Photons and Gravitons in Perturbation Theory: Derivation of Maxwell's and Einstein's Equations*

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The S matrix for photon and graviton processes is studied in perturbation theory, under the restriction that the only creation and annihilation operators for massless particles of spin j allowed in the interaction are those for the physical states with helicity $\pm j$. The most general covariant fields that can be constructed from such operators cannot represent real photon and graviton interactions, because they give amplitudes for emission or absorption of massless particles which vanish as p^j for momentum $p \to 0$. In order to obtain long-range forces it is necessary to introduce noncovariant "potentials" in the interaction, and the Lorentz invariance of the S matrix requires that these potentials be coupled to conserved tensor currents, and also that there appear in the interaction direct current-current couplings, like the Coulomb interaction. We then find that the potentials for $j=1$ and $j=2$ must inevitably satisfy Maxwell's and Einstein's equations in the Heisenberg representation. We also show that although the existence of magnetic monopoles is consistent with parity and time-reversal invariance [provided that P and T are defined to take a monopole into its antiparticle], it is nevertheless impossible to construct a Lorentz-invariant S matrix for magnetic monopoles and charges in perturbation theory.

I. INTRODUCTION

THE classical theories of electromagnetism and gravitation were developed long before physicists discovered quantum mechanics or the S matrix. For this reason, the modern field theorist is generally content to take Maxwell's and Einstein's equations for granted as the starting point of the quantum theory of photons and gravitons.

However, the logical structure of physics is often antiparallel to its historical development. For the purposes of this article, the reader is requested to forget all he knows of electrodynamics and general relativity, and instead to take as his starting point the Lorentz invariance of the S matrix calculated by Feynman-Dyson perturbation theory. That is, we assume the S matrix to be given by

$$S=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}dt_1\cdots dt_n T\{H'(t_1)\cdots H'(t_n)\} \tag{1.1}$$

with $H'(t)$ the interaction Hamiltonian in the interaction representation

$$H'(t)=\exp(iH^f t)H'\exp(-iH^f t)\,, \tag{1.2}$$

where H^f is the free-particle Hamiltonian and H' the interaction. The operator $H'(t)$ is some function of the creation and annihilation operators of free particles, and we know that these operators transform according to the various familiar representations of the inhomogeneous Lorentz group.[1] Among these representations are those characterized by mass $m=0$ and spin $j=1$ or 2, and we accord these the names of photon and graviton, with no implication intended that these particles necessarily have anything to do with gauge invariance or geometry. Our fundamental requirement on the form of $H'(t)$ is that (1.1) must yield a Lorentz-invariant S matrix.

The power of this requirement is only now beginning to be appreciated. There are strong indications,[2,3] (though as yet no careful proof) that it yields all the results usually associated with local field theories, including the existence of antiparticles, crossing symmetry, spin and statistics, CPT, the Feynman rules, etc. The purpose of this article is to explore the consequences of Lorentz invariance in perturbation theory, for the special case of zero mass and integer spin.

We shall find within this perturbative dynamical framework that *Maxwell's theory and Einstein's theory are essentially the unique Lorentz-invariant theories of massless particles with spin $j=1$ and $j=2$.* By "essentially" we mean only that the conserved current $\mathcal{J}^\mu$ and $\mathcal{J}^{\mu\nu}$ to which the photon and graviton are coupled need not be precisely equal to the electric charge current J^μ and the stress-energy tensor $\theta^{\mu\nu}$, since we can always add Pauli-type currents which vanish in the limits of zero momentum transfer, or of long range. In the same sense, we shall also find that there are *no* Lorentz-invariant theories of massless particles with $j=3, 4$ etc., that is, no theories which yield an inverse-square-law macroscopic force.

These conclusions have already been anticipated in an earlier article on pure S-matrix theory.[4] We showed

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[1] E. P. Wigner, Ann. Math. 40, 149 (1939), and *Theoretical Physics* (International Atomic Energy Agency, Vienna, 1963), p. 59.
[2] For the case $m\neq 0$, see S. Weinberg, Phys. Rev. 133, B1318 (1964).
[3] For the case $m=0$, see S. Weinberg, Phys. Rev. 134, B882 (1964).
[4] S. Weinberg, Phys. Rev. 135, B1049 (1964). See also D. Zwanziger, Phys. Rev. 133, B1036 (1964). For a preliminary account of this work and that of the present article, see S. Weinberg, Phys. Letters 9, 357 (1964). A unified treatment of Refs. 2–4 will be published in the lecture notes of the 1964 Brandeis University Summer School on theoretical physics.

there that the Lorentz invariance of the S matrix and a few elementary ideas about pole structure imply that charge is conserved and gravitational mass is equal to inertial mass, the charge and gravitational mass of a particle being defined as its coupling constants for emission of soft photons or gravitons. We also showed that the analogous coupling constants for $j \geq 3$ must vanish. But by using perturbation theory we are able to go much further here, and will in fact derive both the form of the interaction-representation Hamiltonian, and the Maxwell and Einstein field equations in the Heisenberg representation.

We start in Sec. II with a discussion of the $(2j+1)$-component free fields for massless particles of integer spin j. These fields transform according to the $(j,0)$ or $(0,j)$ representations of the homogeneous Lorentz group, and correspond for $j=1$ and $j=2$ to the left- or right-handed parts of the Maxwell field strength tensor $F^{\mu\nu}$ and the Riemann-Christoffel curvature tensor $R^{\mu\nu\lambda\eta}$. They have already been treated in detail in Ref. 3 for general spin, but we concentrate here on the tensor notation appropriate for integer spin, and we also show that *any* covariant free field may be constructed as linear combinations of these simple fields and their derivatives.

However, these simple tensor fields cannot by themselves be used to construct the interaction $H'(t)$, because the coefficients of the operators for creation or annihilation of particles of momentum p and spin j would vanish as p^j for $p \to 0$, in contradiction with the known existence of inverse-square-law forces. We are therefore forced to turn from these tensor fields to the potentials $A^{\mu_1 \cdots \mu_j}(x)$, from which they can be derived by taking a "curl" on each index. But we show in Sec. III that the potentials are not tensor fields; indeed, they cannot be, for we know from a very general theorem[3] that no symmetric tensor field or rank j can be constructed from the creation and annihilation operators of massless particles of spin j. It is for this reason that some field theorists[5] have been led to introduce fictitious photons and gravitons of helicity other than $\pm j$, as well as the indefinite metric that must accompany them.

Preferring to avoid such unphysical monstrosities, we must ask now what sort of coupling we can give our nontensor potentials without losing the Lorentz invariance of the S matrix? And it is here that the failure of manifest covariance turns out to be a blessing in disguise. In Sec. IV we remark that a Lorentz transformation will induce on $A^{\mu_1 \cdots \mu_j}(x)$ a combined tensor and gauge transformation, *so the only interactions allowed by Lorentz invariance are those satisfying gauge invariance*, i.e., those in which the potential is coupled to a conserved current.

For example, the direct photon coupling $(A_\mu A^\mu)^2$ is forbidden, not only by gauge invariance (which we do not assume) but also by Lorentz invariance, because $A_\mu(x)$ is not a four-vector. Actually, we show in Sec. V that even gauge invariance is not sufficient for the Lorentz invariance of the S matrix; as is always the case for spins $j \geqq 1$, we must cancel a noncovariant but temporally local part of the propagator by adding an extra noncovariant interaction to $H'(t)$, which for $j=1$ is the familiar Coulomb interaction, and for $j=2$ we christen the Newton interaction. In Appendix B we present a complete proof[6] that coupling $A^\mu(x)$ to a conserved current and adding a Coulomb term to $H'(t)$ does in fact make the S matrix Lorentz-invariant for $j=1$. (The propagators for photons and gravitons are calculated in Appendix A.)

Having deduced the form of $H'(t)$, we then pass over to the Heisenberg representation, taking care to introduce extra potential components (A^0 for $j=1$; A^{00}, A^{0i}, and $A^i{}_i$ for $j=2$) to represent the effects of the direct Coulomb and Newton interactions. In Sec. VI we show that the Heisenberg representation $A^\mu(x)$ satisfies the Maxwell equations in Coulomb gauge, and in Sec. VII we show that the Heisenberg representation $A^{\mu\nu}(x)$ satisfies the Einstein field equations, in a gauge too ugly to deserve a name.

In Sec. VIII we touch briefly on an old problem: Is it possible to construct a consistent theory of magnetic monopoles? Within the dynamical framework adopted here, the answer is definitely no, because the propagator for a photon linking a charge and a monopole contains noncovariant parts which cannot be cancelled by adding direct terms like the Coulomb interaction to $H'(t)$. The behavior of monopoles under P and T is also discussed.

Although our treatment of electrodynamics is essentially complete, we are not attempting in this article to solve the really difficult problems of quantizing gravitation. In particular, we do not exhibit the conserved energy momentum tensor $\theta^{\mu\nu}$, to which $A^{\mu\nu}$ is coupled, as an explicit function of the gravitation creation and annihilation operators, and we therefore cannot complete the proof that Einstein's equations are sufficient and necessary for Lorentz invariance of the quantized theory. Needless to say, we do not touch upon the ultraviolet-divergence problem either. It is intended that the example of our treatment of photons, together with the beginning made here with gravitons, will serve as the basis for future work on the hard problems of quantum-gravitational theory.

Before setting to work, it may be instructive to compare our development with that of other authors who have also tried to derive electrodynamics or general relativity from first principles. Three different previous approaches may be distinguished.

[5] For photons, see S. N. Gupta, Proc. Phys. Soc. **63**, 681 (1950); **64**, 850 (1951); K. Bleuler, Helv. Phys. Acta **23**, 567 (1950); K. Bleuler and W. Heitler, Progr. Theoret. Phys. (Kyoto) **5**, 600 (1950). For gravitons, see S. N. Gupta, in *Recent Developments in General Relativity* (Pergamon Press, New York, 1962), p. 251, and other references quoted therein.

[6] A similar result was obtained using a different approach, by J. Schwinger, Nuovo Cimento **30**, 278 (1963).

1. Extended Gauge Invariance

We may require the Lagrangian to be invariant under the extended gauge transformation

$$\psi(x) \to e^{iq\Phi(x)}\psi(x) \tag{1.3}$$

[where q is the charge destroyed by $\psi(x)$ and $\Phi(x)$ is an arbitrary c-number function of x^μ]. Then derivatives of $\psi(x)$ must always occur in the form

$$\partial_\mu\psi(x) - iqA_\mu(x)\psi(x) \tag{1.4}$$

the field $A_\mu(x)$ undergoing the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\Phi(x)\,. \tag{1.5}$$

Therefore,

$$\frac{\partial\mathcal{L}(x)}{\partial A_\mu(x)} = \sum_{\text{fields}} -iq\frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\psi(x))}\psi(x) \tag{1.6}$$

and this is the conserved electric current $J^\mu(x)$. Requiring the free field Lagrangian of $A_\mu(x)$ to be gauge-invariant then yields Maxwell's equations, with (1.6) as source. A similar approach has been used to derive Einstein's equations.[7]

The only criticism I can offer to this textbook approach is that no one would ever have dreamed of extended gauge invariance if he did not already know Maxwell's theory. In particular, extended gauge invariance has found no application to the strong or weak interactions, though attempts have not been lacking. In our approach (1.6) is a consequence of Lorentz invariance, and this implies that $A_\mu(x)$ enters in $\mathcal{L}(x)$ only in the combination (1.4), so that invariance under (1.3) and (1.5) appears as an incidental result rather than a mysterious postulate.

2. Geometrization

Einstein's theory rests on the identification of the gravitational field with the metric tensor of Riemannian geometry. Attempts have also been made to include electrodynamics in this geometric approach. The criticism here is, again, that the weak and strong interactions seem to have no more to do with Riemannian geometry than with extended gauge invariance. In our approach the geometric interpretation of the gravitational field arises as an incidental consequence of its coupling to the energy-momentum tensor, though we shall not go into this here.

3. Classical Fields with Definite Spin

A number of attempts[8] have been made to derive Maxwell's and/or Einstein's equations by imposing on classical vector or tensor fields the requirement that they correspond to definite spins, $j=1$ or $j=2$. In criticism of these articles we may say first that they generally seem to be based on specific Lagrangians, and secondly, that there does not seem to be much point in defining the spin of a field without being able to tie the definition to the physically relevant representations of the inhomogeneous Lorentz group, i.e., the one-particle states. In our work everything rests on the known transformation properties of the operators which destroy and create physical particles, and of course we make no use of the Lagrangian formalism.

