T_\parallel/T_\perp$ and $\omega < (T_\perp/T_\parallel)-1$.

It may be mentioned that both inequality (2) and the condition that $\mathrm{Im}(F_C)>0$ when $\omega = l+i\sigma$ or $\omega = l+\frac{1}{2}+i\sigma$, $\sigma>0$, which have been proved explicitly for the case of a bi-Maxwellian distribution of particle velocities,

$$f(v_\parallel, v_\perp) \sim \exp\left(-\frac{mv_\parallel^2}{2\kappa T_\parallel}-\frac{mv_\perp^2}{2\kappa T_\perp}\right)$$

also hold when the distribution of parallel velocities is Lorentzian, viz.,

$$f(v_\parallel, v_\perp) \sim \left(1+\frac{mv_\parallel^2}{2\kappa T_\parallel}\right)^{-1}\exp\left(-\frac{mv_\perp^2}{2\kappa T_\perp}\right).$$