II. THE COVARIANT FIELDS

In this section we shall show that the most general free field for a massless particle of integer helicity $\pm j$ may be constructed from the fundamental field

$$\begin{aligned}F_\pm{}^{[\mu_1\nu_1]\cdots[\mu_j\nu_j]}(x) \equiv (2\pi)^{-3/2}i^j\int d^3p(2|\mathbf{p}|)^{-1/2}&\\ \times[p^{\mu_1}e_\pm{}^{\nu_1}(\mathbf{p}) - p^{\nu_1}e_\pm{}^{\mu_1}(\mathbf{p})]\times\cdots&\\ \times[p^{\mu_j}e_\pm{}^{\nu_j}(\mathbf{p}) - p^{\nu_j}e_\pm{}^{\mu_j}(\mathbf{p})]&\\ \times[a(\mathbf{p},\pm j)e^{ip\cdot x} + b^*(\mathbf{p},\mp j)e^{-ip\cdot x}]&\,,\end{aligned} \tag{2.1}$$

by taking direct sums of $F_\pm$ and/or its derivatives. In Eq. (2.1), $a(\mathbf{p},\lambda)$ and $b(\mathbf{p},\lambda)$ are the annihilation operators for a massless particle and antiparticle of momentum $\mathbf{p}$ and helicity λ; if the particle is its own antiparticle we of course set $a(\mathbf{p},\lambda)=b(\mathbf{p},\lambda)$. The "polarization vectors" $e_\pm{}^\mu(\mathbf{p})$ are defined in Ref. 4, as

$$e_\pm{}^\mu(\mathbf{p}) = R^\mu{}_\nu(\hat{p})e_\pm{}^\nu \tag{2.2}$$

where

$$e_\pm{}^1 = 1/\sqrt{2}\,,\quad e_\pm{}^2 = \pm i/\sqrt{2}\,,\quad e_\pm{}^3 = e_\pm{}^0 = 0, \tag{2.3}$$

and $R^\mu{}_\nu(\hat{p})$ is the pure rotation that takes the z axis into the direction of $\mathbf{p}$. By a "general free field" we mean here any linear combination $\psi_n(x)$ of the $a(\mathbf{p},\lambda)$ and $b^*(\mathbf{p},\lambda)$, such that:

(1) Under an arbitrary Lorentz transformation $x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu$ the field transforms according to some representation $D[\Lambda]$ of the homogeneous Lorentz group

$$U[\Lambda,a]\psi_n(x)U^{-1}[\Lambda,a] = \textstyle\sum_m D_{nm}[\Lambda^{-1}]\psi_m(\Lambda x + a)\,. \tag{2.4}$$

(2) The field commutes with its adjoint at space-like separations

$$[\psi_n(x),\psi_m{}^\dagger(y)] = 0 \quad \text{for} \quad (x-y)^2 > 0\,. \tag{2.5}$$

Our metric has signature $+++-$. (Fields satisfying

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[7] R. Utiyama, Phys. Rev. **101**, 1597 (1956), and T. W. B. Kibble, J. Math. Phys. **2**, 212 (1961).

[8] W. E. Thirring, Ann. Phys. (N.Y.) **16**, 96 (1961); V. I. Ogievetsky and I. V. Polubarinov, Ann. Phys. (N. Y.) **25**, 358 (1963). For earlier work on similar lines see M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939), and M. Fierz, Helv. Phys. Acta **12**, 3 (1939). One sometimes encounters a very simple version of such arguments, to the effect that the current J^μ must obviously be conserved if we *define* it as $J^\mu \equiv -\partial_\mu F^{\mu\nu}$. But this does not say that J^μ is the same as the current $\mathcal{J}^\mu \equiv -\delta H'/\delta A_\mu$ to which A_μ is coupled in the interaction Hamiltonian. In fact, we will see explicitly at the end of Sec. VI that if $\mathcal{J}^\mu$ is not conserved than J^μ is *not* equal to $\mathcal{J}^\mu$.

these two conditions can be coupled together to form a causal scalar Hamiltonian density, yielding a Lorentz-invariant S matrix.)

In order to show that all such free fields may be derived from (2.1), we shall pursue the following line of argument:

(A) We first show that $F_\pm(x)$ are themselves fields, by proving that they are tensors.

(B) We then study the algebraic properties of $F_\pm(x)$, and show that they have at most $(2j+1)$ linearly independent components.

(C) We then use the results of (A) and (B) to show that $F_+(x)$ and $F_-(x)$ are just the simple fields $\chi_\sigma(x)$ and $\varphi_\sigma(x)$ introduced in Ref. 3.

(D) We finally remark that any irreducible free field may be obtained by differentiating these simple fields a suitable number of times.

a. Tensor Behavior of $F_\pm(x)$

The behavior of the polarization vectors $e_\pm{}^\mu(\mathbf{p})$ under an arbitrary Lorentz transformation $\Lambda^\mu{}_\nu$ was shown in Appendix A of Ref. 4 to be

$$\{\Lambda_\nu{}^\mu - p^\mu\Lambda_\nu{}^0/|\mathbf{p}|\}e_\pm{}^\nu(\Lambda\mathbf{p}) = \exp[\pm i\Theta(\mathbf{p},\Lambda)]e_\pm{}^\mu(\mathbf{p}) \quad (2.6)$$

with Θ an angle whose precise definition need not concern us here. Furthermore, the annihilation operator $a(\mathbf{p},\lambda)$ for a massless particle of helicity λ and momentum $\mathbf{p}$ was shown in Ref. 3 to obey the transformation law

$$\begin{aligned} U[\Lambda]a(\mathbf{p},\lambda)U^{-1}[\Lambda] \\ = (|\Lambda\mathbf{p}|/|\mathbf{p}|)^{1/2}\exp[i\lambda\Theta(\mathbf{p},\Lambda)]a(\Lambda\mathbf{p},\lambda) \end{aligned} \quad (2.7)$$

with Θ the same angle as in (2.6). The creation operator $b^*(\mathbf{p},-\lambda)$ transforms like $a(\mathbf{p},\lambda)$. Using (2.6) and (2.7) in (2.1) shows instantly that $F_\pm(x)$ are tensors,

$$\begin{aligned} U(\Lambda,a)F_\pm{}^{[\mu_1\nu_1]\cdots[\mu_j\nu_j]}(x)U^{-1}[\Lambda,a] \\ = \Lambda_{\rho_1}{}^{\mu_1}\Lambda_{\eta_1}{}^{\nu_1}\cdots\Lambda_{\rho_j}{}^{\mu_j}\Lambda_{\eta_j}{}^{\nu_j}F_\pm{}^{[\rho_1\eta_1]\cdots[\rho_j\eta_j]}(\Lambda x+a). \end{aligned} \quad (2.8)$$

The causal character of $F_\pm(x)$ can be deduced directly from the fact that $a(\mathbf{p},\pm j)\exp(ip\cdot x)$ and $b^*(\mathbf{p},\mp j)\times\exp(-ip\cdot x)$ enter with equal coefficients in (2.1).

b. Algebraic Properties of $F_\pm(x)$

It follows from (2.2) and (2.3) that the polarization vectors $e_\pm{}^\mu(\mathbf{p})$ have the algebraic properties

$$p_\mu e_\pm{}^\mu(\mathbf{p}) = 0\,, \quad (2.9)$$

$$e_{\pm\mu}(\mathbf{p})e_\pm{}^\mu(\mathbf{p}) = 0\,, \quad (2.10)$$

$$\epsilon^{\mu\nu\rho\eta}p_\rho e_{\pm\eta}(\mathbf{p}) = \mp i[p^\mu e_\pm{}^\nu(\mathbf{p}) - p^\nu e_\pm{}^\mu(\mathbf{p})]\,, \quad (2.11)$$

where $\epsilon^{\mu\nu\rho\eta}$ is the totally antisymmetric tensor with $\epsilon^{0123}\equiv 1$. [We prefer to use a four-vector notation, even though $e_\pm{}^0(\mathbf{p})\equiv 0$.]

Inspection of (2.1) and (2.9)–(2.11) shows that $F_\pm(x)$ obeys the algebraic conditions:

(*i*) *Symmetry*. $F_\pm(x)$ are symmetric under interchange of any two index pairs $[\mu_r\nu_r]\leftrightarrow[\mu_s\nu_s]$.

(*ii*) *Antisymmetry*. $F_\pm(x)$ are antisymmetric under interchanges $\mu_r\leftrightarrow\nu_r$ within any one index pair:

$$F_\pm{}^{[\mu_1\nu_1]\cdots} = -F_\pm{}^{[\nu_1\mu_1]\cdots}. \quad (2.12)$$

(*iii*) *Duality*. $F_+(x)$ and $F_-(x)$ are, respectively, self-dual or anti-self-dual with respect to each index pair $[\mu_r\nu_r]$:

$$\epsilon^{\mu\nu\mu_1\nu_1}F_{\pm[\mu_1\nu_1]}{}^{[\mu_2\nu_2]\cdots} = \mp 2iF_\pm{}^{[\mu\nu][\mu_2\nu_2]\cdots}. \quad (2.13)$$

(*iv*) *Tracelessness*. The complete contraction of any pair of indices $[\mu_r\nu_r]$, $[\mu_s\nu_s]$ gives zero:

$$g_{\mu_1\mu_2}g_{\nu_1\nu_2}F_\pm{}^{[\mu_1\nu_1][\mu_2\nu_2]\cdots} = 0. \quad (2.14)$$

It is also true that any single trace vanishes:

$$g_{\mu_1\mu_2}F_\pm{}^{[\mu_1\nu_1][\mu_2\nu_2]\cdots} = 0$$

but this follows from (*i*)–(*iv*), and will therefore not be listed as an independent condition. Conditions (*i*) and (*iv*) are of course empty for $j=1$.

These four conditions imply that the $F_\pm(x)$ each have at most $2j+1$ independent components. Condition (*ii*) lowers the number of independent values taken by each index pair $[\mu_r\nu_r]$ from 16 to 6, and (*iii*) lowers it further to 3, so under (*i*), (*ii*), and (*iii*) alone the number of independent components would be the same as for a symmetric tensor of rank j in three dimensions, i.e.,

$$N_j = \binom{j+2}{2} = \tfrac{1}{2}(j+1)(j+2).$$

But condition (*iv*) imposes N_{j-2} further constraints, so the net number of independent components is at most

$$N_j - N_{j-2} = 2j+1.$$

c. Identification of $F_\pm(x)$

In Sec. III of Ref. 3, we showed that the only free fields which can be formed out of the operators

$$a(\mathbf{p},\pm j)e^{ip\cdot x} + b^*(\mathbf{p},\mp j)e^{-ip\cdot x}$$

must transform under the homogeneous Lorentz group as a direct sum of those $(2A+1)(2B+1)$-dimensional irreducible representations (A,B) with

$$B - A = \pm j. \quad (2.15)$$

Indeed, the irreducible fields are determined uniquely by the representation (A,B) under which they transform, as

$$\begin{aligned} \psi_{ab}{}^{AB}(x) = (2\pi)^{-3/2}\int d^3p(2|\mathbf{p}|)^{A+B-1/2} \\ \times D_{a,-A}{}^{(A)}[R(\hat{p})]D_{b,B}{}^{(B)}[R(\hat{p})] \\ \times[a(\mathbf{p},\pm j)e^{ip\cdot x} + b^*(\mathbf{p},\mp j)e^{-ip\cdot x}], \end{aligned} \quad (2.16)$$

where $D^{(J)}[R(\hat{p})]$ is the usual $(2J+1)$-dimensional unitary representation of the rotation $R(\hat{p})$ which takes the z axis into the direction of $\mathbf{p}$, and the indices a and b run by unit steps from $-A$ to $+A$ and $-B$ to $+B$, respectively. [These remarks apply for half-integer j as well as integer j.]

We have learned from Secs. IIa and b above that $F_\pm(x)$ transform according to some reducible or irreducible representation of the homogeneous Lorentz group, with dimensionality at most $2j+1$. But of the irreducible representations (A,B) satisfying (2.15), the ones with the smallest dimensionality are the $2j+1$-dimensional representations $(j,0)$ for helicity $-j$, and $(0,j)$ for helicity $+j$. Hence $F_-(x)$ and $F_+(x)$ must transform purely according to the $(j,0)$ and $(0,j)$ representations, and since the representation uniquely determines the field they must be, respectively, just the $(2j+1)$-component fields $\varphi_\sigma(x)$ and $\chi_\sigma(x)$ introduced in Ref. 3. That is, the components of $F_-^{[\mu_1\nu_1]\cdots}(x)$ and $F_+^{[\mu_1\nu_1]\cdots}(x)$ are linear combinations of those of $\varphi_\sigma(x)$ and $\chi_\sigma(x)$, and vice versa. [The fields $\varphi_a(x)$ and $\chi_b(x)$ can be obtained from the general expression (2.16) by setting $B=0$ or $A=0$.]

d. Derivation of General Fields from $F_\pm(x)$

Let us examine the Lorentz transformation properties of the $2J$th derivatives (J integer or half-integer)

$$\partial_{\lambda_1}\cdots\partial_{\lambda_{2J}}F_\pm^{[\mu_1\nu_1]\cdots[\mu_j\nu_j]}(x). \tag{2.17}$$

This object is a symmetric traceless tensor with respect to the λ indices, so it transforms according to the representation

$$F_+:\ (J,J)\otimes(0,j)=(J,j+J)\oplus\cdots\oplus(J,|j-J|), \tag{2.18}$$

$$F_-:\ (J,J)\otimes(j,0)=(j+J,J)\oplus\cdots\oplus(|j-J|,J). \tag{2.19}$$

The only terms in these Clebsch-Gordan series that satisfy (2.15) are the first, so (2.17) transforms according to the representations $(J, j+J)$ for F_+ and $(j+J, J)$ for F_-. By letting J run over all integers and half-integers we can construct any representation (A,B) satisfying (2.15), so any free field can be built up as direct sums of (2.17). [The only possible flaw in this argument would arise if one of the $(J, j+J)$ or $(j+J, J)$ terms vanished, but then (2.17) would vanish, and this is clearly impossible.]

Incidentally, the same method of proof applies for $m\neq 0$, to show that the most general (A,B) field can be obtained by projecting out the appropriate part of

$$\partial_{\lambda_1}\cdots\partial_{\lambda_{2B}}\varphi_\sigma(x)$$

or

$$\partial_{\lambda_1}\cdots\partial_{\lambda_{2A}}\chi_\sigma(x),$$

where $\varphi_\sigma(x)$ and $\chi_\sigma(x)$ are the $(2j+1)$-component free fields constructed for massive particles in Ref. 2. [In this case the (A,B) part cannot vanish because $\varphi_\sigma(x)$ and $\chi_\sigma(x)$ cannot obey any homogeneous field equations.]

In contrast with the massive particle case, the massless free fields $F_\pm(x)$ obey homogeneous field equations which just express the absence of those terms in (2.18) and (2.19) which do not satisfy (2.15). The simplest such equation may be deduced directly from (2.1) and (2.9):

$$\partial_{\mu_1}F_\pm^{[\mu_1\nu_1]\cdots}=0. \tag{2.20}$$

For instance, for $j=1$ the algebraic properties noted under Sec. IIb let us write

$$F_\pm^{[ij]}=\epsilon_{ijk}(E^k\pm iB^k),$$
$$F_\pm^{[0k]}=\pm i(E^k\pm iB^k),$$

so (2.20) gives

$$\nabla\times[\mathbf{E}\pm i\mathbf{B}]=\pm i\frac{\partial}{\partial t}[\mathbf{E}\pm i\mathbf{B}],$$

$$\nabla\cdot[\mathbf{E}\pm i\mathbf{B}]=0,$$

or

$$\nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t,\quad \nabla\times\mathbf{B}=\partial\mathbf{E}/\partial t,$$
$$\nabla\cdot\mathbf{E}=0,\quad \nabla\cdot\mathbf{B}=0,$$

justifying the identification of $\mathbf{E}$ and $\mathbf{B}$ with the free electric and magnetic fields. A similar argument allows us to identify the five independent components of $F_\pm^{[\mu\nu][\lambda\eta]}$ with the left- or right-handed parts of the source-free Riemann-Christoffel tensor.

It should perhaps be stressed that *up to this point* we have done little but put the work of Ref. 3 into tensor notation.

III. POTENTIALS

After having shown that any free field can be constructed from $F_\pm(x)$ and its derivatives, we might feel justified in trying to construct the interaction Hamiltonians for photons and gravitons out of $F_\pm^{[\mu\nu]}$ and $F_\pm^{[\mu\nu][\lambda\eta]}$. *But this does not work.* Inspection of (2.1) shows that the amplitude for emitting or absorbing a massless particle of spin j by a field $F_\pm(x)$ will vanish like $p^{j-1/2}$ for momentum $p\to 0$. Hence an interaction built out of $F_\pm(x)$ could never give rise to the phenomena most closely associated with electromagnetism and gravitation, i.e., long-range forces and infrared divergences.[9] [Using other free fields would be even worse; we can see from (2.16) that the amplitudes yielded by a field of type (A,B) would vanish as $p^{A+B-1/2}$ for $p\to 0$, and (2.15) gives $A+B\geqq j$. This is of course because such fields can be written as the $2A$th derivative of $F_+(x)$ or the $2B$th derivative of $F_-(x)$.]

Instead of yielding to despair at this point, let us ignore the results of Sec. II for a moment, and try to strip away the objectionable factor of p^j in $F_\pm(x)$, by writing these fields as jth derivatives of other objects. We note by inspection of Eq. (2.1) that the $F_\pm(x)$ can

[9] By "infrared divergence" here we mean that the amplitude for internal bremsstrahlung of a soft photon or graviton is dominated by a term that behaves like ω^{-1} for $\omega\to 0$. See e.g., Ref. 4.

be written as generalized curls of the potentials

$$A_{\pm}^{\mu_1\cdots\mu_j}(x)\equiv(2\pi)^{-3/2}\int d^3p(2|\mathbf{p}|)^{-1/2}e_{\pm}^{\mu_1}(\mathbf{p})\cdots e_{\pm}^{\mu_j}(\mathbf{p})$$
$$\times[a(\mathbf{p},\pm j)e^{ip\cdot x}+(-)^j b^*(\mathbf{p},\mp j)e^{-ip\cdot x}]. \quad (3.1)$$

That is,

$$F_{\pm}^{[\mu\nu]}=\partial^\mu A_{\pm}^{\nu}-\partial^\nu A_{\pm}^{\mu}, \quad (3.2)$$

$$F_{\pm}^{[\mu\nu][\lambda\eta]}=\partial^\mu\partial^\lambda A_{\pm}^{\nu\eta}-\partial^\mu\partial^\eta A_{\pm}^{\nu\lambda}-\partial^\nu\partial^\lambda A_{\pm}^{\mu\eta}+\partial^\nu\partial^\eta A_{\pm}^{\mu\lambda}, \quad (3.3)$$

and so on. The potentials $A_{\pm}(x)$ which are symmetric in their μ indices, vanish if any one μ index is zero

$$A_{\pm}^{0\mu_2\cdots\mu_j}=0 \quad (3.4)$$

and have zero trace

$$g_{\mu_1\mu_2}A_{\pm}^{\mu_1\mu_2\cdots\mu_j}=0. \quad (3.5)$$

They satisfy the free-field equations

$$\partial_{\mu_1}A_{\pm}^{\mu_1\cdots\mu_j}=0 \quad (3.6)$$

$$\pm i\epsilon^{\nu_1\mu_1\nu\mu}\partial_\nu A_{\pm\mu}{}^{\mu_2\cdots\mu_j}=\partial^{\nu_1}A_{\pm}^{\mu_1\mu_2\cdots\mu_j}-\partial^{\mu_1}A_{\pm}^{\nu_1\mu_2\cdots\mu_j} \quad (3.7)$$

and of course

$$\Box^2 A_{\pm}^{\mu_1\cdots\mu_j}=0. \quad (3.8)$$

The discussion of Sec. III makes it clear that the $A_{\pm}(x)$ cannot be fields, in the sense of (2.4) and (2.5). Indeed, we can see that they are not even tensors, because their time-like components vanish; if they were tensors then they would transform according to the $(\frac{1}{2}j,\frac{1}{2}j)$ representation of the homogeneous Lorentz group, and this representation does not satisfy the fundamental condition (2.15) for fields constructed from the operators $a(\mathbf{p},\pm j)$ and $b^*(\mathbf{p},\mp j)$.

But $F_{\pm}(x)$ are tensors, so the noncovariance of $A_{\pm}$ must be manifested in the appearance of gradient terms in the Lorentz transformation law for the potentials,[10] which do not show up when we take curls to obtain $F_{\pm}(x)$. In fact, this is the case. A simple calculation using (3.1), (2.6), and (2.7) shows that

$$U[\Lambda]A_{\pm}^{\mu_1\cdots\mu_j}(x)U^{-1}[\Lambda]$$
$$=\Lambda_{\nu_1}{}^{\mu_1}\cdots\Lambda_{\nu_j}{}^{\mu_j}(2\pi)^{-3/2}\int d^3p(2|\mathbf{p}|)^{-1/2}$$
$$\times[e_{\pm}^{\nu_1}(\mathbf{p})-p^{\nu_1}f_{\pm}(\mathbf{p},\Lambda)]\cdots[e_{\pm}^{\nu_j}(\mathbf{p})-p^{\nu_j}f_{\pm}(\mathbf{p},\Lambda)]$$
$$\times[a(\mathbf{p},\pm j)e^{ip\cdot\Lambda x}+(-)^j b^*(\mathbf{p},\mp j)e^{-ip\cdot\Lambda x}] \quad (3.9)$$

with

$$f_{\pm}(\mathbf{p},\Lambda)=\Lambda_\nu{}^0 e_{\pm}^{\nu}(\mathbf{p})/|\Lambda^{-1}\mathbf{p}|. \quad (3.10)$$

[10] See e.g., J. Schwinger, Phys. Rev. **74**, 1439 (1948); **127**, 324 (1964).

This can be written

$$U[\Lambda]A_{\pm}^{\mu_1\cdots\mu_j}(x)U^{-1}[\Lambda]=\Lambda_{\nu_1}{}^{\mu_1}\cdots\Lambda_{\nu_j}{}^{\mu_j}A_{\pm}^{\nu_1\cdots\nu_j}(\Lambda x)$$
$$+\sum_{r=1}^{j}\partial^{\mu_r}\Phi_{\pm}^{\mu_1\cdots\mu_{r-1}\mu_{r+1}\cdots\mu_j}(x;\Lambda). \quad (3.11)$$

We will fortunately not need the rather complicated explicit formulas for $\Phi_{\pm}$. However, for an infinitesimal Lorentz transformation $\Lambda^\mu{}_\nu=\delta^\mu{}_\nu+\omega^\mu{}_\nu$ the functions $f_{\pm}(\mathbf{p},\Lambda)$ are infinitesimal

$$f_{\pm}(\mathbf{p},1+\omega)=\omega_{0i}e_{\pm}^{i}(\mathbf{p})/|\mathbf{p}| \quad (3.12)$$

and $\Phi_{\pm}$ is then given by the simple expression

$$\Phi_{\pm}^{\nu_1\cdots}(x;1+\omega)=i(2\pi)^{-3/2}\int d^3p(2|\mathbf{p}|)^{-1/2}$$
$$\times f_{\pm}(\mathbf{p},1+\omega)e_{\pm}^{\nu_1}(\mathbf{p})\cdots$$
$$\times[a(\mathbf{p},\pm j)e^{ip\cdot x}-(-)^j b^*(\mathbf{p},\mp j)e^{-ip\cdot x}]. \quad (3.13)$$

We will find it convenient from now on to shift our attention from $A_{\pm}(x)$ to the potentials $A(x)$ and $B(x)$, defined by

$$A^{\mu_1\cdots\mu_j}(x)\equiv A_{+}^{\mu_1\cdots\mu_j}(x)+A_{-}^{\mu_1\cdots\mu_j}(x) \quad (3.14)$$

$$iB^{\mu_1\cdots\mu_j}(x)\equiv A_{+}^{\mu_1\cdots\mu_j}(x)-A_{-}^{\mu_1\cdots\mu_j}(x). \quad (3.15)$$

A particle that interacts with left- and right-handed particles with the same coupling constant will be coupled only to $A(x)$, while one that has coupling constants of opposite sign to left- and right-handed quanta will interact only with $B(x)$. Hence an ordinary charge will couple only to $A^\mu(x)$, while a magnetic monopole will couple to $B^\mu(x)$. The two fields can be distinguished by their different behavior [11]under parity (P) and time-reversal (T):

$$PA^{\mu_1\cdots\mu_j}(x)P^{-1}=(-)^j A^{\mu_1\cdots\mu_j}(-\mathbf{x},t), \quad (3.16)$$

$$PB^{\mu_1\cdots\mu_j}(x)P^{-1}=-(-)^j B^{\mu_1\cdots\mu_j}(-\mathbf{x},t), \quad (3.17)$$

$$TA^{\mu_1\cdots\mu_j}(x)T^{-1}=(-)^j A^{\mu_1\cdots\mu_j}(\mathbf{x},-t), \quad (3.18)$$

$$TB^{\mu_1\cdots\mu_j}(x)T^{-1}=-(-)^j B^{\mu_1\cdots\mu_j}(\mathbf{x},-t). \quad (3.19)$$

In Sec. VIII we will discuss reasons why nature has not made use of $B(x)$ in forming the interactions of massless particles.

Since the photon and graviton are both purely neutral, we will surrender a little of our extreme generality, by

[11] These are derived using the P and T behavior of the operators $a(\mathbf{p},\lambda)$ and $b^*(\mathbf{p},\lambda)$, as worked out in Sec. IX of Ref. 3. In order to obtain the particular sign changes given here for P it is necessary to adjust the relative phases of $a(\mathbf{p},+j)$ and $a(\mathbf{p},-j)$, while the sign change under T can be obtained by adjusting the over-all phase of both operators. The important thing is that the P and T sign changes are both opposite for $A(x)$ and $B(x)$, because P interchanges helicities $\pm j$, and because T is antiunitary.

restricting ourselves to the case of particles identical with their antiparticles. It will be convenient to define phases so that

$$b(\mathbf{p},\lambda)=(-)^j a(\mathbf{p},\lambda). \tag{3.20}$$

We note that $e_\pm{}^{\mu*}=e_\mp{}^\mu$, so now we have

$$A_\pm{}^{\mu_1\cdots\mu_j\dagger}=A_\mp{}^{\mu_1\cdots\mu_j}. \tag{3.21}$$

Therefore (3.14) and (3.15) give $A(x)$ and $B(x)$ as Hermitian operators:

$$A^{\mu_1\cdots\mu_j}(x)=A^{\mu_1\cdots\mu_j\dagger}(x)=(2\pi)^{-3/2}\int d^3p(2|\mathbf{p}|)^{-1/2}\sum_\pm e_\pm{}^{\mu_1}(\mathbf{p})\cdots e_\pm{}^{\mu_j}(\mathbf{p})[a(\mathbf{p},\pm j)e^{ip\cdot x}+a^*(\mathbf{p},\mp j)e^{-ip\cdot x}] \tag{3.22}$$

$$B^{\mu_1\cdots\mu_j}(x)=B^{\mu_1\cdots\mu_j\dagger}(x)=-i(2\pi)^{-3/2}\int d^3p(2|\mathbf{p}|)^{-1/2}\sum_\pm \pm e_\pm{}^{\mu_1}(\mathbf{p})\cdots e_\pm{}^{\mu_j}(\mathbf{p})[a(\mathbf{p},\pm j)e^{ip\cdot x}+a^*(\mathbf{p},\mp j)e^{-ip\cdot x}]\,. \tag{3.23}$$

IV. LORENTZ INVARIANCE AND CURRENT CONSERVATION

The potentials $A(x)$ and $B(x)$ are not tensors, and cannot be made into tensors by redefiniton of the polarization vectors, because condition (2.15) does not allow us to construct $(\frac{1}{2}j,\frac{1}{2}j)$ tensor fields of rank j for massless particles of helicity $\pm j$. On the other hand, the true tensor fields $F_\pm(x)$ of rank $2j$ (and any other truly covariant fields) give amplitudes for emission and absorption of soft quanta which vanish at least as fast as $p^{j-1/2}$ for momentum $p\to 0$, in contradiction to our everyday experience with photons and gravitons. There seem to be just two available methods for the circumvention of this difficulty:

1. The traditional approach[5] is to introduce operators for fictitious particles of helicity other than $\pm j$ (some with negative probabilities) in such a way that $A(x)$ and $B(x)$ become true tensor fields.

2. Alternatively, we can resign ourselves to the non-tensor character of $A(x)$ and $B(x)$, but construct the interaction Hamiltonian so that the S matrix is nevertheless Lorentz-invariant.

I will follow the second path. One reason is that no one likes unphysical particles, or the indefinite metric and subsidiary state-vector conditions that they entail. But, more significant, the lack of manifest Lorentz covariance in the second approach means that we must impose powerful restrictions on the interaction Hamiltonian in order to obtain a Lorentz-invariant S matrix. This feature is a minor nuisance if we are sure we already know the correct theory of photons and gravitons, but it becomes all-important if what we want is an *a priori* derivation of electrodynamics and general relativity.

So we must ask what sort of couplings we can give $A(x)$ and $B(x)$ without violating the Lorentz invariance of the S matrix. For the moment, we will assume that only $A(x)$ enters in the interaction, (e.g., no magnetic monopoles) and will return to the more general case in Sec. VIII.

A Lorentz transformation $\Lambda^\mu{}_\nu$ induces on the potential $A(x)$ a combined tensor and "gauge" transformation

$$U[\Lambda]A^{\mu_1\cdots\mu_j}(x)U^{-1}[\Lambda]=\Lambda_{\nu_1}{}^{\mu_1}\cdots\Lambda_{\nu_j}{}^{\mu_j}A^{\nu_1\cdots\nu_j}(\Lambda x)+\sum_{r=1}^{j}\partial^{\mu_r}\Phi^{\mu_1\cdots\mu_{r-1}\mu_{r+1}\cdots\mu_j}(x;\Lambda) \tag{4.1}$$

with $\Phi=\Phi_++\Phi_-$. [See Eq. (3.11)]. The potential appears in the interaction Hamiltonian $H'(t)$ coupled to a current

$$\mathcal{J}_{i_1\cdots i_j}(x)\equiv-\delta H'(t)/\delta A^{i_1\cdots i_j}(x)\,, \tag{4.2}$$

but when we sum to all orders of perturbation theory the matrix elements for creation or annihilation of real or virtual massless particles are determined by the current in the Heisenberg representation

$$\mathcal{J}^H{}_{i_1\cdots i_j}(x)\equiv\exp(iHx^0)\mathcal{J}_{i_1\cdots i_j}(0)\exp(-iHx^0)\,. \tag{4.3}$$

The form of the two terms in (4.1) then leads us to guess that the Lorentz invariance of the S matrix requires $\mathcal{J}$ to have the properties

(a): $\mathcal{J}_{i_1\cdots i_j}(x)$ is the spatial part of a symmetric tensor $\mathcal{J}_{\mu_1\cdots\mu_j}(x)$,

$$U[\Lambda]\mathcal{J}_{\mu_1\cdots\mu_j}(x)U^{-1}[\Lambda]=\Lambda_{\mu_1}{}^{\nu_1}\cdots\Lambda_{\mu_j}{}^{\nu_j}\mathcal{J}_{\nu_1\cdots\nu_j}(\Lambda x)\,. \tag{4.4}$$

(b): $\mathcal{J}^H{}_{\mu_1\cdots\mu_j}$ is conserved

$$\partial^{\mu_1}\mathcal{J}^H{}_{\mu_1\cdots\mu_j}(x)=0\,. \tag{4.5}$$

We will remove most of the guesswork in the next section, but let us accept (A) and (B) for the moment as necessary requirements for Lorentz invariance.

There are two familiar types of conserved symmetric tensor: for $j=1$ there are the currents J^μ of additively conserved quantities such as charge and baryon number, and for $j=2$ there is the symmetric stress-energy tensor $\theta^{\mu\nu}$. In addition, it is easy to construct conserved currents of the "Pauli"-type for any j:

$$\mathcal{J}_{\text{Pauli}}{}^{\mu_1\cdots\mu_j}(x)=\partial_{\nu_1}\cdots\partial_{\nu_j}\Sigma^{[\mu_1\nu_1]\cdots[\mu_j\nu_j]}(x)\,, \tag{4.6}$$

where Σ is any tensor antisymmetric within each index pair $[\mu,\nu]$ and symmetric between different index pairs. A familiar example for $j=1$ is the Pauli-moment current

$$\mathcal{J}^\mu{}_{\text{Pauli}}\propto\partial_\nu(\bar\psi\sigma^{\mu\nu}\psi)\,.$$

However, coupling the potential $A(x)$ to the current (4.6) is equivalent to coupling the tensor field $F(x)$ to $\Sigma(x)$, and cannot by itself give finite amplitudes for producing or absorbing very soft massless particles. In particular, the "charge" carried by (4.6) vanishes, i.e.,

$$\int d^3x\,\mathcal{J}_{\text{Pauli}}{}^{0\,\mu_2\cdots\mu_j}(x)=0\,.$$

The only currents which avoid this criticism are the charge (or baryon number, etc.) current J^μ and the stress-energy tensor $\theta^{\mu\nu}$. Hence we conclude that *Lorentz invariance forces the photon potential $A^\mu(x)$ to be coupled to $J^\mu(x)$, and the graviton potential $A^{\mu\nu}(x)$ to be coupled to $\theta^{\mu\nu}(x)$*, except that in both cases there is the possibility of adding extra terms like (4.6) to J^μ and $\theta^{\mu\nu}$, or equivalently, of adding interactions involving the covariant fields $F^{[\mu\nu]}(x)$ or $F^{[\mu\nu][\lambda\eta]}(x)$.

In fact, nature does not seem to take its option of using terms like (4.6) in the interaction currents of massless particles. For the photon we have clear evidence of this in the success of Dirac's calculation of the magnetic moment of the electron. And also, the very absence of massless particles with $j\geqq 3$ is symptomatic of nature's abhorrence of Pauli-type currents, since these are the only currents with which such particles could interact. For photons the absence of Pauli couplings is sometimes referred to as the "principle of minimal electromagnetic coupling," but it remains a mystery nonetheless. Perhaps the solution will be found in considerations of high-energy behavior, since the Pauli currents are worse in this respect than J^μ and $\theta^{\mu\nu}$, and, in particular, can never give renormalizable interactions.

It seems fairly obvious that the statements that A^μ couples only to J^μ and $A^{\mu\nu}$ couples only to $\theta^{\mu\nu}$ (except in both cases for possible Pauli terms) are equivalent, respectively, to gauge invariance of the second kind and to Einstein's equivalence principle. We will not pursue this point further here, as it would lead us into the Lagrangian formalism, which we have been so far successful in avoiding. Instead, we will give a direct derivation of Maxwell's and Einstein's equations in the Heisenberg representation, in Secs. VI and VII.

V. LORENTZ INVARIANCE OF THE FEYNMAN RULES

In order to understand better what conditions are actually necessary and sufficient for the Lorentz invariance of the S matrix, we will now examine the Feynman rules generated by formula (3.22) for the potentials $A(x)$. We have already remarked in Ref. 4 that the requirements (4.4), (4.5) for a conserved tensor current are sufficient for the Lorentz invariance of S-matrix elements with external massless particle lines (provided that the covariance of matrix elements of J^μ is not spoiled by the internal massless particle lines) and that these conditions are also necessary at least on the light cone in momentum space. Our remaining task is to examine the Lorentz transformation properties of the internal massless particle lines.

The coordinate-space propagator of the field $A(x)$ is easily calculated as

$$\langle T\{A^{\mu_1\cdots\mu_j}(x)A^{\nu_1\cdots\nu_j}(y)\}\rangle_0 = (2\pi)^{-3}\int\frac{d^3p}{2|\mathbf{p}|}\Pi^{\mu_1\cdots\mu_j\nu_1\cdots\nu_j}(\mathbf{p}) \times[\theta(x-y)e^{ip\cdot(x-y)}+\theta(y-x)e^{ip\cdot(y-x)}] \quad (5.1)$$

with

$$\Pi^{\mu_1\cdots\mu_j\nu_1\cdots\nu_j}(\mathbf{p}) =\sum\nolimits_\pm e_\pm^{\mu_1}(\mathbf{p})\cdots e_\pm^{\mu_j}(\mathbf{p})e_\pm^{\nu_1}(\mathbf{p})^*\cdots e_\pm^{\nu_j}(\mathbf{p})^*. \quad (5.2)$$

In momentum space the propagator is

$$\Delta_c^{\mu_1\cdots\mu_j\nu_1\cdots\nu_j}(q)\equiv i\int d^4x e^{-iq\cdot x}\langle T\{A^{\mu_1\cdots\mu_j}(x),A^{\nu_1\cdots\nu_j}(y)\}\rangle_0 =\Pi^{\mu_1\cdots\mu_j\nu_1\cdots\nu_j}(\mathbf{q})/(q^2-i\epsilon). \quad (5.3)$$

For $j=1$ we easily calculate (in Appendix A)

$$\Pi^{\mu\nu}(\mathbf{q})=g^{\mu\nu}+n^\mu\hat{q}^\nu+n^\nu\hat{q}^\mu-\hat{q}^\mu\hat{q}^\nu$$
$$n^\mu=\{0,0,0,1\} \quad (5.4)$$
$$\hat{q}^\mu=\{\hat{q},1\}.$$

In order to express (5.4) in terms of a non-light-like q^μ, we set

$$\hat{q}^\mu=[q^\mu+n^\mu(|\mathbf{q}|-q^0)]/|\mathbf{q}| \quad (5.5)$$

and we obtain

$$\Pi^{\mu\nu}(q)=g^{\mu\nu}+((n^\mu q^\nu+n^\nu q^\mu)q^0/|\mathbf{q}|^2) -(q^\mu q^\nu/|\mathbf{q}|^2)+(q^2n^\mu n^\nu/|\mathbf{q}|^2). \quad (5.6)$$

Hence the propagator may be written as the sum of three terms

$$\Delta_c^{\mu\nu}(q)=\Delta^{\mu\nu}{}_{\mathrm{cov}}(q)+\Delta^{\mu\nu}{}_{\mathrm{grad}}(q)+\Delta^{\mu\nu}{}_{\mathrm{loc}}(q). \quad (5.7)$$

The first term $\Delta^{\mu\nu}{}_{\mathrm{cov}}(q)$ is the usual covariant tensor propagator

$$\Delta^{\mu\nu}{}_{\mathrm{cov}}(q)=g^{\mu\nu}/(q^2-i\epsilon). \quad (5.8)$$

The second term $\Delta^{\mu\nu}{}_{\mathrm{grad}}$ is not covariant, but it is proportional to factors q^μ or q^ν which give zero[12] when multiplied into the conserved currents connected by $\Delta_c^{\mu\nu}$. The final term $\Delta^{\mu\nu}{}_{\mathrm{loc}}$ is also not covariant, but it is characterized by the absence of the pole at $|q^0|=|\mathbf{q}|$:

$$\Delta^{\mu\nu}{}_{\mathrm{loc}}(q)=n^\mu n^\nu/|\mathbf{q}|^2. \quad (5.9)$$

Hence it gives a coordinate-space propagator that is

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[12] This is easy to prove in electrodynamics, where the current does not involve the potential; see R. P. Feynman, Phys. Rev. **101**, 769 (1949), Sec. 8. [This result is also implicit in the theorem proved here in Appendix B.] The situation is enormously more complicated in the case of gravitation, where the "current" must involve the potential $A^{\mu\nu}$; we will not attempt a treatment of this highly nontrivial problem here.

local in time:

$$\frac{1}{(2\pi)^4}\int d^4q\, e^{iq\cdot(x-y)}\Delta^{\mu\nu}{}_{\mathrm{loc}}(q) = \delta(x^0-y^0)\mathfrak{D}(\mathbf{x}-\mathbf{y})n^\mu n^\nu, \quad (5.10)$$

$$\mathfrak{D}(\mathbf{x})=(2\pi)^{-3}\int d^3q\, \exp(i\mathbf{q}\cdot\mathbf{x})|\mathbf{q}|^{-2}=1/4\pi|\mathbf{x}-\mathbf{y}| \quad (5.11)$$

and it may therefore be cancelled by the addition to $H'(t)$ of the familiar Coulomb interaction

$$H_{\mathrm{Coul}}'(t)=\frac{1}{2}\int d^3x\int d^3y\, \mathcal{J}^0(\mathbf{x},t)\mathfrak{D}(\mathbf{x}-\mathbf{y})\mathcal{J}^0(\mathbf{y},t). \quad (5.12)$$

Note that this cancellation is only possible because (5.12) is temporally local; the interaction *must* be local in time because of its definition as

$$H'(t)=e^{iH^ft}H'(0)e^{-iH^ft}. \quad (5.13)$$

In particular, $\Delta^{\mu\nu}{}_{\mathrm{grad}}(q)$ does not have a temporally local Fourier transform so it cannot be cancelled by adding a term to $H'(t)$, and it must be eliminated by requiring the current $\mathcal{J}_\mu{}^H$ to be conserved.

All this is familiar for $j=1$, and it works out much the same for $j\geqq 2$, because the general polarization sum $\Pi(q)$ is built up out of the $\Pi^{\mu\nu}$. For example

$j=2$ [see Appendix A]

$$\Pi^{\mu_1\mu_2\nu_1\nu_2}(q)=\tfrac{1}{2}[\Pi^{\mu_1\nu_1}(q)\Pi^{\mu_2\nu_2}(q)+\Pi^{\mu_1\nu_2}(q)\Pi^{\mu_2\nu_1}(q)-\Pi^{\mu_1\mu_2}(q)\Pi^{\nu_1\nu_2}(q)], \quad (5.14)$$

$j=3$

$$\begin{aligned}\Pi^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}=\tfrac{1}{3}[&\Pi^{\mu_1\nu_1}\Pi^{\mu_2\nu_2}\Pi^{\mu_3\nu_3}+\Pi^{\mu_1\nu_2}\Pi^{\mu_2\nu_3}\Pi^{\mu_3\nu_1}+\Pi^{\mu_1\nu_3}\Pi^{\mu_2\nu_1}\Pi^{\mu_3\nu_1}+\Pi^{\mu_1\nu_2}\Pi^{\mu_2\nu_1}\Pi^{\mu_3\nu_3}+\Pi^{\mu_1\nu_3}\Pi^{\mu_1\nu_2}\Pi^{\mu_3\nu_1}\\&+\Pi^{\mu_1\nu_1}\Pi^{\mu_2\nu_3}\Pi^{\mu_3\nu_2}]-\tfrac{1}{6}[\Pi^{\mu_1\nu_1}\Pi^{\mu_2\mu_3}\Pi^{\nu_2\nu_3}+\Pi^{\mu_2\nu_1}\Pi^{\mu_1\mu_3}\Pi^{\nu_2\nu_3}+\Pi^{\mu_3\nu_1}\Pi^{\mu_1\mu_2}\Pi^{\nu_2\nu_3}+\Pi^{\mu_1\nu_2}\Pi^{\mu_2\mu_3}\Pi^{\nu_1\nu_3}\\&+\Pi^{\mu_1\nu_2}\Pi^{\mu_1\mu_3}\Pi^{\nu_1\nu_2}+\Pi^{\mu_3\nu_2}\Pi^{\mu_1\mu_2}\Pi^{\nu_1\nu_3}+\Pi^{\mu_1\nu_2}\Pi^{\mu_2\mu_3}\Pi^{\nu_1\nu_2}+\Pi^{\mu_2\nu_2}\Pi^{\mu_1\mu_2}\Pi^{\nu_1\nu_2}+\Pi^{\mu_3\nu_3}\Pi^{\mu_1\mu_2}\Pi^{\nu_1\nu_2}],\end{aligned} \quad (5.15)$$

and so on. Evidently the propagator for any integral j can be decomposed as in (5.7), into a covariant part Δ_{cov} built up out of the $g_{\mu\nu}$, plus a noncovariant part Δ_{grad} proportional to one or more factors of q_μ, plus a noncovariant part Δ_{loc} which lacks the pole at $q^2=0$. The last term is to be cancelled by adding a temporally local term to $H'(t)$. The second term Δ_{grad} is not temporally local, so it must be eliminated by requiring that $A(x)$ be coupled to a conserved current.

For instance, Eqs. (5.3), (5.6), and (5.14) give the three parts of the $j=2$ propagator as

$$\Delta_{\mathrm{cov}}{}^{\mu_1\mu_2\nu_1\nu_2}(q)=[g^{\mu_1\nu_1}g^{\mu_2\nu_2}+g^{\mu_1\nu_2}g^{\mu_2\nu_1}-g^{\mu_1\mu_2}g^{\nu_1\nu_2}]/2(q^2-i\epsilon), \quad (5.16)$$

$$\Delta_{\mathrm{grad}}{}^{\mu_1\mu_2\nu_1\nu_2}(q)=\left[g^{\mu_1\nu_1}+\frac{n^{\mu_1}q^{\nu_1}+n^{\nu_1}q^{\mu_1}}{2|\mathbf{q}|^2}q^0+\frac{q^2n^{\mu_1}n^{\nu_1}}{|\mathbf{q}|^2}\right]\times\frac{(n^{\mu_2}q^{\nu_2}+n^{\nu_2}q^{\mu_2})}{2|\mathbf{q}|^2(q^2-i\epsilon)}q^0 \quad (5.17)$$

$\pm$ five similar terms.

$$\begin{aligned}\Delta_{\mathrm{loc}}{}^{\mu_1\mu_2\nu_1\nu_2}(q)=[&g^{\mu_1\nu_1}n^{\mu_2}n^{\nu_2}+g^{\mu_2\nu_2}n^{\mu_1}n^{\nu_1}\\&+g^{\mu_1\nu_2}n^{\mu_2}n^{\nu_1}+g^{\mu_2\nu_1}n^{\mu_1}n^{\nu_2}-g^{\mu_1\mu_2}n^{\nu_1}n^{\nu_2}\\&-g^{\nu_1\nu_2}n^{\mu_1}n^{\mu_2}]/2|\mathbf{q}|^2+n^{\mu_1}n^{\nu_1}n^{\mu_2}n^{\nu_2}q^2/2|\mathbf{q}|^4.\end{aligned} \quad (5.18)$$

The gradient term (5.17) does not contribute if we require the "current" $\mathcal{J}_{\mu\nu}{}^H(x)$ to be conserved.[12] The term (5.18) gives a temporally local contribution to the propagator

$$\begin{aligned}(2\pi)^{-4}\int d^4q\, &e^{iq\cdot(x-y)}\Delta_{\mathrm{loc}}{}^{\mu_1\mu_2\nu_1\nu_2}(q)\\=\tfrac{1}{2}[&g^{\mu_1\nu_1}n^{\mu_2}n^{\nu_2}+g^{\mu_2\nu_2}n^{\mu_1}n^{\nu_1}+g^{\mu_1\nu_2}n^{\mu_2}n^{\nu_1}\\&+g^{\mu_2\nu_1}n^{\mu_1}n^{\nu_2}-g^{\mu_1\mu_2}n^{\nu_1}n^{\nu_2}-g^{\nu_1\nu_2}n^{\mu_1}n^{\mu_2}\\&+n^{\mu_1}n^{\nu_1}n^{\mu_2}n^{\nu_2}]\delta(x^0-y^0)\mathfrak{D}(\mathbf{x}-\mathbf{y})\\&+\tfrac{1}{2}n^{\mu_1}n^{\nu_1}n^{\mu_2}n^{\nu_2}\ddot{\delta}(x^0-y^0)\mathcal{E}(\mathbf{x}-\mathbf{y}),\end{aligned} \quad (5.19)$$

where $\mathfrak{D}(\mathbf{x})$ is given by (5.11), and

$$\mathcal{E}(\mathbf{x})\equiv(2\pi)^{-3}\int d^3q\, \exp(i\mathbf{q}\cdot\mathbf{x})|\mathbf{q}|^{-4}=\mathcal{E}(0)-\frac{|\mathbf{x}|}{8\pi}. \quad (5.20)$$

[We will see that the divergent constant $\mathcal{E}(0)$ gives no trouble.] In order to cancel (5.19) we must add to the Hamiltonian a "Newtonian" term:

$$\begin{aligned}H'_{\mathrm{Newt}}(t)=\frac{1}{2}\int d^3xd^3y[&2\mathcal{J}^\mu{}_0(\mathbf{x},t)\mathcal{J}_{\mu 0}(\mathbf{y},t)\\&-\tfrac{1}{2}\mathcal{J}^\mu{}_\mu(\mathbf{x},t)\mathcal{J}_{00}(\mathbf{y},t)-\tfrac{1}{2}\mathcal{J}_{00}(\mathbf{x},t)\mathcal{J}^\mu{}_\mu(\mathbf{y},t)\\&+\tfrac{1}{2}\mathcal{J}_{00}(\mathbf{x},t)\mathcal{J}_{00}(\mathbf{y},t)]\mathfrak{D}(\mathbf{x}-\mathbf{y})\\&+\frac{1}{2}\int d^3xd^3y\mathcal{J}_{00}(\mathbf{x},t)\ddot{\mathcal{J}}_{00}(\mathbf{y},t)\mathcal{E}(\mathbf{x}-\mathbf{y}).\end{aligned} \quad (5.21)$$

In Sec. VII we shall see that this term, ugly as it seems, is precisely what is needed to generate Einstein's field equations when we pass to the Heisenberg representation.

The conclusion suggested by the above is that the conservation and covariance of the current plus the presence of direct-interaction terms like H'_{Coul} and H'_{Newt}, are together the necessary and sufficient conditions for the Lorentz invariance of the S matrix. In Appendix B we show that these conditions do in fact imply the Lorentz invariance of the S matrix in quantum electrodynamics.[6] Our proof of their sufficiency makes their necessity rather evident, and can also obviously be extended to any massless particle theory in which the potential does not itself appear in the current. The rigorous treatment of Lorentz invariance in cases like the gravitational or the Yang-Mills field where the potential must appear in the current requires a much more elaborate discussion, and I reserve this for a future paper. [The problem has lost some of its urgency, because we have already seen in Ref. 4 that very simple and general arguments insure that any Lorentz-invariant theory of massless particles with $j=1$ or $j=2$ must possess the most striking dynamical features of photons and gravitons, to wit, the conservation of charge and the equality of gravitational and inertial mass.]

VI. DERIVATION OF MAXWELL'S EQUATIONS

The space-components of the vector potential $A_H{}^\mu(x)$ in the Heisenberg representation are defined, as usual, by

$$A_H{}^i(x)\equiv U(x^0)A^i(x)U^{-1}(x^0)\,, \tag{6.1}$$

$$U(t)\equiv\exp(iHt)\exp(-iH^f t) \tag{6.2}$$

with H^f the free-particle Hamiltonian and $H=H^f+H'$ the total Hamiltonian. The interaction-representation potential $A^i(x)$ is explicitly given by

$$A^i(x)=(2\pi)^{-3/2}\int d^3p(2|\mathbf{p}|)^{-1/2}\sum_\pm e_\pm{}^i(\mathbf{p})\times[a(\mathbf{p},\pm1)e^{ip\cdot x}+a^*(\mathbf{p},\mp1)e^{-ip\cdot x}] \tag{6.3}$$

so it satisfies the field equations

$$\Box^2A^i(x)=0\,, \tag{6.4}$$

$$\partial_iA^i(x)=0\,, \tag{6.5}$$

and the commutation reltions

$$[A^i(x),A^j(y)]=0\,, \tag{6.6}$$

$$[\dot{A}^i(x),\dot{A}^j(y)]=0\,, \tag{6.7}$$

$$[A^i(x),\dot{A}^j(y)]=i\mathfrak{D}^{ij}(\mathbf{x}-\mathbf{y})\,, \tag{6.8}$$

with

$$\mathfrak{D}^{ij}(\mathbf{x}-\mathbf{y})=(2\pi)^{-3}\int d^3p\ \Pi^{ij}(p)\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})]=\delta_{ij}\delta^3(\mathbf{x}-\mathbf{y})+\partial_i\partial_j\mathfrak{D}(\mathbf{x}-\mathbf{y})\,. \tag{6.9}$$

The lemma proved in Appendix C thus allows us immediately to write down the field equation for A^i in the Heisenberg representation:

$$\Box^2A_H{}^i(\mathbf{x},t)=-\int d^3\mathbf{y}\ \mathfrak{D}^{ij}(\mathbf{x}-\mathbf{y})\mathcal{J}_H{}^j(\mathbf{y},t)=-\mathcal{J}_H{}^i(\mathbf{x},t)-\partial_i\partial_j\int d^3\mathbf{y}\ \mathfrak{D}(\mathbf{x}-\mathbf{y})\mathcal{J}_H{}^j(\mathbf{y},t)\,. \tag{6.10}$$

However, the response of one system of charges to another system cannot be described solely in terms of the three-vector field $A_H{}^i(x)$, because there is also a direct Coulomb interaction (5.12) between the two systems. For instance, it is easy to show that the S matrix for a transition $\alpha\to\beta$ caused by an infinitesimal c-number current $\delta\mathcal{J}^\mu(x)$ is, to first order in $\delta\mathcal{J}$

$$S_{\beta\alpha}=i\int d^4x\langle\beta\text{ out}|A_H{}^i(x)|\alpha\text{ in}\rangle\delta\mathcal{J}_i(x)-i\int_{-\infty}^{\infty}dt\int d^3x\int d^3y\langle\beta\text{ out}|\mathcal{J}_H{}^0(\mathbf{x},t)|\alpha\text{ in}\rangle\times\mathfrak{D}(\mathbf{x}-\mathbf{y})\delta\mathcal{J}^0(\mathbf{y},t)\,. \tag{6.11}$$

Therefore we invent a fourth component of $A_H{}^\mu(x)$

$$A_H{}^0(\mathbf{x},t)\equiv\int d^3\mathbf{y}\ \mathfrak{D}(\mathbf{x}-\mathbf{y})\mathcal{J}_H{}^0(\mathbf{y},t) \tag{6.12}$$

which enables us to write an expression like (6.11) compactly as

$$S_{\beta\alpha}=i\int d^4x\langle\beta\text{ out}|A_H{}^\mu(x)|\alpha\text{ in}\rangle\delta\mathcal{J}_\mu(x)\,. \tag{6.13}$$

The field $A_H{}^0$ obeys the Poisson equation

$$\nabla^2A_H{}^0(x)=-\mathcal{J}_H{}^0(x)\,. \tag{6.14}$$

Also, (6.12) and the current conservation condition (4.5) let us write (6.10) as

$$\Box^2A_H{}^i(x)=-\mathcal{J}_H{}^i(x)+\partial_0\partial_iA_H{}^0(x)\,. \tag{6.15}$$

Together (6.14) and (6.15) yield Maxwell's equations

$$\partial_\mu F_H{}^{\mu\nu}(x)=-\mathcal{J}_H{}^\nu(x) \tag{6.16}$$

$$F_H{}^{\mu\nu}(x)\equiv\partial^\mu A_H{}^\nu(x)-\partial^\nu A_H{}^\mu(x)\,. \tag{6.17}$$

The particular form of (6.14) and (6.15) arises because (6.1) and (6.5) impose on $A_H{}^\mu(x)$ the Coulomb gauge condition

$$\partial_iA_H{}^i(x)=0\,. \tag{6.18}$$

It may be of interest to note that in the absence of current conservation (6.15) would become

$$\Box^2A_H{}^i(x)=-\mathcal{J}_H{}^i(x)+\partial_0\partial_iA_H{}^0(x)-\partial_i\int d^3\mathbf{y}\ \mathfrak{D}(\mathbf{x}-\mathbf{y})\partial_\mu\mathcal{J}_H{}^\mu(\mathbf{y},t)$$

and Maxwell's equations would read

$$\partial_\mu F_H{}^{\mu i}(x)=-\mathcal{J}_H{}^i(x)+\partial_i\int d^3\mathbf{y}\ \mathfrak{D}(\mathbf{x}-\mathbf{y})\partial_\mu\mathcal{J}_H{}^\mu(\mathbf{y},t)$$

$$\partial_\mu F_H{}^{\mu0}(x)=-\mathcal{J}_H{}^0(x)\,.$$

The crucial importance of current conservation for Lorentz invariance is apparent again in these field equations.

VII. DERIVATION OF EINSTEIN'S EQUATIONS

The traceless part of the spatial components of the gravitational field $A_H{}^{\mu\nu}(x)$ in the Heisenberg representation are defined in the same way as we have defined $A_H{}^i$ in (6.1), i.e.,

$$A_H{}^{ij}(x)-\tfrac{1}{3}\delta^{ij}\delta_{kl}A_H{}^{kl}(x)\equiv U(x^0)A^{ij}(x)U^{-1}(x^0) \quad (7.1)$$

with the interaction representation potential $A^{ij}(x)$ given explicitly by

$$A^{ij}(x)=(2\pi)^{-3/2}\int d^3p(2|\mathbf{p}|)^{-1/2}\sum_{\pm} e_{\pm}{}^i(\mathbf{p})e_{\pm}{}^j(\mathbf{p})$$
$$\times[a(\mathbf{p},\pm2)e^{ip\cdot x}+a^*(\mathbf{p},\mp2)e^{-ip\cdot x}]. \quad (7.2)$$

Note that A^{ij} is traceless because $(\mathbf{e}_{\pm})^2=0$. Also, $A^{ij}(x)$ satisfies the field equations

$$\Box^2A^{ij}(x)=0, \quad (7.3)$$

$$\partial_iA^{ij}(x)=0, \quad (7.4)$$

and the commutation relations

$$[A^{ij}(x),A^{kl}(y)]=0, \quad (7.5)$$

$$[\dot{A}^{ij}(x),\dot{A}^{kl}(y)]=0, \quad (7.6)$$

$$[A^{ij}(x),\dot{A}^{kl}(y)]=i\mathfrak{D}^{ij,kl}(\mathbf{x}-\mathbf{y}), \quad (7.7)$$

with

$$\mathfrak{D}^{ij,kl}(\mathbf{x}-\mathbf{y})=(2\pi)^{-3}\int d^3p\,\Pi^{ijkl}(\mathbf{p})\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})]. \quad (7.8)$$

Equation (5.14) gives the polarization sum for $j=2$ as

$$\Pi^{ijkl}(\mathbf{p})=\tfrac{1}{2}[(\delta^{ik}-\hat{p}^i\hat{p}^k)(\delta^{jl}-\hat{p}^j\hat{p}^l)+(\delta^{il}-\hat{p}^i\hat{p}^l)(\delta^{jk}-\hat{p}^j\hat{p}^k)-(\delta^{ij}-\hat{p}^i\hat{p}^j)(\delta^{kl}-\hat{p}^k\hat{p}^l)]$$

so

$$\mathfrak{D}^{ij,kl}(\mathbf{x}-\mathbf{y})=\tfrac{1}{2}[\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}-\delta^{ij}\delta^{kl}]\delta^3(\mathbf{x}-\mathbf{y})$$
$$+\tfrac{1}{2}[\partial^i\partial^k\delta^{jl}+\partial^j\partial^l\delta^{ik}+\partial^i\partial^l\delta^{jk}+\partial^j\partial^k\delta^{il}-\partial^i\partial^j\delta^{kl}-\partial^k\partial^l\delta^{ij}]\mathfrak{D}(\mathbf{x}-\mathbf{y})$$
$$+\tfrac{1}{2}\partial^i\partial^j\partial^k\partial^l\mathcal{E}(\mathbf{x}-\mathbf{y}), \quad (7.9)$$

$$\mathfrak{D}(\mathbf{x})=1/4\pi|\mathbf{x}|,$$
$$\mathcal{E}(\mathbf{x})=\mathcal{E}(0)-|\mathbf{x}|/8\pi.$$

The lemma of Appendix C thus gives us the field equations satisfied by (7.1)

$$\Box^2[A_H{}^{ij}(\mathbf{x},t)-\tfrac{1}{3}\delta^{ij}\delta_{kl}A_H{}^{kl}(\mathbf{x},t)]=-\int d^3y\,\mathfrak{D}^{ij,kl}(\mathbf{x}-\mathbf{y})\mathcal{J}_{H,kl}(\mathbf{y},t) \quad (7.10)$$

or more explicitly

$$\Box^2[A_H{}^{ij}(\mathbf{x},t)-\tfrac{1}{3}\delta^{ij}\delta_{kl}A_H{}^{kl}(\mathbf{x},t)]=-\mathcal{J}_H{}^{ij}(\mathbf{x},t)+\tfrac{1}{2}\delta^{ij}\delta_{kl}\mathcal{J}_H{}^{kl}(\mathbf{x},t)-\int d^3y[\mathcal{J}_H{}^{ik}(\mathbf{y},t)\partial_k\partial^j+\mathcal{J}_H{}^{jk}(\mathbf{y},t)\partial_k\partial^i$$
$$-\tfrac{1}{2}\mathcal{J}^k{}_{Hk}(\mathbf{y},t)\partial_i\partial_j-\tfrac{1}{2}\mathcal{J}_H{}^{kl}(\mathbf{y},t)\delta^{ij}\partial_k\partial_l]\mathfrak{D}(\mathbf{x}-\mathbf{y})-\tfrac{1}{2}\partial^i\partial^j\partial_k\partial_l\int d^3y\,\mathcal{J}_H{}^{kl}(\mathbf{y},t)\mathcal{E}(\mathbf{x}-\mathbf{y}). \quad (7.11)$$

[The current $\mathcal{J}_{H,kl}$ in the Heisenberg representation is related to the interaction representation current $\mathcal{J}_{kl}=-\delta H'/\delta A^{kl}$ by the same unitary operator $U(t)$ as appears in (6.1) and (7.1).]

Just as the space components of the vector potential $A_H{}^\mu(x)$ had to be supplemented by a time component to represent the direct-Coulomb interaction, it is necessary now to invent auxiliary components of the Heisenberg representation gravitational field $A_H{}^{\mu\nu}(x)$ in order to take account of the direct Newtonian interaction (5.21). This interaction can be written

$$H'_{\mathrm{Newt}}(t)=\frac{1}{2}\int d^3xd^3y[2\mathcal{J}^i{}_0(\mathbf{x},t)\mathcal{J}_{i0}(\mathbf{y},t)-\tfrac{1}{2}\mathcal{J}^i{}_i(\mathbf{x},t)\mathcal{J}_{00}(\mathbf{y},t)-\tfrac{1}{2}\mathcal{J}_{00}(\mathbf{x},t)\mathcal{J}^i{}_i(\mathbf{y},t)-\tfrac{1}{2}\mathcal{J}_{00}(\mathbf{x},t)\mathcal{J}_{00}(\mathbf{y},t)]\mathfrak{D}(\mathbf{x}-\mathbf{y})$$
$$+\frac{1}{2}\int d^3xd^3y\,\mathcal{J}_{00}(\mathbf{x},t)\ddot{\mathcal{J}}_{00}(\mathbf{y},t)\mathcal{E}(\mathbf{x}-\mathbf{y}).$$

Therefore, we define

$$A_H{}^{i0}(\mathbf{x},t)\equiv\int d^3\mathbf{y}\,\mathcal{J}_H{}^{i0}(\mathbf{y},t)\mathfrak{D}(\mathbf{x}-\mathbf{y}), \quad (7.12)$$

$$A_{Hi}{}^i(\mathbf{x},t)\equiv\frac{3}{2}\int d^3\mathbf{y}\,\mathcal{J}_H{}^{00}(\mathbf{y},t)\mathfrak{D}(\mathbf{x}-\mathbf{y}), \quad (7.13)$$

$$A_H{}^{00}(\mathbf{x},t)\equiv\frac{1}{2}\int d^3\mathbf{y}[\mathcal{J}_{Hi}{}^i(\mathbf{y},t)+\mathcal{J}_H{}^{00}(\mathbf{y},t)\mathfrak{D}(\mathbf{x}-\mathbf{y})]-\frac{1}{2}\int d^3\mathbf{y}\,\ddot{\mathcal{J}}_H{}^{00}(\mathbf{y},t)\mathcal{E}(\mathbf{x}-\mathbf{y}). \quad (7.14)$$

With these definitions, the S matrix for a transition $\alpha\to\beta$ due to an infinitesimal c number $\delta\mathcal{J}_{\mu\nu}(x)$ is

$$S_{\beta\alpha}=-i\int d^4x\langle\beta\text{ out}|A_H{}^{\mu\nu}(x)|\alpha\text{ in}\rangle\delta\mathcal{J}_{\mu\nu}(x).$$

These synthetic field components obey the field equations

$$\nabla^2A_H{}^{i0}(x)=-\mathcal{J}_H{}^{i0}(x), \quad (7.15)$$

$$\nabla^2A_H{}^i{}_i(x)=-\tfrac{3}{2}\mathcal{J}_H{}^{00}(x), \quad (7.16)$$

$$\nabla^2A_H{}^{00}(x)=-\tfrac{1}{2}\mathcal{J}_{Hi}{}^i(x)-\tfrac{1}{2}\mathcal{J}_H{}^{00}(x)+\tfrac{1}{3}\ddot{A}_{Hi}{}^i(x). \quad (7.17)$$

Using the current conservation condition (4.5) let us write (7.11) as

$$\Box^2 A_H{}^{ij}(x) = -\mathcal{J}_H{}^{ij}(x) + \tfrac{1}{2}\delta^{ij}\mathcal{J}_H{}^{\mu}{}_{\mu}(x) + \partial_0\partial^j A_H{}^{i0}(x) + \partial_0\partial^i A_H{}^{j0}(x) + \partial^i\partial^j[A_H{}^{00}(x) - \tfrac{1}{3}A_H{}^k{}_k(x)]. \quad (7.18)$$

To the field equations (7.15)–(7.18) we must append two first-order equations, which remind us that we have defined the traceless part (7.1) of $A_H{}^{ij}$ to be divergenceless

$$\partial_i A_H{}^{ij}(x) = \tfrac{1}{3}\partial_j A_H{}^i{}_i(x) \quad (7.19)$$

and have defined $A_H{}^{i0}$ and $A_H{}^i{}_i$ in (7.12) and (7.13) so that the conservation of $\mathcal{J}_H{}^{\mu\nu}$ relates them by

$$\partial_i A_H{}^{i0}(x) = -\tfrac{2}{3}\partial_0 A_H{}^i{}_i(x). \quad (7.20)$$

Equations (7.15)–(7.20) can be put together compactly as

$$R_H{}^{\mu\nu}(x) = -\mathcal{J}_H{}^{\mu\nu}(x) + \tfrac{1}{2}g^{\mu\nu}\mathcal{J}_H{}^{\lambda}{}_{\lambda}(x), \quad (7.21)$$

where

$$R_H{}^{\mu\nu}(x) \equiv \Box^2 A_H{}^{\mu\nu}(x) - \partial^\mu\partial_\lambda A_H{}^{\lambda\nu}(x) - \partial^\nu\partial_\lambda A_H{}^{\mu\lambda}(x) + \partial^\mu\partial^\nu A_H{}^\lambda{}_\lambda(x). \quad (7.22)$$

The complicated form of (7.15)–(7.18) just arises from the fact that we happen to have defined $A_H{}^{\mu\nu}$ in the peculiar gauge characterized by (7.19) and (7.20). We might have avoided some algebra along the way had we chosen a different polarization "tensor" in forming the potential (7.2), but the choice we made was the most obvious generalization of the Coulomb gauge used for $j=1$, and at any rate has brought us safely to our goal.

Equation (7.21) can also be put in the familiar form

$$R_H{}^{\mu\nu}(x) - \tfrac{1}{2}g^{\mu\nu}R_H{}^\lambda{}_\lambda(x) = -\mathcal{J}_H{}^{\mu\nu}(x). \quad (7.23)$$

If $\mathcal{J}_H{}^{\mu\nu}(x)$ were proportional to the energy-momentum tensor of matter alone then (7.23) would be identical with Einstein's equations in the weak field limit, where we set the Einstein metric tensor equal to the Minkowski $g^{\mu\nu}$ plus our $A_H{}^{\mu\nu}$, and keep only terms of first order in $A_H{}^{\mu\nu}$. However, such a theory would not be Lorentz invariant, because Lorentz invariance requires that $\partial_\mu\mathcal{J}_H{}^{\mu\nu}=0$, and this condition is fulfilled only if the current $\mathcal{J}_H{}^{\mu\nu}$ contains terms involving $A_H{}^{\mu\nu}$, representing the energy and momentum density of gravitation. If we therefore identify $\mathcal{J}_H{}^{\mu\nu}$ with the full-energy momentum tensor $\theta^{\mu\nu}$ of matter plus gravitation, Eq. (7.23) becomes highly nonlinear. As remarked by Gupta,[13] there is obviously one choice of a conserved $\theta^{\mu\nu}$ which makes (7.23) equivalent to Einstein's nonlinear equations, namely that obtained by identifying the nonlinear terms on the left-hand-side of Einstein's equations with the negative of the gravitational part of $\theta^{\mu\nu}$. In fact, Feynman[14] has shown that this is the only choice which works. In Feynman's Lagrangian approach, Lorentz invariance is built in, but other desiderata of perturbation theory such as unitarity can be lost by making the wrong choice of $\theta^{\mu\nu}$, while in our approach unitarity and the particle interpretation are built in, and only Lorentz invariance can go wrong; therefore we may presume that the sort of covariance proof given in Appendix B for photons will only work for gravitons if we choose $\theta^{\mu\nu}$ in agreement with Einstein's theory. However, this still leaves an ambiguity in the matter part of $\theta^{\mu\nu}$, because we can always add "Pauli" terms such as (4.6).

[13] S. N. Gupta, Proc. Phys. Soc. **A65**, 608 (1952).

[14] R. P. Feynman (private communication). I am indebted to Professor Feynman for a discussion of this point.

VIII. MAGNETIC AND OTHER MONOPOLES

We saw in Sec. III that the particle operators for mass zero and integer spin j can be used to construct two different Hermitian potentials, a normal one $A^{\mu_1\cdots\mu_j}(x)$ with parity and time-reversal-phase $(-)^j$, and an abnormal one $B^{\mu_1\cdots\mu_j}(x)$ with P and T phases equal to $-(-)^j$. [See Eqs. (3.14)–(3.19).] Both $A(x)$ and $B(x)$ must be coupled to conserved tensor currents. However, the Hermitian current J^μ of charge (or baryon number, etc.) and the Hermitian energy-momentum tensor $\theta^{\mu\nu}(x)$ both have normal P and T phases, by which we mean that their spatial components obey the same P and T transformation rules (3.16) and (3.18) as for $A^i(x)$ and $A^{ij}(x)$. Therefore, both P and T invariance do not allow $B^\mu(x)$ and $B^{\mu\nu}(x)$ to be coupled to $J^\mu(x)$ or $\theta^{\mu\nu}(x)$. We could, of course, couple $B(x)$ to a Pauli current (4.6), but such interactions can be rewritten in terms of $A(x)$; for instance the coupling $B^\mu\partial^\nu[\bar\psi\gamma_5\sigma_{\mu\nu}\psi]$ is equivalent [using (3.7)] to $A^\mu\partial^\nu[\bar\psi\sigma_{\mu\nu}\psi]$. Hence, we would normally conclude from P or T invariance that all interactions may be expressed in terms of the normal potential $A(x)$, and in particular that there can be no magnetic monopoles.[15]

But there *is* one way that magnetic monopoles can occur without violating P or T. Suppose there is a particle which turns into its antiparticle under the operation of either parity[16] or time-reversal, and that the number of such particles is conserved. Then the Hermitian current $M^\mu(x)$ of the particle would undergo an extra sign change under P and T, and hence could be coupled to $B^\mu(x)$. Note that in this case P or T would forbid $A^\mu(x)$ from being coupled to $M^\mu(x)$; that is, a magnetic monopole cannot also carry a normal charge. Note also that we are *defining* P and T so that they act as usual on familiar particles like electrons and photons,

[15] The apparent violation of time-reversal invariance by magnetic monopoles has been noted by L. I. Schiff, Am. J. Phys., **32**, 812 (1964).

[16] This is sometimes expressed in the statement that the true symmetry is not P but PM, where M changes the sign of all magnetic monopole moments. See N. F. Ramsey, Phys. Rev. **109**, 225 (1959). We would prefer to say that M takes magnetic monopoles into their antiparticles, and include this in the definition of C, P, and T. (The product CPT takes all particles into their antiparticles, including magnetic monopoles.) This redefinition of T resolves the contradiction noted in Ref. 15.

and it is these ordinary inversions that take magnetic monopoles into their antiparticles; if all particles had this abnormal behavior under P and T we would just interchange the definitions of P and CP, T and CT, $A^{\mu}(x)$ and $B^{\mu}(x)$, charge and magnetic pole strength, etc.

In contrast, P or T do not allow the abnormal gravitational potential $B^{\mu\nu}(x)$ to interact with anything. Even if there were magnetic monopoles which went into their antiparticles under P and T, they would still make a contribution to the energy-momentum tensor $\theta^{\mu\nu}(x)$ which behaved normally under P and T, and which therefore could only be coupled to the normal potential $A^{\mu\nu}(x)$.

Since magnetic monopoles are allowed by C, P, and T, but are not observed in nature, we must ask if there is any other reason why they should not exist. Zwanziger[17] has noted that their existence would give the charge-monopole scattering amplitude $A(s,t)$ two very peculiar branch points in s near $t=0$. This suggests that field theories of photons, charges, and monopoles might be unavoidably acausal, and therefore, not Lorentz invariant. We now show that this is the case, at least within the interaction-representation dynamical framework used here.

The trouble arises in diagrams in which a photon is exchanged between a charge and monopole. Since the charge current $J_\mu(x)$ is coupled to $A^{\mu}(x)$ and the monopole current $M_\nu(y)$ is coupled to $B^{\nu}(y)$, the photon propagator will be

$$-i\Delta_{AB}{}^{\mu\nu}(q)=\int d^4x\, e^{-iq\cdot(x-y)}\langle T\{A^{\mu}(x),B^{\nu}(y)\}\rangle_0. \quad (8.1)$$

This can be easily calculated using (3.22) and (3.23) and the results of Appendix A we find

$$\Delta_{AB}{}^{\mu\nu}(q)=\frac{\Xi^{\mu\nu}(q)(q^0/|\mathbf{q}|)}{q^2-i\epsilon} \quad (8.2)$$

$$\Xi^{\mu\nu}(q)=i\sum\nolimits_{\pm}(\pm)e_{\pm}{}^{\mu}(\mathbf{q})e_{\pm}{}^{\nu}(\mathbf{q})^* = \epsilon^{\mu\nu\lambda\rho}q_\lambda n_\rho/|\mathbf{q}|\,. \quad (8.3)$$

This is not covariant, but more important, it cannot be split up as in (5.7) into a covariant part, a noncovariant gradient part which vanishes between conserved currents, and a noncovariant "local" part which can be cancelled by adding a temporally local term to $H'(t)$.

To see that this crucial decomposition is impossible for (8.3), note that the one-photon-exchange matrix element for scattering of a charge, with conserved current J_μ, and a monopole, with conserved current M_μ, is

$$J_\mu\Delta_{AB}{}^{\mu\nu}M_\nu=(q^0/|\mathbf{q}|)[(J_\mu J^\mu+\alpha J_0{}^2)(M_\mu M^\mu+\alpha M_0{}^2) -(J_\mu M^\mu+\alpha J_0M_0)^2]^{1/2}/(q^2-i\epsilon) \quad (8.4)$$

with

$$\alpha\equiv q^2/|\mathbf{q}|^2. \quad (8.5)$$

This may be compared with one-photon-exchange between two charges (or two monopoles)

$$J_\mu\Delta_{AA}{}^{\mu\nu}J_\nu'=[J_\mu J^{\mu\prime}+\alpha J_0J_0']/(q^2-i\epsilon). \quad (8.6)$$

In both cases the matrix element is invariant for q^μ precisely on the light cone ($\alpha=0$) but not otherwise. The great difference between (8.4) and (8.6) is that the α term in (8.6) can be cancelled by a temporally local interaction $J_0J_0'/|\mathbf{q}|^2$, while no similar cancellation is possible in (8.4).

Incidentally, the square root in (8.4) would yield Zwanziger's branch points[17] if we set $\alpha=0$. But the failure of analyticity is academic if the theory of monopoles isn't even Lorentz invariant.

There is one possible hope for saving Lorentz invariance. According to Dirac,[18] the coupling constant ge for charge-monopole interactions must be an integer or a half-integer. Perhaps the exact S matrix is Lorentz-invariant for these particular large values of ge, though not in any finite order of perturbation theory. However, preliminary examination of the ladder series by A. Goldhaber indicates that this is unlikely.

There is a possibility that time-reversal as well as parity is violated by the weak interactions. In this case, some of the conclusions reached earlier in this section might need revision. In particular, CPT alone would not prevent a particle from carrying a magnetic monopole moment as well as an ordinary charge, or in other words, of coupling with different strength to the left- and and right-handed parts of the electromagnetic field. And in the same way, all particles might respond with different coupling constants $f_\pm$ (the ratio of gravitational to inertial mass) to the left and right-handed parts of the gravitational field. However, this still would not produce observable anomalies in gravitational interactions, for Lorentz invariance tells us[4] that all particles must have the same f_+ and the same f_- (perhaps $\neq f_+$). The contribution of virtual graviton lines in Feynman diagrams would therefore be proportional to

$$\sum\nolimits_{\pm} f_\pm f_\mp\langle T\{A_\pm{}^{\mu}(x),A_\mp{}^{\nu}(y)\}\rangle_0=f_+f_-\langle T\{A^{\mu}(x),A^{\nu}(y)\}\rangle_0$$

and this has the same form as if the coupling constants $f_\pm$ for right- and left-handed gravitons were the same.

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APPENDIX A: THE POLARIZATION SUMS

We wish to evaluate the sums

$$\Pi^{\mu\nu}(\hat{q})=\sum\nolimits_{\pm} e_\pm{}^{\mu}(\hat{q})e_\pm{}^{\nu*}(\hat{q})\,, \quad (A.1)$$

$$\Xi^{\mu\nu}(\hat{q})=i\sum\nolimits_{\pm}(\pm)e_\pm{}^{\mu}(\hat{q})e_\pm{}^{\nu*}(\hat{q})\,, \quad (A.2)$$

$$\Pi^{\mu_1\mu_2\nu_1\nu_2}(\hat{q})=\sum\nolimits_{\pm} e_\pm{}^{\mu_1}(\hat{q})e_\pm{}^{\mu_2}(\hat{q})e_\pm{}^{\nu_1*}(\hat{q})e_\pm{}^{\nu_2*}(\hat{q})\,. \quad (A.3)$$

[17] D. Zwanziger (to be published).

[18] P. A. M. Dirac, Proc. Roy. Soc. (London) **133**, 60 (1931).

These are the numerators, respectively, of the photon propagator linking two charges or two monopoles, the photon propagator linking a charge and a monopole, and the graviton propagator.

First take $\hat{q}=\hat{k}$, defined as the unit vector in the z direction. Then the polarization is

$$e_\pm{}^1(\hat{k})=1/\sqrt{2}\,,\quad e_\pm{}^2(\hat{k})=\pm i/\sqrt{2}\,,\quad e_\pm{}^3(\hat{k})=e_\pm{}^0(\hat{k})=0\,,$$

so the only nonvanishing components of (A.1)–(A.3) are

$$\Pi^{11}=\Pi^{22}=1\,, \tag{A.4}$$

$$\Xi^{12}=-\Xi^{21}=1\,, \tag{A.5}$$

$$\Pi^{1111}=\Pi^{2222}=\Pi^{1212}=\Pi^{2112}=\Pi^{2121}=\Pi^{1221}=-\Pi^{1122}=-\Pi^{2211}=1/2\,. \tag{A.6}$$

For $\hat{q}=\hat{k}$, (A.4) agrees with (5.4) or (5.6), so it agrees with them for all $\hat{q}$, because $\Pi^{\mu\nu}(\hat{q})$ is related to $\Pi^{\mu\nu}(\hat{k})$ by the rotation $R^\mu{}_\lambda(\hat{q})$ which takes $\hat{k}$ into $\hat{q}$:

$$\Pi^{\mu\nu}(\hat{q})=R^\mu{}_\lambda(\hat{q})R^\nu{}_\rho(\hat{q})\Pi^{\lambda\rho}(\hat{k})\,. \tag{A.7}$$

A similar argument verifies Eq. (5.14) for $\Pi^{\mu_1\mu_2\nu_1\nu_2}(\hat{q})$ and verifies Eq. (8.3) for $\Xi^{\mu\nu}(\hat{q})$.

APPENDIX B: LORENTZ INVARIANCE OF THE QUANTUM-ELECTRODYNAMICAL S MATRIX

We shall show in a separate paper that if the interaction is translation and rotation invariant then the S matrix will be Lorentz invariant *if and only if* the behavior of the interaction under infinitesimal "boosts" takes the form

$$[\mathbf{K}^f,H'(t)]=-[\mathbf{K}'(t),H^f+H'(t)]\,. \tag{B.1}$$

Here $\mathbf{K}^f$ is the generator of pure Lorentz transformations on the free-particle states; H^f is the free-particle Hamiltonian, and $H'(t)$ is the interaction in the interaction representation

$$H'(t)=\exp(iH^ft)H'\exp(-iH^ft)\,.$$

The operator $\mathbf{K}'(t)$ is unrestricted, except that it must have the same t dependence as $H'(t)$:

$$\mathbf{K}'(t)=\exp(iH^ft)\mathbf{K}'\exp(-iH^ft) \tag{B.2}$$

with the free-particle matrix elements of K' sufficiently smooth functions of energy so that, effectively,

$$\mathbf{K}'(t)\to 0 \text{ for } t\to\pm\infty \tag{B.3}$$

this limit being understood in the same sense as the usual "adiabatic switching on and off" of $H'(t)$.

We will prove here that (B.1) is satisfied in the simplest case, i.e., quantum electrodynamics with an A-independent current:

$$H'(t)=-\int d^3x\,J_i(\mathbf{x},t)A^i(\mathbf{x},t)+\frac{1}{2}\int d^3x d^3y\,J^0(\mathbf{x},t)\mathfrak{D}(\mathbf{x}-\mathbf{y})J^0(\mathbf{y},t) \tag{B.4}$$

provided that the current is a vector and conserved in both the interaction and Heisenberg representations, i.e.,

$$\partial_\mu J^\mu(x)=0 \tag{B.5}$$

$$[H'(t),J^0(\mathbf{x},t)]=0\,. \tag{B.6}$$

(This is the case in spinor electrodynamics, and it can always be arranged by introducing enough auxiliary fields to make the free-field Lagrangian linear in space-time derivatives.)

The interaction (B.4) is manifestly translation- and rotation-invariant, so we need only check that it satisfies (B.1). The product $\mathbf{J}\cdot\mathbf{A}=J_\mu A^\mu$ is scalar except for the extra Φ term in Eq. (4.1), which for infinitesimal Lorentz transformations is given by (3.12) and (3.13) as

$$\Phi(x)=\Phi_+(x)+\Phi_-(x)=-i\omega_{i0}C^i(x)$$

$$C^i(x)\equiv(2\pi)^{-3/2}\int d^3p(2|\mathbf{p}|^3)^{-1/2}\sum_\pm e_\pm{}^i(\mathbf{p})\times[a(\mathbf{p},\pm1)e^{ip\cdot x}-a^*(\mathbf{p},\mp1)e^{-ip\cdot x}]\,. \tag{B.7}$$

Hence $\mathbf{J}\cdot\mathbf{A}$ transforms under infinitesimal boosts according to

$$i[\mathbf{K}^f,J_i(x)A^i(x)]=(x_0\nabla-\mathbf{x}\partial_0)J_i(x)A^i(x)+iJ_\mu(x)\partial^\mu\mathbf{C}(x)\,. \tag{B.8}$$

Also, since $J^\mu(x)$ is a vector we have

$$i[\mathbf{K}^f,J^0(x)]=\mathbf{J}(x)+(x_0\nabla-\mathbf{x}\partial_0)J^0(x)\,. \tag{B.9}$$

The ∇ terms drop out when we integrate over 3 space, leaving us with

$$\begin{aligned}[\mathbf{K}^f,H'(t)]=&-i\partial_0\int d^3x\,\mathbf{x}J_i(\mathbf{x},t)A^i(\mathbf{x},t)\\&-\int d^3x\,J_\mu(\mathbf{x},t)\partial^\mu\mathbf{C}(\mathbf{x},t)\\&-i\int d^3xd^3y\,\mathbf{J}(\mathbf{x},t)\mathfrak{D}(\mathbf{x}-\mathbf{y})J^0(\mathbf{y},t)\\&+i\int d^3xd^3y[\partial_0J^0(\mathbf{x},t)]\mathbf{x}\mathfrak{D}(\mathbf{x}-\mathbf{y})J^0(\mathbf{y},t)\,.\end{aligned}$$

Using (B.5), and writing $\mathbf{x}$ in the last term as $\frac{1}{2}(\mathbf{x}+\mathbf{y})+\frac{1}{2}(\mathbf{x}-\mathbf{y})$, we can put this in the form

$$[\mathbf{K}^f,H'(t)]=-i(d\mathbf{K}'(t)/dt)-L(t) \tag{B.10}$$

with

$$\mathbf{K}'(t)\equiv\int d^3x\,\mathbf{x}J_\mu(\mathbf{x},t)A^\mu(\mathbf{x},t)-i\int d^3x\,J^0(\mathbf{x},t)\mathbf{C}(\mathbf{x},t)-\frac{1}{2}\int d^3xd^3y\,J^0(\mathbf{x},t)\mathbf{x}\mathfrak{D}(\mathbf{x}-\mathbf{y})J^0(\mathbf{y},t)\,, \tag{B.11}$$

$$L_i(t)\equiv-i\int d^3xd^3y\,J_j(\mathbf{x},t)\mathfrak{F}_{ij}(\mathbf{x}-\mathbf{y})J^0(\mathbf{y},t)\,, \tag{B.12}$$

$$\mathfrak{F}_{ij}(\mathbf{x})\equiv\delta_{ij}\mathfrak{D}(\mathbf{x})-\tfrac{1}{2}\partial_j(x_i\mathfrak{D}(\mathbf{x}))\,. \tag{B.13}$$

Because of (B.6), the only term in $\mathbf{K}'(t)$ that does not commute with $H'(t)$ is that containing $\mathbf{C}(x)$, and therefore

$$[\mathbf{K}'(t),H'(t)]=i\int d^3xd^3y\,J_j(\mathbf{x},t)J^0(\mathbf{y},t)\times[A_j(\mathbf{x},t),\mathbf{C}(\mathbf{y},t)]\,. \tag{B.14}$$

But we can calculate directly from (B.7) and (6.3) that

$$[A_j(\mathbf{x},t),C_i(\mathbf{y},t)]=(2\pi)^{-3}\int d^3p\,|\mathbf{p}|^{-2}(\delta_{ij}-\hat{p}_i\hat{p}_j)\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})]$$

$$=\delta_{ij}\mathfrak{D}(\mathbf{x}-\mathbf{y})+\partial_i\partial_j\mathcal{E}(\mathbf{x}-\mathbf{y})=\mathfrak{F}_{ij}(\mathbf{x}-\mathbf{y}) \tag{B.15}$$

so (B.14) gives

$$[\mathbf{K}'(t),H'(t)]=-L(t)\,. \tag{B.16}$$

Also, $\mathbf{K}'(t)$ evidently has the t dependence (B.2), so (B.10) and (B.16) give (B.1). Condition (B.3) is satisfied because (B.11) is as "smooth" an operator as the interaction (B.4).

APPENDIX C: DERIVATION OF FIELD EQUATIONS IN THE HEISENBERG REPRESENTATION

Suppose the interaction representation fields $\phi_n(x)$ have the properties

$$\Box^2\phi_n(x)=0\,, \tag{C.1}$$

$$[\phi_n(\mathbf{x},t),\phi_m(\mathbf{y},t)]=0\,, \tag{C.2}$$

$$[\dot\phi_m(\mathbf{x},t),\dot\phi_m(\mathbf{y},t)]=0\,, \tag{C.3}$$

$$[\phi_n(\mathbf{x},t),\dot\phi_m(\mathbf{y},t)]=i\mathfrak{D}_{nm}(\mathbf{x}-\mathbf{y})\,. \tag{C.4}$$

[This is the case for the potentials $A(x)$ defined for general j in Sec. III.] Suppose that the interaction $H'(t)$ does not involve any derivatives of ϕ_n higher than the first, and define the "partial currents" $\mathcal{S}_n$, $\mathcal{S}_n{}^\mu$ by the statement

$$\delta H'(t)=-\int d^3x\sum_n[\mathcal{S}_n(\mathbf{x},t)\delta\phi_n(\mathbf{x},t)+\mathcal{S}_n{}^\mu(\mathbf{x},t)\partial_\mu\delta\partial_\mu\phi_n(\mathbf{x},t)] \tag{C.5}$$

where $\delta\phi_n$ and $\delta\partial_\mu\phi_n$ are arbitrary infinitesimal c-number variations of ϕ_n and $\partial_\mu\phi_n$. [If $H'(t)$ is the space integral of a local $\mathcal{H}(x)$, then $\mathcal{S}_n\equiv\partial\mathcal{H}/\partial\phi_n$ and $\mathcal{S}_n{}^\mu\equiv\partial\mathcal{H}/\partial(\partial_\mu\phi_n)$.] Define the Heisenberg representation field $\phi_{nH}(x)$ by

$$\phi_{nH}(\mathbf{x},t)\equiv U(t)\phi_n(\mathbf{x},t)U^{-1}(t) \tag{C.6}$$

$$U(t)\equiv\exp(iHt)\exp(-iH^ft)\,. \tag{C.7}$$

Then ϕ_H will obey the field equation

$$\Box^2\phi_{nH}(\mathbf{x},t)=-\int d^3\mathbf{y}\;\mathfrak{D}_{nm}(\mathbf{x}-\mathbf{y})J_{mH}(\mathbf{y},t) \tag{C.8}$$

with J_H defined as the total current

$$J_{nH}(x)\equiv\mathcal{S}_{nH}(x)-\partial_\mu\mathcal{S}_{nH}{}^\mu(x)\,, \tag{C.9}$$

$$\mathcal{S}_{nH}(\mathbf{x},t)\equiv U(t)\mathcal{S}_n(x)U^{-1}(t)\,, \tag{C.10}$$

$$\mathcal{S}_{nH}{}^\mu(\mathbf{x},t)\equiv U(t)\mathcal{S}_n{}^\mu(x)U^{-1}(t)\,. \tag{C.11}$$

Proof: We note first that

$$dU(t)/dt=iU(t)H'(t)$$

$$dU^{-1}(t)/dt=-iH'(t)U^{-1}(t)\,.$$

Therefore the time derivative of (C.6) gives

$$\dot\phi_{nH}(\mathbf{x},t)=U(t)\{\dot\phi_n(\mathbf{x},t)+i[H'(t),\phi_n(\mathbf{x},t)]\}U^{-1}(t)\,.$$

But (C.5), (C.2), and (C.4) give the commutator

$$[H'(t),\phi_n(\mathbf{x},t)]=+i\int d^3y\sum_m\mathcal{S}_m{}^0(\mathbf{y},t)\mathfrak{D}_{nm}(\mathbf{x}-\mathbf{y}) \tag{C.12}$$

so

$$\dot\phi_{nH}(\mathbf{x},t)=U(t)\dot\phi_n(\mathbf{x},t)U^{-1}(t)-\sum_m\int d^3y\;\mathcal{S}_{mH}{}^0(\mathbf{y},t)\mathfrak{D}_{nm}(\mathbf{x}-\mathbf{y})\,.$$

A second time derivative gives

$$\ddot\phi_{nH}(\mathbf{x},t)=U(t)\{\ddot\phi(\mathbf{x},t)+i[H'(t),\dot\phi_n(\mathbf{x},t)]\}U^{-1}(t)-\sum_m\int d^3y\;\partial_0\mathcal{S}_{mH}{}^0(\mathbf{y},t)\mathfrak{D}_{nm}(\mathbf{x}-\mathbf{y})\,. \tag{C.13}$$

But (C.5), (C.3), and (C.4) give the commutator

$$[H'(t),\dot\phi_n(\mathbf{x},t)]=-i\int d^3y\sum_m[\mathcal{S}_m(\mathbf{y},t)\mathfrak{D}_{nm}(\mathbf{y}-\mathbf{x})+\mathcal{S}_m{}^i(\mathbf{y},t)\partial_i\mathfrak{D}_{mn}(\mathbf{y}-\mathbf{x})]\,. \tag{C.14}$$

Using (C.14) and (C.1) and integrating by parts let us write (C.13) as

$$\ddot\phi_{nH}(\mathbf{x},t)=\nabla^2\phi_{nH}(\mathbf{x},t)+\int d^3y\sum_m[\mathfrak{D}_{mn}(\mathbf{y}-\mathbf{x})\mathcal{S}_{mH}(\mathbf{y},t)-\mathfrak{D}_{mn}(\mathbf{y}-\mathbf{x})\partial_i\mathcal{S}_{mH}(\mathbf{y},t)\,\mathfrak{D}_{nm}(\mathbf{x}-\mathbf{y})\partial_0\mathcal{S}_{mH}{}^0(\mathbf{y},t)]\,. \tag{C.15}$$

But differentiating (C.2) with respect to t gives

$$\mathfrak{D}_{nm}(\mathbf{y}-\mathbf{x})=\mathfrak{D}_{mn}(\mathbf{x}-\mathbf{y}) \tag{C.16}$$

so (C.15) and (C.9) yield the desired Eq. (C.8).